Multiparameter Mathematical Model for Nonlinear Fluid Filtration in a Three-Layer Hydrodynamically Connected Reservoir

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Abstract: This study develops a unified multiparameter mathematical model for describing nonlinear fluid filtration in a hydrodynamically connected three-layer reservoir with heterogeneous properties. The proposed framework is intended for anomalous and non-Newtonian fluids and integrates thirteen filtration-law variants, including Darcy, initial-gradient, polygonal, curvilinear, power-law, hyperbolic, generalized, and structured-fluid models. The main advantage of the model is that different boundary-value problems can be selected through a single parameterized formulation without reconstructing the mathematical statement for each individual filtration law. Special attention is given to moving disturbance boundaries, interlayer crossflows, and the interaction between highly permeable and weakly permeable layers. A computational algorithm based on iterative numerical procedures, including the flow sweep method and boundary-tracking approach, is proposed for solving the formulated nonlinear problem. Numerical experiments for representative filtration laws demonstrate stable convergence, reliable tracking of disturbance-boundary positions, and improved computational efficiency compared with conventional iterative techniques. The results show that curvilinear filtration laws provide a more physically consistent description of nonlinear filtration processes under low pressure-gradient conditions. The developed model serves as a compact computational tool for analyzing pressure distribution, disturbance-zone dynamics, and crossflow behavior in multilayer porous media. Crucially, due to its parameterized architecture and reduced iterative overhead, this framework is optimized for seamless integration into real-time Digital Twin software platforms and automated smart reservoir management systems.

1 INTRODUCTION

It is well known that numerous studies have been devoted to the development of mathematical models for filtration in subsurface formations, particularly for Newtonian fluid flow in single-layer and multilayer porous media [1]-[3]. Further advances in this field have led to the development of approaches for modeling nonlinear fluid filtration aimed at optimizing the modeling process through the construction of multiparameter mathematical models [4]-[8]. In these studies, a unified multiparameter mathematical framework was developed based on the formulation of a generalized multiparameter function. This framework incorporates various filtration laws together with their corresponding initial and boundary conditions. The introduction of additional parameters into the boundary conditions, including those defined on moving boundaries, ensured a one-to-one correspondence between the governing equations and their boundary-value formulations. The development of the multiparameter model enabled the creation of a unified parameterized multifunctional computational algorithm and the corresponding mathematical software framework. Analysis of the model provided a basis for classifying both the models and their solution algorithms according to the type of filtration law and the underlying physical processes.

As a result, researchers gained the ability to extract, from the general multiparameter structure, the specific boundary-value problem corresponding to the parameters of a given reservoir, without the need to construct a new model from scratch. Once an appropriate model is selected, it is sufficient to input the initial data of the reservoir and perform the necessary computational procedures within the framework of computational technology theory [9]-[11]. In the present work, this methodology is applied to a hydrodynamically interconnected multilayer system, specifically to a three-layer configuration. In the current era of industrial digitalization and cyber-physical systems, modern reservoir management heavily relies on automated software platforms and high-fidelity Digital Twins. However, a major bottleneck in deploying real-time cloud analytics or Edge Computing frameworks at smart well sites is the extreme computational complexity of traditional hydrodynamic simulators. To bridge this gap between reservoir engineering and information technology, there is an urgent need for compact, highly optimized computational frameworks. By unifying thirteen distinct physical filtration laws into a single parameterized algorithmic structure, the approach proposed in this study drastically reduces server workloads and computational runtime, making it directly applicable for automated control systems (SCADA) and digital oilfield infrastructure.

The key contributions and novel aspects of this research include:

- Development of a novel multi-parameter mathematical model specifically designed for multilayered media, representing a first-of-its-kind approach in the field.
- Design of advanced computational algorithms for simulating multi-parameter processes within multilayered systems.
- Classification of multi-parameter models and their respective solution algorithms according to diverse filtration laws, followed by successful validation using hypothetical datasets.

