Mathematical Modeling and Dynamic Analysis of the Structural Response of Two-Dimensional Frames under Variable Loads

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Abstract: This study presents a mathematical model for analyzing the dynamic response of two-dimensional (2D) frame structures subjected to dynamic loads such as wind and seismic forces, due to their significant impact on structural safety and performance. The main objective is to develop a numerical framework to investigate the behavior of these structures when modeled as multi-degree-of-freedom (MDOF) systems. The governing equations of motion are solved numerically using the fourth-order Runge–Kutta method in MATLAB to compute the time-history displacement response, which is evaluated at both peak and steady-state conditions. In addition, linear regression is used to examine the relationship between applied load and displacement. The results of the coefficient of determination (R^2) indicate a strong correlation between the variables. The findings demonstrate the effectiveness and accuracy of the proposed model in capturing structural behavior under dynamic excitations. This approach provides an efficient and practical tool for estimating structural response while reducing computational effort, making it suitable for preliminary analysis and engineering design applications.

1 INTRODUCTION

Mathematical model mathematical modeling is the process of using mathematical concepts to represent real-world phenomena and address important scientific and practical problems. In this process, a real-world scenario is converted into a mathematical formulation, analyzed using appropriate methods, and the results are then interpreted in the context of everyday life. To predict future outcomes, the solution of the mathematical model is translated back into real-life concepts and conditions [1].

Structural engineering is a basic field of civil engineering concerned with studying and modeling structures such as buildings, bridges, and tunnels. Analysis of the response to vibration is important in the design of entities subjected to dynamic disturbances, where the response can, in some cases, cause large stresses and damage. Its primary aim is to ensure the structural integrity of structures against multiple loads, while mathematical modeling is an essential service to engineers, as they can make sound predictions of structural responses, cutting time and cost before any type of construction begins [2].

Structural vibrations are a common phenomenon in engineering, with various applications aimed at either mitigating or enhancing these vibrations. In fact, the majority of human activities are influenced directly or indirectly by vibrations and their effects [3].

Most complex engineering structures were simplified into multi-degree-of-freedom (MDOF) systems, which required mathematics to explain as MDOF systems are composed of small interconnected masses which constantly interact with each other. Humar, J. L., “Dynamics of Structures” is a book intended to serve as a primary reference for this approach, providing practical steps for dealing with such systems and their application for earthquake and vibration field applications [2].

Multi-degree-of-freedom (MDOF) systems involve two or more independent coordinates to describe the motion of a structure. Continuous systems, which theoretically have an infinite number of degrees of freedom, are not included in this category. However, the widespread use of the finite element (FE) method allows structures that were once treated as continuous to be represented as MDOF systems, MDOF models now account for the majority of practical problems in structural dynamics [4].

The results in [5] show that simplified MDOF models can effectively represent the essential characteristics of structural vibration when validated with numerical simulations.

According to Kasimali in 2011, rigid frames refer to stable structural systems composed of elements linked by joints that resist motion. This study applied a two-dimensional (2D) planar frame with all structural members and applied dynamic loads in a single plane with horizontal motion only. It allows one to accurately analyze horizontal forces, the main drivers of the building's structural response [6].

Data were described, informed by the science of statistics and provided for scientific inference, and structural displacements were treated as continuous quantitative variables. The correlation coefficients for the dynamic loads and the corresponding response were almost perfectly linear (R^2 = 0.9110 and R^2 = 0.9985) and thus statistically significant. The accuracy of this model was confirmed, as well as its high predictive significance, suggesting an effective means of inferring the structural response under different dynamic conditions [7].

The main objective of this study is to develop a mathematical and numerical framework for analyzing the dynamic response of two-dimensional frame structures represented as a multi-degree-of-freedom (MDOF) system. The study also aims to investigate the relationship between dynamic loads and displacements using numerical simulations, and to assess the efficiency and limitations of regression-based prediction models.

2 STRUCTURAL LOADS

Structural loads can be broadly classified into four groups.

2.1 Dead Loads

Dead loads are structural loads that remain constant over time. They include the self-weight of structural elements such as walls, beams, and columns (e.g., concrete slabs and steel girders), in addition to permanent fixtures. Since the dimensions of these elements are initially determined based on architectural drawings before the final structural analysis, dead loads can be estimated with a relatively high degree of accuracy [8].

