Analysis of Nonlinear Reaction-Diffusion Processes in Atmospheric Media with Self-Similar Behavior

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Abstract: This paper presents a thorough analytical and numerical investigation of a strongly nonlinear, coupled reaction-diffusion system formulated for modeling atmospheric dust transport in heterogeneous media. The governing equations represent a doubly nonlinear parabolic system of the Fisher-KPP type, containing generalized p-Laplacian diffusion operators, nonlinear interphase exchange kinetics, and coefficients that are dependent on space and time. To analyze the qualitative behavior of the system, we use a self-similar reduction, a comparison of the corresponding behavior with compact-support Barenblatt-type profiles. Using variational methods and comparison principles, it is shown that weak solutions exist and are unique in the right Sobolev spaces. Moreover, to properly represent the intricate spatiotemporal dynamics and the limited propagation speed property of the degenerate diffusion terms, we employ an effective numerical method. Our numerical simulations establish the theoretical similarity solutions and demonstrate the model's ability to represent localized nonlinear diffusion effects, complex phase interactions, and actual dust concentration patterns, thereby offering an effective mathematical and predictive structure for environmental transport mechanisms.

1 INTRODUCTION

Atmospheric particulate transport should be treated as a multiscale process in which diffusion, interphase exchange, and nonlinear interactions jointly determine the system dynamics. In particular, the evolution of fine particles such as $PM_{2.5}$ and PM_{10} cannot be adequately described without accounting for their simultaneous presence in airborne and surface-deposited states [1].

These particles are of specific concern due to their long residence time in the atmosphere and their capacity for large-scale spatial propagation. As a result, even weak nonlinear effects may accumulate and significantly alter the overall transport behavior [2].

Classical diffusion models based on Fick's law provide only a first-order approximation and are typically valid under restrictive assumptions such as homogeneity of the medium and low concentration levels [3]. However, in heterogeneous environments, the diffusion coefficient often depends explicitly on both concentration and spatial variables. This dependence leads to degenerate behavior and introduces nonlinear transport mechanisms that are not captured within the linear framework [4].

In addition, the coupling between airborne particles and surface deposition creates a bidirectional exchange process. This interaction modifies the effective transport rates and may induce nontrivial feedback effects, particularly in regions with strong gradients or variable boundary conditions [5].

For these reasons, it becomes necessary to consider coupled nonlinear reaction-diffusion systems with variable coefficients. Such models allow one to incorporate concentration-dependent diffusivity, interphase mass transfer, and nonstandard growth conditions, thereby providing a more adequate description of realistic particulate transport processes [6].

In this work, we consider a nonlinear two-component reaction-diffusion system describing the evolution of airborne and surface-bound concentrations. The model incorporates generalized diffusion operators of p -Laplacian type, nonlinear interphase exchange terms, and space-time dependent parameters [7]. Such a formulation allows the analysis of qualitative phenomena including self-similar behavior, finite-time blow-up, and long-time stabilization [8].

The main contributions of this study are threefold [9]. First, a self-similar analysis of the proposed system is performed, revealing possible scaling regimes of solutions [10]. Second, the existence of solutions is established within an appropriate functional framework. Third, an efficient numerical scheme based on an alternating direction implicit (ADI) approach is constructed to simulate the model and capture its complex spatiotemporal dynamics.

2 MATHEMATICAL MODELING AND SOLUTIONS

Dust transport in atmospheric environments is governed by nonlinear diffusion mechanisms and interphase interactions between airborne and surface-bound particles. In this study, we consider two coupled concentrations: the airborne phase $C_a(x,t)$ and the surface-bound phase $C_s(x,t)$. Their interaction reflects deposition and resuspension processes, which play a dominant role in the overall dynamics [11].

Nonlinear diffusion is introduced to account for concentration-dependent transport arising from heterogeneous media and particle interactions [4]. In contrast to classical linear models, this approach allows for degenerate behavior and nonstandard growth conditions [1]. The coupling between phases is modeled through nonlinear reaction terms, while space-time variability is incorporated via heterogeneous coefficients [12].