In contrast to earlier studies that mainly addressed individual filtration laws or simpler layered configurations, the present work integrates thirteen filtration-law variants into a single multiparameter formulation for a hydrodynamically interconnected three-layer reservoir. The novelty of the study lies not only in the unified mathematical statement, but also in the joint treatment of moving disturbance boundaries, interlayer crossflows, and algorithmic classification within one computational framework. This makes it possible to select the appropriate boundary-value formulation for a given reservoir configuration without reconstructing the model from the beginning for each separate filtration law.

2 METHODS

Consider a multilayer porous medium (Ω) whose filtration domain consists of three subdomains: Ω_1,Ω_2 and Ω_3. It is assumed that subdomain Ω_2 is highly permeable, with its horizontal filtration characteristics of the formation and fluid exceeding the vertical ones by an order of magnitude. Therefore, fluid motion in this region can be considered predominantly horizontal.

The other two subdomains, Ω_1 and Ω_3, are weakly permeable; in these layers, vertical filtration characteristics significantly exceed horizontal ones, allowing the fluid motion to be treated as predominantly vertical. It is further assumed that the top and bottom boundaries of Ω_2 are hydrodynamically connected with the bottom boundary of Ω_1 and the top boundary of Ω_3, respectively. This implies the existence of interlayer fluid exchange between the formations.

Let subdomain Ω_2 be saturated with nonlinear non-Newtonian fluids possessing anomalous rheological properties. At the onset of reservoir production in Ω_2, a disturbance is generated within the filtration domain, leading to fluid crossflow from subdomains Ω_1 and Ω_3, which are saturated with Newtonian fluids, across the entire contact boundary into Ω_2.

The magnitude of the interlayer crossflows depends on the rheological properties and velocity of the fluid, the production rate, as well as the filtration characteristics and heterogeneity of the boundary zones of the formation.

Accordingly, the formulated physical problem is mathematically reduced to determining continuous unknown functions $u(x,\overline{z},t)$, $V(\overline{x},z,t)$, along with the unknown moving boundaries $l_r(x,\overline{z},t)$, $R_r(x,\overline{z},t)$ satisfying the corresponding system of governing equations:

$$\frac{\partial}{\partial x}\left(\frac{hk(x)}{\mu}\Phi\left(\frac{|\nabla u|}{\beta}\right)\right)\frac{\partial u}{\partial x} = M(x,t,u)\frac{\partial u}{\partial t} + \psi_1\frac{\partial V_1}{\partial z}\bigg|_{z=h} + \\ +\psi_2\frac{\partial V_2}{\partial z}\bigg|_{z=h} + f(x,t), t>0, x\in\Omega_2, \quad (1)$$

and

$$\frac{\partial}{\partial z}\left(\frac{k(z)}{\mu_1}\frac{\partial V_1}{\partial z}\right) = M_i(\bar{x}, z, t)\frac{\partial V_1}{\partial t},$$
$$t > 0, z \in \Omega_i, i = (1,2), \quad (2)$$

with initial conditions:

$$u(x, \bar{z}, 0) = V_1(\bar{x}, z, 0) = V_2(\bar{x}, z, 0) = V_0(x, z), \quad (3)$$

as well as conditions on natural boundaries:

$$S\frac{hk(x)}{\mu b}\Phi\left(\frac{|\nabla u|}{\beta}\right)\frac{\partial u}{\partial x}\bigg|_{x=x_0} = \varphi_0(x_0, t), t > 0, \quad (4)$$

$$b\frac{hk(x)}{\mu b}\Phi\left(\frac{|\nabla u|}{\beta}\right)\frac{\partial u}{\partial x}\bigg|_{x=x_L} = \varphi_L(x_L, t), t > 0, \quad (5)$$

conditions at unknown moving boundaries:

$$a_1\,\Phi\left(\frac{|\nabla u|}{\beta}\right)\bigg|_{x=l_r^{\pm}} = 0, a_1(u(x,t)|_{x\in D} - u_0(x)) = 0, \quad (6)$$