2.2 Live Loads

Live loads are variable in magnitude and position due to the use of a structure. Their design values are defined in building codes, and structural elements are designed for the load arrangement that produces the maximum stress. Examples include occupants, furniture, and movable equipment [8].

2.3 Wind Loads

Wind loads are caused by air movement around a structure and are affected by location, nearby obstacles, structural geometry, and dynamics. Most building codes estimate them using the relationship between wind velocity and dynamic pressure q on exposed surfaces [8].

2.4 Earthquake Loads

An earthquake is a sudden ground vibration. Horizontal motion, more than vertical, causes structural damage. As foundations move, the building’s upper part resists inertia, creating horizontal vibrations and shear forces. For accurate prediction of stress levels, a dynamic analysis must be conducted, taking into account both the building's mass and stiffness [9].

The loads acting on the building are as shown in Figure 1.

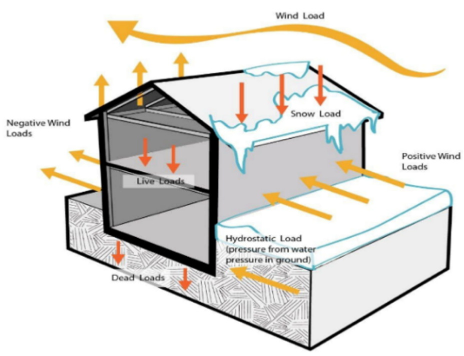

Figure 1: The effect of several types of loads on a building [10].

3 SUBJECT

In this study, we investigate the dynamic response of multi-degree-of-freedom (MDOF) structures under periodic loads, focusing on the interaction of mass, stiffness, and damping, and aim to develop accurate predictive models for the structural response, including the following time-dependent variables:

- Displacement: $x(t)$;
- Velocity: $\dot{x}(t)$;
- Acceleration: $\ddot{x}(t)$.

In the accompanying figures, the structure is represented by a (Node), which is a specific point in the structure illustrating the movement of a portion of the building or frame. The motion of the Node reflects how the structure responds to applied loads and helps to understand the effects of mass, stiffness, and damping on the overall structural response in a simple and clear manner.

4 MODEL

In this model, the equations of motion are adopted for each node, where two nodes are used to describe the dynamic behavior of the structure [2]:

Equation of Node (1):

$$m_1\ddot{x}_1 + c_1(\dot{x}_1 - \dot{x}_2) + k_1(x_1 - x_2) = F_1(t). \quad (1)$$

Equation of Node (2):

$$m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 - c_1(\dot{x}_1 - \dot{x}_2) - k_1(x_1 - x_2) = F_2(t). \quad (2)$$

Where m_1, m_2 mass matrix of nodes 1 and 2, $\ddot{x}_1$, $\ddot{x}_2$ acceleration vector matrix of nodes 1 and 2, c_1 , c_2 damping coefficient matrix of nodes 1 and 2, $\dot{x}_1, \dot{x}_2$ velocity vector matrix of nodes 1 and 2, k_1, k_2 stiffness matrix of nodes 1 and 2, x_1, x_2 displacements vector matrix of nodes 1 and 2 and F_1 (t), $F_2(t)$ applied external load vector matrix on nodes 1 and 2.

The system can be written in matrix form as follows:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} - \begin{bmatrix} c_1 & -c_1 \\ c_1 & c_1 + c_2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ k_1 & k_1 + k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}.$$

Example of the system: $m_1 = 1, m_2 = 1$ and $c_1 = 0.2, c_2 = 0.1$ and $k_1 = 10, k_2 = 5$,

$$m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, c = \begin{bmatrix} 0.2 & -0.2 \\ -0.2 & 0.3 \end{bmatrix},$$

$$k = \begin{bmatrix} 10 & -10 \\ -10 & 15 \end{bmatrix}, F = \begin{bmatrix} 5000\sin(2\pi t) \\ 0 \end{bmatrix}.$$

Initial values: $x = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$ and $\dot{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

5 SOLUTION METHODOLOGY

The proposed solution methodology was developed to analyze the dynamic response of the structural system and evaluate the relationship between applied forces and resulting displacements. The procedure involves numerical modeling, time-domain simulation, response analysis, and statistical validation of the obtained results using regression-based performance indicators. The main steps of the adopted methodology are summarized as follows:

- Define the structural properties, including mass, stiffness, and damping matrices.
- Formulate the equations of motion and convert them into a first-order system.
- Solve the system numerically using the Runge–Kutta method.
- Compute nodal displacements over time.
- Determine peak and steady-state responses.
- Apply linear regression to establish the relationship between forces and displacements.
- Evaluate the model performance using the coefficient of determination (R^2)
- Mean Absolute Error (MAE)
- Root Mean Square Error (RMSE)
- These metrics were calculated based on the residuals between actual and predicted values.