Let $\Omega \subset \mathcal{R}^2$ be a bounded domain with boundary $\partial\Omega$ and define the space-time domain $Q = \{(t,x,y)\colon t > 0,\ (x,y) \in \Omega\}$. The evolution of the concentrations $C_a(x,t)$ and $C_s(x,t)$ is governed by the following nonlinear coupled system:

$$\begin{cases} \rho_1(x)\frac{\partial C_a}{\partial t} = \nabla \cdot \left(D_a C_a^{m_1-1}\left|\nabla C_a^{k_1}\right|^{p-2}\nabla C_a^{l_1}\right) + \\ +\lambda_{12}C_aC_s - \rho_2(x)e^{\lambda_{1,2}\alpha_1 t}C_s^{k_1}, \\ \rho_1(x)\frac{\partial C_s}{\partial t} = \nabla \cdot \left(D_s C_s^{m_2-1}\left|\nabla C_s^{k_2}\right|^{p-2}\nabla C_s^{l_2}\right) - \\ -\lambda_{12}C_aC_s + \rho_2(x)e^{\lambda_{1,2}\alpha_2 t}C_a^{k_2}. \end{cases} \quad (1)$$

The initial conditions are prescribed as spatially localized distributions with oscillatory perturbations:

$$\begin{cases} C_a(0,x,y) = C_{a,0}(x,y) = C\, exp({}_{1_0}{}^{2}{}_{a,max} \\ \alpha_2(y-y_0)^2)\cdot(1+\mu_1 sin(\omega_1 x)\, cos(\omega_2 y)) \\ C_s(0,x,y) = C_{s,0}(x,y) = C\, exp({}_{1_1}{}^{2}{}_{s,max} \\ \beta_2(y-y_1)^2)\cdot(1+\mu_2 cos(\omega_3 x)\, sin(\omega_4 y))\{ \end{cases} \quad (2)$$

The system (1) represents a class of nonlinear degenerate parabolic equations with p -Laplacian type diffusion and strong interphase coupling [13]. The presence of nonlinear gradient-dependent fluxes and exponential source terms allows for the emergence of complex qualitative behaviors, including self-similar regimes, finite-time blow-up, and long-time stabilization [2].

In the subsequent analysis, attention is focused on the qualitative properties of solutions, including self-similar structures and existence within an appropriate functional framework, as well as their numerical approximation [14].

For analytical convenience, we briefly note that system (1)-(2) admits a natural non-dimensional formulation [15]. By introducing characteristic scales for concentration, spatial variables, and time, the governing system can be reformulated in dimensionless form [9]. This transformation reduces the number of independent parameters and makes it possible to isolate the relative contributions of nonlinear diffusion, interphase exchange, and source terms [16]. Despite these advantages, the dimensional formulation is retained in what follows in order to preserve direct physical interpretability and facilitate comparison with experimental or observational data [2].

From a mathematical standpoint, system (1) can be classified as a coupled nonlinear degenerate parabolic system with variable coefficients and nonstandard growth conditions [17]. Such a structure introduces several analytical difficulties. In particular, the appearance of generalized p-Laplacian-type operators, combined with nonlinear reaction terms, leads to a highly nonuniform diffusion mechanism that depends on the gradient magnitude [18].

Moreover, the interaction between gradient-dependent fluxes and exponential-type source terms substantially enriches the qualitative behavior of solutions [19]. Depending on the balance between diffusion and reaction, one may observe the formation of self-similar regimes, finite-time blow-up, or, alternatively, convergence to stable long-time states. These regimes are highly sensitive to parameter values and initial conditions [20].

As a consequence, the analysis of such systems requires the use of advanced analytical tools adapted to degenerate operators, together with carefully constructed numerical schemes that ensure stability and consistency in the presence of strong nonlinearities [3].

Degenerate parabolic equations involving p-Laplacian operators typically take the form:

$$\frac{\partial u}{\partial t} = \nabla \cdot (|\nabla u|^{p-2}\nabla u),$$

provide a natural framework for describing nonlinear gradient-driven diffusion and share structural similarities with the flux terms in system (1).

Coupled reaction-diffusion systems of the form

$$\begin{cases} \dfrac{\partial C_1}{\partial t} = \nabla \cdot (D_1 \nabla C_1) + R(C_1, C_2), \\ \dfrac{\partial C_2}{\partial t} = \nabla \cdot (D_2 \nabla C_2) - R(C_1, C_2), \end{cases}$$

are widely used to model interphase interactions and mass exchange processes [21]. In contrast to these classical formulations, system (1) incorporates nonlinear degenerate diffusion, strong coupling between C_a and C_s, and space-time dependent coefficients, which significantly enriches the model structure [2].