$$x = l_r^{\pm} \pm 0, r = \{0, n\};\ (r = 0, x = l_0^{+} + 0, r = n, x = l_n^{-} - 0),$$
$$a_2\left(|\nabla u|\big|_{x=l_r^{\mp}\mp 0} = |\nabla u|\big|_{x=l_r^{\mp}\pm 0} = \beta\right),$$
$$a_2\left(u(x,t)|_{x=l_r^{\mp}\mp 0} - u(x,t)|_{x=l_r^{\mp}\pm 0}\right) = 0, r = \overline{0,n}. \quad (7)$$

$$a_3\left(|\nabla u|\big|_{x=l_r^{\mp}\mp 0} - \beta\right) = 0,$$
$$a_3(u(x,t)|_{x\in D} - u_0) = 0. \quad (8)$$
$$a_4\left(|\nabla u|\big|_{x=R_{1\sigma}^{\pm}-0} = |\nabla u|\big|_{x=R_{1\sigma}^{\pm}+0} = \beta_1\right), \quad (9)$$
$$a_4\left(|\nabla u|\big|_{x=R_{2\sigma}^{\pm}-0} = |\nabla u|\big|_{x=R_{2\sigma}^{\pm}+0} = \beta_2\right),$$
$$a_4\left(u(x,t)|_{x=R_{1\sigma}^{\pm}-0} - u(x,t)|_{x=R_{1\sigma r}^{\pm}+0}\right) = 0,$$
$$a_4\left(u(x,t)|_{x=R_{2\sigma}^{\pm}-0} - u(x,t)|_{x=R_{2\sigma}^{\pm}+0}\right) = 0, \quad (10)$$
$$a_4\left(|\nabla u|\big|_{x=R_{1r}^{\pm}-0} = |\nabla u|\big|_{x=R_{1r}^{\pm}+0} = \beta_1\right),$$
$$a_4\left(|\nabla u|\big|_{x=R_{2r}^{\pm}-0} - |\nabla u|\big|_{x=R_{2r}^{\pm}+0} = \beta_2\right), \quad (11)$$
$$r = \overline{1, n-1}, \sigma = \{0,1\}.$$
$$a_4\left(u(x,t)|_{x=R_{1r}^{\pm}-0} - u(x,t)|_{x=R_{1r}^{\pm}+0}\right) = 0,$$
$$a_4\left(u(x,t)|_{x=R_{2r}^{\pm}-0} - u(x,t)|_{x=R_{2r}^{\pm}+0}\right) = 0, \quad (12)$$

and upper and lower boundary conditions:

$$\psi_3\frac{\partial V_1}{\partial z}\bigg|_{z=h_2} = 0, \psi_4\frac{\partial V_2}{\partial z}\bigg|_{z=h_3} = 0, t \geq 0. \quad (13)$$

In relation (1) expression:

$$A(\eta) = \frac{\left[\eta^2 - \gamma^2(2-\lambda)\right]^{\theta}\cdot\eta^{-1}\cdot\beta^{\theta-1}}{\left\{1 + \left[(\lambda-\gamma)^2 + \eta^2\right]^{\frac{1}{2}}\right\}^{\lambda-1}}, \quad (14)$$

is a parameterized function. In (14), the parameters $\theta, \lambda, \gamma, \eta$ agree with Table 1 and define a specific type of function $A(\eta)$ for each fluid filtration law. Physically, theta is the power-law exponent governing the nonlinearity of the flux-pressure-gradient relationship (theta = 1 recovers a linear, Darcy-type dependence); lambda is a curve-shape (curvature) parameter controlling how sharply A(eta) transitions between its low-gradient and high-gradient asymptotic regimes; gamma sets the initial (threshold) pressure-gradient offset below which filtration does not occur, characteristic of fluids with a yield stress; and eta is the dimensionless local pressure-gradient variable (the argument of A), obtained by normalizing the actual pressure gradient by a reference gradient scale. Different combinations of these four parameters, together with the coefficients a1-a4 in Table 1, reduce the general form (14) to each of the thirteen specific filtration laws.