6 NUMERICAL SOLUTION

Numerical analysis is a branch of mathematics that focuses on designing methods to obtain numerical solutions for complex problems, especially when finding analytical solutions is difficult or impossible. For instance, solving millions of linear equations involving thousands of unknowns requires numerical approximations. With the increasing power and availability of digital computers, it has become possible to use precise mathematical models in science and engineering, enabling researchers and engineers to advance their projects across various fields [11].

7 MATHEMATICAL TECHNIQUE

This study focuses on examining the effect of applied loads (F) while keeping both stiffness (K) and damping (C) constant, with the aim of analyzing the relationship between the applied load and structural displacement over time. The Runge–Kutta method was employed using MATLAB to solve the coupled second-order differential equations, allowing for an accurate time history of displacements during both the transient and steady-state phases. Subsequently, linear regression was applied to the extracted data to develop predictive models linking force amplitudes to the maximum structural response. Accordingly, the coefficient of determination (R^2) was adopted as a key metric for evaluating model accuracy, due to its ability to measure the agreement between numerical results and observed values, while also efficiently assessing the estimation of structural response.

8 SOLUTIONS (RESULTS)

Case 1: Load applied to Node 1 $F_1(t) \neq 0$, $F_2(t) = 0$.

The external load vector is given by:

$$F = \begin{bmatrix} 5000\sin(2\pi t) \\ 0 \end{bmatrix}.$$

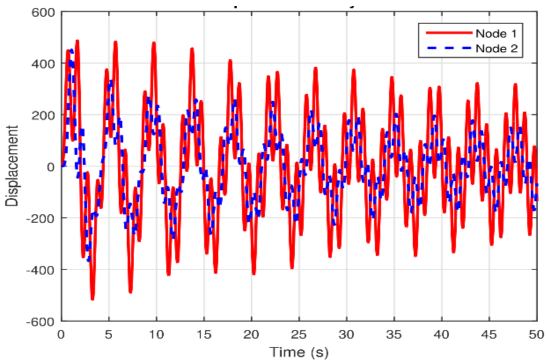

Figure 2: The time response of the horizontal displacement under the sin dynamic load one, solution is produced by using MATLAB.

Where the Node 1 (red) a clear oscillation appears due to the direct external load, and the displacement does not decay because of the continuous periodic excitation. The Node 2 (blue) a response is observed despite the absence of a direct load, as the vibration results from dynamic coupling with the first Node the system is dynamically stable but not in a static equilibrium state.

Case 2: Lode applied to Node 2 $F_1(t) = 0$, $F_2(t) \neq 0$.

The external load vector is given by:

$$F = \begin{bmatrix} 0 \\ 300\sin(2\pi t) \end{bmatrix}.$$

Where the Node 1 (red) a response is observed despite the absence of a direct external load. The vibration results from dynamic coupling with the second node. The Node 2 (blue) a clear oscillation appears due to the external load, exhibiting a larger vibration amplitude that later stabilizes. The system is dynamically stable but not in a static equilibrium state.

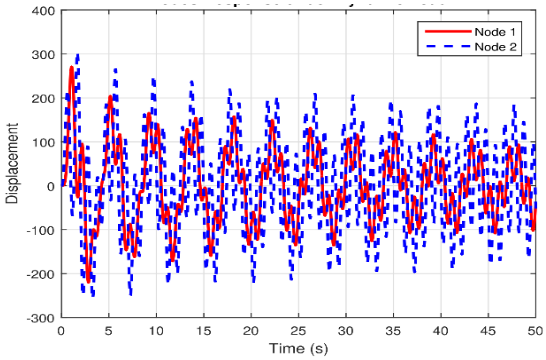

Figure 3: The time response of the horizontal displacement under the sin dynamic load one, solution is produced by using MATLAB.

9 ANALYSIS

Experiments were conducted for 15 different force loads, and the results were recorded at two specific time instants: $t_1 = 1.2s$ corresponding to the peak of the vibration, and $t_2 = 50s$ corresponding to the final response. for the load applied on the first node, corresponding to the (red) case in Figure 2.