In order to derive the self-similar solution, a preliminary structure is formulated in the following manner:

$$\begin{cases} C_a(t,x) = \bar{C}_a(t) w(\tau(t), x) \\ C_s(t,x) = \bar{C}_s(t) z(\tau(t), x) \end{cases}, \tag{4}$$

where $\bar{C}_a(t) = X_1(T_0 - t)^{-\alpha_1}, \bar{C}_s(t) = X_2(T_0 - t)^{-\alpha_2}$,

$$X_i = (\alpha_1/\alpha_2)^{\frac{(-1)^i}{k_i - 1}}, \alpha_i = \frac{k_i + 1}{k_1 k_2 - 1}, i = 1,2, \tag{5}$$

represents a solution to the system labeled

$$\begin{cases} \frac{d\bar{C}_a}{dt} = \varepsilon_1 C_s(t)^{k_1} \\ \frac{d\bar{C}_s}{dt} = \varepsilon_1 C_a(t)^{k_2} \end{cases} \tag{6}$$

After applying a self-similar transformation to system (1)-(2) and assuming radial symmetry system, labeled

$$\begin{cases} \dfrac{\partial w}{\partial \tau} = \varphi^{1-s} \dfrac{\partial}{\partial \varphi}\left(\varphi^{s-1} w^{m_1 - 1} \left|\dfrac{\partial w^{k_1}}{\partial \varphi}\right|^{p-2} \dfrac{\partial w^{l_1}}{\partial \varphi}\right) + \\ \frac{1}{\tau} \frac{X_2^{k_1}}{X_1(1 - \alpha_1(m_1 + l_1 + k_1(p-2) - 2))} (\rho_2/\rho_1 z^{k_1} + \alpha_1 w) \\ \dfrac{\partial z}{\partial \tau} = \varphi^{1-s} \dfrac{\partial}{\partial \varphi}\left(\varphi^{s-1} z^{m_2 - 1} \left|\dfrac{\partial z^{k_2}}{\partial \varphi}\right|^{p-2} \dfrac{\partial z^{l_2}}{\partial \varphi}\right) + \\ \frac{1}{\tau} \frac{X_1^{k_2}}{X_2(1 - \alpha_2(m_2 + l_2 + k_2(p-2) - 2))} (\rho_2/\rho_1 w^{k_2} + \alpha_2 z) \end{cases} \tag{7}$$

If the assumed form of the function corresponds to

$$\tau(t) = \begin{cases} D_i(T_0 + t)^{d_i}, \text{if } \lambda_i = m_i + l_i + k(p-2) - 2 \neq 0, \\ T + t, \text{if } \lambda_i = 0, \end{cases}$$

$$\varphi(r) = \begin{cases} \frac{s}{N+l} r^{\frac{N+l}{s}}, \text{if } n \neq p + l \\ \ln r, \text{if } n = p + l \end{cases}, s = \frac{p(N+l)}{p+l-n}, \alpha_i = \frac{k_i + 1}{k_1 k_2 - 1}, D_i = \frac{A_i^{\lambda_i}}{d_i}, d_i = d_{3-i} = 1 - \alpha_i \lambda_i, i = 1,2.$$

The system is observed to admit an approximately self-similar solution, characterized by the structure given in

$$\begin{cases} w(\tau, \varphi) = f(\xi) \\ z(\tau, \varphi) = \psi(\xi) \end{cases} \Rightarrow \xi = \varphi(r) \tau^{-1/p},$$

and the corresponding functions $f(\xi)$, $\psi(\xi)$ fulfill the following nonlinear degenerate system of equations that exhibit approximate self-similarity, referred to as

$$\begin{cases} \xi^{1-s} \frac{d}{d\xi}\left(\xi^{s-1} f^{m_1 - 1} \left|\frac{df^{k_1}}{d\xi}\right|^{p-2} \frac{df^{l_1}}{d\xi}\right) + \frac{\xi}{p}\frac{df}{d\xi} + \\ \frac{X_2^{k_1}/X_1}{1 - \alpha_1(m_1 + l_1 + k_1(p-2) - 2)} (\rho_2/\rho_1 \psi^{k_1} + \alpha_1 f) = 0 \\ \xi^{1-s} \frac{d}{d\xi}\left(\xi^{s-1} \psi^{m_2 - 1} \left|\frac{d\psi^{k_2}}{d\xi}\right|^{p-2} \frac{d\psi^{l_2}}{d\xi}\right) + \frac{\xi}{p}\frac{d\psi}{d\xi} + \\ \frac{X_1^{k_2}/X_2}{1 - \alpha_2(m_2 + l_2 + k_2(p-2) - 2)} (\rho_2/\rho_1 f^{k_2} + \alpha_2 \psi) = 0 \end{cases} \tag{8}$$