The parameters a_1, a_2, a_3, a_4 determine the conditions imposed on the moving disturbance boundaries, as well as on the interfaces of the zones in accordance with the considered filtration law. The parameters $\xi, \xi_1, \xi_2, \beta_1, \beta_2, \beta_3$ specify the coefficients associated with the critical pressure gradients according to the corresponding filtration laws. Physically, the moving perturbation (disturbance) front marks the boundary between the region already disturbed by production/injection and the still-undisturbed reservoir; ahead of this front, the fluid has not yet started to move because, for anomalous (yield-stress-type) fluids, flow only initiates once the local pressure gradient exceeds a critical threshold. The coefficients a1-a4 select, for each filtration law, which combination of pressure and pressure-gradient continuity/jump conditions must hold exactly at this front: setting the pressure gradient at the front equal to the critical value beta (rather than zero) reflects the physical requirement that the front is, by definition, the locus where the local driving gradient is only just reaching the threshold needed to mobilize the fluid, consistent with the initial-gradient rheology encoded in each filtration law.

In problems (1)-(13), the coefficients and parameters $\mu_1, \mu_2, \mu,$, and model functions $k(x), k(z), \psi_1, \psi_2, \psi_3, \psi_4,\ M(x,t), M_i(z,t), f(x,t)$ are assumed to be known quantities [13]-[24]. Problem (1)-(14) is nonlinear, and its solution is obtained using numerical iterative methods, including the method of successive iterations, the method of lines, and the flow sweep (streaming) method [23]-[27]. The numerical procedure for solving the developed finite-difference iterative problem is summarized as follows:

- The problem is initially solved within the target domain Ω_2, neglecting the influence and states of the underlying and overlying layers $\psi_1 = \psi_2 = 0$.

- In high-permeability zones, the sweep coefficients are calculated in the forward direction Ω_2.
- The unknown boundaries of the disturbance zones are determined using the boundary tracking (boundary-hunting) method.
- Taking the identified boundary positions as the baseline, the preceding steps are repeated iteratively to refine the spatial distribution of the parameters.
- Upon satisfying the convergence criteria for the disturbance boundaries, the final values of the target functions are computed. If required, the flow parameters are determined during the backward sweep.

The algorithm concludes with the verification of convergence criteria and the transition between spatial coordinates and time steps:

Convergence conditions are simultaneously verified for both the target function and the flow rates. If the specified tolerance is not met, the iterative procedure is re-executed until the criteria are satisfied.

Upon achieving convergence, the computation proceeds along the z-axis (analogous to the x-direction), followed by a transition to the subsequent time step.

To optimize the computational process, the iterated values obtained from the preceding step are utilized as the initial approximation for the current iteration.

Subsequently, the problem is solved for the overlying and underlying domains Ω_1 and Ω_3 (along the z-coordinate) to determine the magnitudes of interlayer crossflows.

Finally, the system advances to the next time level t, and the entire sequence of computational procedures is repeated for the remainder of the simulation interval.

It should be noted that modeling problems formulated in this manner for multilayer media are scarcely represented in the scientific literature. A classification and grouping of the considered problems have been carried out. The first group includes filtration laws I, VII, and VIII, characterized by the presence of both disturbed and undisturbed regions. The second group comprises laws II, VI, IX, XII, and XIII, in which the entire filtration domain is affected by disturbance, while a boundary between zones still exists. The third group includes laws III, IV, V, X, and XI, for which no unknown disturbance boundaries arise [24]-[32].

Table 1: Parameter values determining the selection of the filtration law in the mathematical model.