Until the applied load reached $F = \begin{bmatrix} 5000\sin(2\pi t) \\ 0 \end{bmatrix}$, experiments were conducted as shown in Figure 2 at the x-axis at $t = 1.2s$ and $t = 50s$ the following results were obtained as shown in Table 1.

Table 1: Results for case 1.

F	t_1	t_2
120	11.2	-1.5
160	15.2	-6.4
200	19.2	-7.5
600	58.5	-21.4
1000	96.5	-34.2
1400	138.4	-48.6
1800	178.6	-62.4
2200	218.7	-78.4
2600	254.3	-92.8
3000	282.4	-106.7
3400	319.8	-118.5
3800	343.2	-129.6
4200	411.5	-146.6
4600	456.3	-164.8
5000	490.3	-169.3

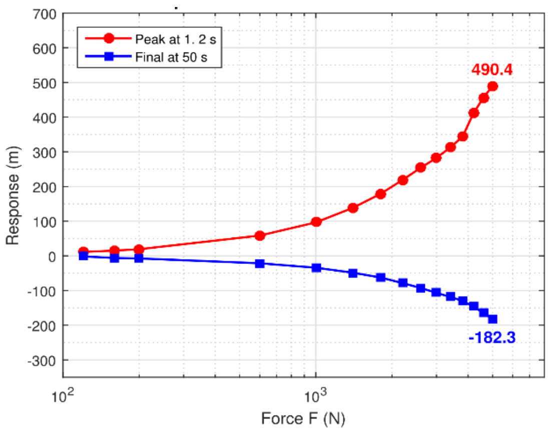

Figure 4: Force response relationship for peak response at 1.2 s and final response at 50 s, solution is produced by using MATLAB.

Similarly, experiments were conducted for 15 different force lodes and the results were recorded at the same two times instants for the load applied on the two nodes, corresponding to the (blue) case in Figure 3.

Until the applied load reached $F = \begin{bmatrix} 0 \\ 3000\sin(2\pi t) \end{bmatrix}$, experiments were conducted as shown in Figure 3 at the x-axis at $t = 1.2s$ and $t = 50s$, the following results were obtained as shown in Table 2.

Table 2: Results for case 2.

F	t_1	t_2
200	20.4	-1.2
400	39.7	-2.6
600	60.3	-4.8
800	79.6	-6.2
1000	99.4	-8.3
1200	121.4	-10.5
1400	140.8	-11.2
1800	161.2	-13.4
2000	184.2	-15.6
2200	201.4	-18.7
3400	223.8	-20.4
2400	243.6	-22.1
2600	264.8	-23.9
2800	281.4	-25.8
3000	312.5	-28.4

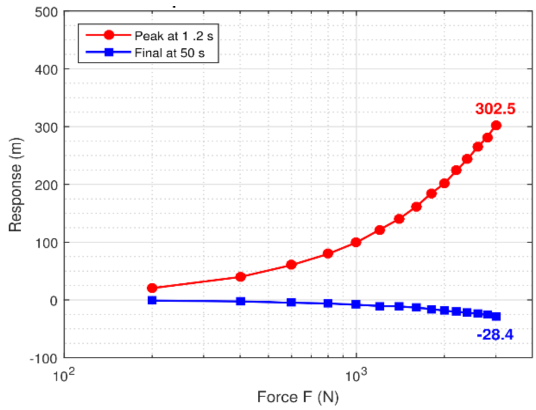

Figure 5: Force response relationship for peak response at 1.2 s and final response at 50 s, solution is produced by using MATLAB.

For all cases, and from the figures that illustrated the behavior, the curves represent exponential functions describing the relationship between force and response.

10 RESULTS AND DISCUSSION

This section analyzes the relationship between the applied force and the structural response based on the results presented in Figures 4 and 5. Linear regression models are obtained to describe the relationship between the applied force and the structural displacement. In addition, the coefficient of determination (R^2) is calculated to evaluate the strength of these relationships.

A study found that conventional methods are insufficient for accurately assessing seismic structural damage, and that using machine learning algorithms such as linear regression and KNN can effectively replicate the damage index [12].

Linear regression equation and corresponding figure from (Fig. 4):

$$y_{peak} = 0.096700F - 0.089261 \ (R^2 = 0\ .9973)$$
$$y_{final} = -0.034812F - 0.153463 \ (R^2 = 0.9985)^{\cdot}$$

The Figure 6 shows the above relation.