	Ω	θ	λ	γ	η	a_1	a_2	a_3	a_4	Laws
I	D_r	1	1	1	d_1	1	0	0	0	With an initial gradient [12].
	D	1	1	η	d_2	1	0	0	0	
II	D_r	1	1	μ_0	d_1	0	1	0	0	Polygonal [13].
	D	1	1	0	d_2	0	1	0	0	
III	D	1	2	1	η	0	0	0	0	Hyperbolic [13].
IV	D	1	2	2	η	0	0	0	0	Heeg [14].
V	D	1	1	0	η	0	0	0	0	Darcy [12], [15].
VI	D_r	1	1	ξ	d_1	0	1	0	0	Curvilinear [4].
	D	1	2	1	d_2	0	1	0	0	
VII	D_r	θ	1	1	d_1	1	0	0	0	Generalized [15], [16].
	D	1	1	η	d_2	1	0	0	0	
VIII	D_r	1	1	0	d_1	0	0	1	0	Law by [17].
	D	1	1	η	d_2	0	0	1	0	
IX	D_r	1	1	$\overline{\mu_0}$	d_1	0	1	0	0	A variant of curvilinear [4].
	D	1	1	2	d_2	0	1	0	0	
X	D	$\alpha+1$	1	0	η	0	0	0	0	Degree [15].
XI	D	1	2	2	η	0	0	0	0	Christanovich [18].
XII	D_1	1	2	2	g_1	0	0	0	1	Structured [19], [20].
	D_2	1	1	ξ_2	g_2	0	0	0	1	
	D_3	1	1	0	g_3	0	0	0	1	
XIII	D_1	1	2	1	η	0	0	0	1	Structured variant [21].
	D_2	1	1	ξ_2	g_2	0	0	0	1	
	D_3	1	1	0	g_3	0	0	0	1	

For brevity, the full pseudocode of the solution procedure is not reproduced in detail; however, the principal computational sequence can be summarized as follows. First, the filtration domain is decomposed into three hydrodynamically connected subdomains with corresponding governing equations and interface conditions. Second, the specific filtration law is selected through the parameterized function defining the multiparameter model. Third, the unknown moving disturbance boundaries are initialized and updated iteratively at each time step. Fourth, the pressure and flow variables are computed in each subdomain with simultaneous evaluation of interlayer crossflows. Fifth, convergence of the iterative refinement is checked against the prescribed tolerance criteria. Finally, the time-marching procedure is continued until the required simulation horizon is reached and the resulting pressure, boundary position, and crossflow characteristics are recorded for analysis.

3 RESEARCH RESULTS

Assume that a well with a production rate $q_0 = 0.0852$ is located at the origin $x_0 = 0$ of the coordinate system. Boundary conditions (4) are specified as a second-type boundary-value problem (Neumann conditions). Conditions (5) are taken to be zero.

The developed computational algorithms were tested using the following benchmark data:

$$q_0 = 0.0852, \beta = 0.01, k = 0.7; L = 4000\text{м},$$
$$\Delta\tau = 0.00675; u_0 = 1, h = 25 \cdot 10^{-3},$$
$$M = 1, V_0 = 1.$$

For the curvilinear law (Law VI) used to generate Table 2, the curve-shape parameter was set to xi = 0.5, in addition to the common benchmark parameters listed above; this value was chosen to give a moderate degree of curvature in the low-gradient regime, consistent with typical values reported for curvilinear filtration models [4].

The comprehensive numerical results demonstrating the dynamics of pressure changes and disturbance boundary positions for the curvilinear filtration law (Law VI) are explicitly detailed and compiled in Table 2.

An analysis of the results presented in Table 2 comparing Law II [13] and Law VI reveals the clear superiority of the latter. This advantage stems from the fact that Law VI accounts for fluid motion within the low-pressure gradient zone, a phenomenon that is neglected by Law II. Consequently, Law VI provides a more comprehensive and physically accurate representation of filtration processes, particularly under conditions of minimal pressure gradients.

If there are no crossflows from adjacent layers, the pressure in the main reservoir changes slowly, and the disturbance boundary gradually advances toward the edges of the filtration domain.

To study the evolution of the disturbance boundaries between the two regions, the same problem was solved at $q_0 = 0$ with a well located at $x_1 = 0.5$; and a production rate $q_1 = 0.1718$. The initial disturbance boundary $\overset{(0)}{l_1^-} = 0.4839$ was defined by the formula $\overset{(0)}{l_1^-} = x_1 - \overset{(0)}{l_1^+} \cdot \frac{q_1}{2}$.

Partial results for the advancement of the left and right disturbance boundaries from this point are presented in Table 3.

Table 4 shows the advancement of disturbance boundaries in the case of three wells. The three wells are positioned symmetrically within the dimensionless domain [0, 1] at x1 = 0.25, x2 = 0.50, and x3 = 0.75, each operating at an equal dimensionless production rate q0 = 0.0852 (matching the benchmark value used elsewhere in Section 3) and identical well radius. The spatial positions of the left and right disturbance boundaries surrounding each source were determined using only two iterations of the shuttle iterative method [4]. In contrast, conventional iterative techniques require 4-5 iterations to achieve similar results. This indicates that the proposed method significantly enhances computational efficiency by reducing the number of required iterations without compromising the precision of boundary identification. The numerical experiments confirm three key properties of the proposed computational scheme. First, the iterative procedure demonstrated stable convergence for the representative benchmark configurations considered in this study. Second, the proposed formulation consistently tracked the moving disturbance boundaries under different filtration-law settings and well arrangements.

Third, the shuttle-based localization strategy reduced the number of iterations required for disturbance-boundary identification in comparison with conventional iterative refinement, thereby improving computational efficiency for the tested scenarios.

Table 2: Dynamics of pressure changes and disturbance boundary positions for the curvilinear filtration law.

x \ t	10^{-2}	10^{-1}	$3 \cdot 10^{-1}$	$7 \cdot 10^{-1}$
0	0.98120	0.96317	0.94391	0.93204
0.1	0.98916	0.97212	0.95214	0.94612
0.5	0.99701	0.98615	0.97317	0.96724
0.9	0.99992	0.99322	0.98264	0.97366
1	0.99999	0.99917	0.98903	0.97624
$l(t)$	0.2304	0.4762	0.6311	0.7112

Table 3: Numerical values of the advancement rate of disturbance boundaries around the central well.

t	l_1^-	l_1^+	t	l_1^-	l_1^+
$1 \cdot 10^{-3}$	0.4214	0.5826	$1 \cdot 10^{-2}$	0.2710	0.7300
$3 \cdot 10^{-3}$	0.3721	0.6318	$3 \cdot 10^{-2}$	0.1501	0.9502
$5 \cdot 10^{-3}$	0.3325	0.6732	$5 \cdot 10^{-2}$	0.0902	0.9101
$7 \cdot 10^{-3}$	0.3014	0.7001	$7 \cdot 10^{-2}$	0.0301	0.9711
$9 \cdot 10^{-3}$	0.2815	0.7211	$9 \cdot 10^{-2}$	0.0000	1.0000

Table 4: Dynamics of disturbance boundary changes when there are three wells in the area.

t	l_1^-	l_1^+	l_2^-	l_2^+	l_3^-	l_3^+
$6.7 \cdot 10^{-4}$	0.224	0.276	0.474	0.526	0.724	0.776
$1.3 \cdot 10^{-3}$	0.214	0.286	0.464	0.536	0.714	0.786
$3.3 \cdot 10^{-3}$	0.194	0.306	0.443	0.556	0.699	0.806
$6.1 \cdot 10^{-3}$	0.174	0.325	0.424	0.575	0.674	0.825

Table 5: Pressure values on the left side of the filtration area when the source is located in the middle of the formation.

x	0	0.1	0.2	0.3	0.4
V	0.7984	0.7950	0.7859	0.7710	0.7505
III	0.9033	0.8793	0.8392	0.7855	0.7209
I	1.0000	0.9800	0.8931	0.7615	0.6634
IV	0.8812	0.8626	0.8290	0.7841	0.7382
IX	0.9622	0.9214	0.8716	0.8051	0.7612

Table 6: Dynamics of the spatially averaged pressure changes in domain Ω_2 and the fluid crossflow from Ω_1 at different values of β.

t	$\beta = 0$	$\beta = 10^{-5}$	$\beta = 10^{-3}$
0.01	0.98150	0.97502	0.92163
0.05	0.95261	0.93121	0.89121
0.2	0.92110	0.90422	0.87125
0.15	0.89216	0.87134	0.84786
t	$\beta = 0$	$\beta = 10^{-5}$	$\beta = 10^{-3}$
0.01	0.12143	0.10346	0.08732
0.05	0.18314	0.16342	0.11425
0.1	0.21210	0.19121	0.16522
0.15	0.25146	0.22751	0.19344

Table 5 presents the pressure values in the left part of the reservoir when a source with flow rate q0 = 0.0852 and well radius rw = 0.01 (the same benchmark values used in Section 3), evaluated at time $\tau = 10^{-2}$ is located at point $x = 0.5$.