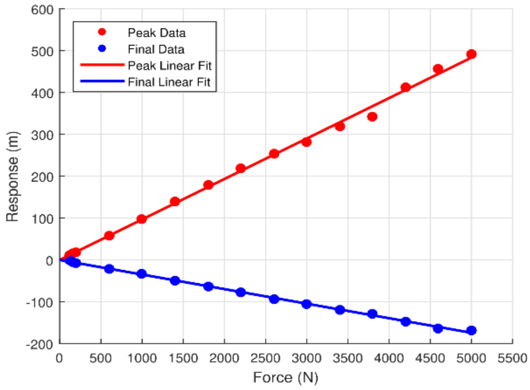

Figure 6: Force response relationship for peak response at 1.2 s and final response at 50 s using linear regression, solution are produced by using MATLAB.

Linear regression equation and corresponding Figure 5 from:

$$y_{peak} = 0.086880F + 12.866535\ (R^2 = 0.9087)$$
$$y_{final} = -0.008268F + 0.014773\ (R^2 = 0.9110)^{\cdot}$$

The Figure 7 shows the above relation.

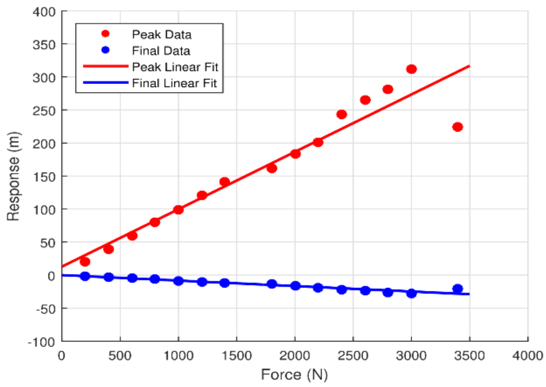

Figure 7: Force response relationship for peak response at 1.2 s and final response at 50 s using linear regression, solution are produced by using MATLAB.

Peak Response: The analysis shows that the slope of the straight line at the peak time (t = 1.2s) is the highest. This indicates that the structure experiences the maximum displacement during the initial stage of loading.

Steady State Response: As time approaches the final time (t = 50 s) the slope decreases due to damping, which absorbs part of the vibration energy and leads the system to a stable state.

The coefficient of determination R^2 in Figures 6 and 7 is the most important indicator of the goodness of fit between the extracted results and the mathematical model. An R^2 value of 0.9985 and 0.9087 means that the proposed linear model can explain 99.85 % of the variation in the structural response data. This high value indicates that almost all data points lie very close to the ideal straight line. From an engineering perspective, this suggests that the system does not exhibit random behavior or numerical disturbances during the solution process using the Runge–Kutta method.

Once this relationship is obtained, there is no longer a need to run the complex MATLAB code to determine the displacement at a specific force (for example, 2500 N) as an alternative, the value can be directly substituted into the equation derived from the figures to obtain an immediate and highly accurate result. This approach is referred to in recent research as Reduced-Order Modeling (ROM).

10.1 Error Analysis

Residuals are extracted to compute MAE and RMSE, to assess the accuracy of the regression model.

Figure 4 (Node1):

$$\text{Residual} = y_{\text{actual}} - y_{\text{predicted}}$$

$$F_{\text{peak}} = 5000\sin(2\pi \cdot 1.2) = 4727.5$$

$$y_{\text{predicted}} = 0.096700 \cdot 4727.5 - 0.089261 \approx 457.11$$

$$y_{\text{actual}} = 490.3$$

$$\text{Residual}_{\text{peak}} = 490.3 - 457.11 \approx 33.19$$

$$F_{\text{final}} = 5000\sin(2\pi \cdot 50) \approx 0$$

$$y_{\text{predicted}} = -0.034812 \cdot 0 - 0.153463 \approx -0.153$$

$$y_{\text{actual}} = -169.3$$

$$\text{Residual}_{\text{final}} = -169.3 - (-0.153) = -169.147$$

$$\text{MAE} = \frac{|33.19| + |-169.147|}{2} = \frac{33.19 + 169.147}{2} = 101.17$$

$$\text{RMSE} = \sqrt{\frac{(33.19)^2 + (-169.147)^2}{2}} = \sqrt{\frac{1099.6 + 28598.6}{2}} = \sqrt{14849.1} \approx 121.88$$