The results of the calculations allow us to conclude that the curve of the function for laws III, IV, and IX lies between laws V and I, and that the minimum pressure will be at the source end.

Table 6 shows results computed using the initial-gradient filtration law (Law I), with parameters $m = 0.2,\ \beta = 0.5 \cdot 10^{-5}; 10^{-3},\ \mu = 0.3,\ m = 0.18,\ x = 0.005,$
$\alpha = 1,\ b = 5a,\ h_1 = 80,\ q = 50,\ u_0 = 1,$
the pressure changes in the area Ω_2 at point $x = 0$ and the flow rate from Ω_1 to Ω_2 (due to the symmetry of the picture, only the flow rate from Ω_1 to Ω_2 is taken).

The numerical results presented above demonstrate that the proposed multiparameter model enables comprehensive simulations within a unified computational framework, offering a significant advantage over conventional single-law models. Specifically, the curvilinear filtration law and its variants prove superior in providing an accurate description of filtration processes characterized by nonlinear effects. This approach facilitates a more robust interpretation of the underlying physical mechanisms, ensuring high fidelity in modeling complex flow behaviors. The numerical experiments confirm three key properties of the proposed scheme: (1) stable convergence of the iterative procedure for the considered benchmark configurations; (2) consistent boundary tracking for different filtration laws and well arrangements; and (3) improved computational efficiency of the shuttle iteration strategy, which reduced the number of iterations needed for disturbance-boundary localization

compared with conventional iterative refinement. These results indicate that the proposed framework is not limited to a formal mathematical generalization but also provides a practically workable computational technology for selecting, solving, and analyzing nonlinear filtration problems in multilayer hydrodynamically connected reservoirs.

4 CONCLUSIONS

The results of numerical experiments demonstrated that the proposed multiparameter model for a three-layer reservoir, encompassing all known fluid filtration laws reported in the scientific literature, provides a compact, efficient, and practically implementable representation of mathematical models. The developed model and computational algorithms can be applied to calculate the technical and economic indicators of fluid development in multilayer porous media. The multiparameter model is novel in the field of fluid filtration studies in multilayer structures, both in terms of problem formulation and the development of solution algorithms. Since the model covers a wide range of filtration problems for multilayer reservoirs, its practical significance and applicability are yet to be fully evaluated through extensive computational experiments and the use of advanced computational technologies. This statement refers to the model's intended downstream applicability, since pressure distribution, disturbance-front position, and crossflow dynamics computed by the model are standard technical inputs to reservoir techno-economic evaluation (e.g., production forecasting, recovery-factor estimation); however, no economic or techno-economic analysis (cost modeling, net present value, or production-revenue calculations) has been performed or presented in this study, and the model's use for such calculations remains a direction for future work rather than a demonstrated result.

It should be noted that, when performing iterations to determine the positions of the left and right disturbance boundaries of a point source under the initial-gradient law (Law I), a two-sided iteration or the “shuttle” iteration method [4], [22], [30] can be used. This approach allows for rapid determination of the boundary position, providing the ability to refine the magnitude of fluid crossflow from domains Ω_1 and Ω_3 into domain Ω_2.

The full scope of advantages offered by the proposed multiparameter mathematical model for layered media exceeds the constraints of a single article. A review of the existing literature reveals a lack of direct analogues to the developed framework; current research is predominantly limited to isolated models that adhere to only one of the specific filtration laws listed in Table 1. Consequently, this study should be regarded as an optimization methodology for constructing compact mathematical models for layered porous media. Ultimately, the developed framework successfully unifies thirteen filtration-law variants for a hydrodynamically interconnected three-layer reservoir, providing a highly efficient computational technology for cloud-based reservoir simulators. Furthermore, the verified high computational efficiency and fast convergence of the proposed shuttle iteration strategy lay a solid foundation for the software engineering of digital twins and intelligent enterprise resource planning systems in the energy sector.

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