Figure 4 (Node1)

$$F_{\text{peak}} = 3000\sin(2\pi \cdot 1.2) = 2836.5$$

$$y_{\text{predicted}} = 0.08688 \cdot 2836.5 + 12.866535 \approx 259.17$$

$$y_{\text{actual}} = 312.5$$

$$\text{Residual}_{\text{peak}} = 312.5 - 259.17 = 53.33$$

$$\mathrm{F}_{\text{final}} = 3000 \sin(2\pi \cdot 50) \approx 0$$

$$y_{predicted} = -0.008268 \cdot 0 + 0.014773 \approx 0.0148$$

$$y_{actual} = -28.4$$

$$Residual_{final} = -28.4 - 0.0148 = -28.4148$$

$$MAE = \frac{|53.33| + |-28.4148|}{2} = \frac{53.33 + 28.4148}{2} = 40.87$$

$$RMSE = \sqrt{\frac{(53.33)^2 + (-28.4148)^2}{2}} = \sqrt{\frac{2844.0 + 807.7}{2}} = \sqrt{1825.85} \approx 42.74$$

Error analysis using residuals, the Mean Absolute Error (MAE), and Root Mean Square Error (RMSE), provides a more reliable assessment for the model performance. Although the regression models show a high level of agreement at the peak response, slight differences may appear in the steady-state condition due to the effect of damping and changes in system behavior over time, and that combining regression analysis with accuracy measures yields more precise and realistic results for the structural behavior under applied loading.

Metrics such as the Mean Absolute Error (MAE) and the Root Mean Square Error (RMSE) are widely used to evaluate model performance. However, as noted by Chicco et al. [13], these metrics are subject to a common limitation: their values range from 0 to infinity, which prevents them from accurately reflecting regression performance relative to the true values.

10.2 Validation of the Model Using Cross-Validation

As an additional step to verify the performance of the proposed model, cross-validation was applied. The results are presented in Table 3, which compares the obtained R^2 values with the original values. The following results were obtained as shown in Table 3.

Table 3: Original and Cross-Validated R^2 Results.

Case / Response	R^2 (Original)	R^2 (Cross-Validated)
Case 1 -Peak	0.9973	0.9933 ± 0.0077
Case 1 - Final	0.9985	0.9964 ± 0.0031
Case 2 - Peak	0.9087	0.8964 ± 0.1328
Case 2 - Final	0.9110	0.8944 ± 0.1175

The cross-validation results showed that Case 1 exhibited a high degree of stability, with R^2 values remaining close to the original values. In contrast, Case 2 demonstrated a relative decline in R^2 values and increased variability, indicating that its predictive performance is less stable across different data subsets.

The cross-validated MAE and RMSE values for each case are presented in Table 4. The following results were obtained as shown in Table 4.

Table 4: Cross-validated MAE and RMSE Values for Each Case.

Case / Response	MAE (CV)	RMSE (CV)
Case 1 -Peak	6.285	9.373
Case 1 - Final	2.034	2.877
Case 2 - Peak	18.072	31.849
Case 2 - Final	1.763	2.957

The MAE and RMSE results support the cross-validation findings for the coefficient of determination (R^2). As shown in Table 4, prediction errors varied among the different cases, with the highest error values observed for Case 2–Peak, while the remaining cases achieved lower error levels. This may be attributed to the greater variability and complexity of the peak response, which makes accurate prediction more challenging.

11 CONCLUSIONS

The Runge–Kutta method demonstrated high efficiency in representing the dynamic response of two-dimensional frame structures, with the results showing a clear linear relationship between applied loads and nodal displacements within the studied range. It was observed that the maximum response occurs during the initial loading stages, followed by a gradual decrease as the system transitions from its initial transient response toward the steady-state amplitude of the sustained sinusoidal load, with damping progressively dissipating the transient component of the response.

Although the coefficient of determination (R^2) indicated a high agreement between numerical and expected results, this metric does not fully capture the system's dynamic behavior. Therefore, relying solely on R^2 may not be sufficient for assessing model accuracy, highlighting the need to support the findings with additional verification techniques such as residual analysis or comparisons with finite element models.

The proposed approach provides an effective tool for estimating structural response while reducing computational effort, making it suitable for preliminary analysis and engineering design. Future work is recommended to extend the study to include nonlinear behavior and experimental validation to enhance model accuracy and improve prediction reliability.

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