To investigate the mathematical properties of the self-similar formulation of system (1)-(2), we analyze the reduced system satisfied by the similarity profiles [5]. The presence of nonlinear degenerate diffusion operators, coupled interaction terms, and nonlinear boundary fluxes requires a precise functional framework to justify existence and uniqueness results [4]. Throughout the analysis, the self-similar problem is considered on a bounded spatial domain $(0, R)$ for some $R > 0$, and over a finite time interval $(0, T)$, with $T > 0$. The unknown profiles are sought in the Sobolev space $W^{1,p}(0, R)$, while time-dependent approximations are constructed in the Bochner space $L^p(0, T; W^{1,p}(0, R))$. Under appropriate assumptions on the nonlinear exponents, diffusion coefficients, and initial data, uniform a priori estimates are established [6]. These estimates imply boundedness of the approximate solutions in $L^p(0, T; W^{1,p}(0, R))$ and of their time derivatives in the dual space $L^{p'}(0, T; W^{-1,p'}(0, R))$, which enables the application of the Aubin-Lions compactness lemma [7]. The main existence and uniqueness result is stated below.

Theorem 1. Let the self-similar formulation of system (1)-(2) be considered on the space-time domain $Q_T = (0,T) \times (0,R)$, where $T > 0$ and $R > 0$ are finite. Assume that all nonlinear exponents are positive, the diffusion coefficients, weight functions, and velocity field are continuous and bounded in Q_T and the initial data $C_{a,0}, C_{s,0} \in L^p(0,R)$ are nonnegative, smooth, and compactly supported in $(0,R)$. Then, there exist $T > 0$and $R > 0$ such that the self-similar problem admits a unique weak solution pair $(f_a, f_s) \in L^p(0,T; W^{1,p}(0,R))$ which satisfies the transformed coupled system almost everywhere in Q_T together with the associated boundary conditions in the weak sense [22].

Proof. We prove the existence and uniqueness of weak solutions via the Galerkin approximation method [4]. Let $\{\phi_k(x)\}_{k=1}^{\infty}$ be an orthonormal basis in the Sobolev space $W_0^{1,p}(0,R)$. Consider approximate solutions of the form:

$$\begin{cases} f_a^N(x,t) = \sum_{k=1}^{N} a_k(t)\phi_k(x), \\ f_s^N(x,t) = \sum_{k=1}^{N} b_k(t)\phi_k(x). \end{cases}.$$

Substituting these into the weak formulation leads to the following system of ordinary differential equations for coefficients $\{a_k(t)\}$ and $\{b_k(t)\}$:

$$\begin{cases} \frac{d}{dt}(f_a^N, \phi_j) + A_a^N(f_a^N, f_s^N, \phi_j) = F_a(f_a^N, f_s^N, \phi_j), \\ \frac{d}{dt}(f_s^N, \phi_j) + A_s^N(f_s^N, f_a^N, \phi_j) = F_s(f_s^N, f_a^N, \phi_j), \end{cases}$$
$$\text{for} j = 1, \dots, N,$$

where A_a^N and A_s^N represent the nonlinear diffusion and reaction terms [23]. Since all nonlinearities are continuous and bounded, the classical Cauchy-Peano theorem ensures the existence of a local-in-time solution for the finite-dimensional system [7]. We derive energy estimates by multiplying each equation by $a_k(t)$ and $b_k(t)$ respectively, summing over k , and integrating in time [24]. Using the Poincaré and Young inequalities, we obtain:

$$\begin{cases} \sup_{t\in[0,T]} \| f_a^N(t) \|_{L^p}^p + \int_0^T \| \nabla f_a^N(t) \|_{L^p}^p \, dt \le C, \\ \sup_{t\in[0,T]} \| f_s^N(t) \|_{L^p}^p + \int_0^T \| \nabla f_s^N(t) \|_{L^p}^p \, dt \le C, \end{cases}$$

where C is a constant independent of N. Hence,

f_a^N, f_s^N areuniformly boundedin$L^p(0,T; W^{1,p}(0,R))$.

From the Galerkin system, for any test function $\varphi \in W_0^{1,p}(0,R)$, we have

$$\langle \partial_t f_a^N, \varphi \rangle = -A_a^N(f_a^N, f_s^N, \varphi) + F_a(f_a^N, f_s^N, \varphi),$$

and similarly for f_s^N. Using the previously obtained energy bounds and the growth assumptions on the nonlinear terms, we deduce that $\partial_t f_a^N, \partial_t f_s^N$ are unifromly bounded in $L^{p'}(0,T; W^{-1,p'}(0,R))$, where p' is the conjugate exponent of p. The uniform bounds in $L^p(0,T; W^{1,p}(0,R))$ together with the bounds on the time derivatives allow us to apply the Aubin-Lions compactness lemma. Consequently, there exist subsequences (still denoted by f_a^N, f_s^N) such that $f_a^N \to f_a$ stronglyin$L^p(0,T; L^p(0,R))$, $f_s^N \to f_s$ stronglyin$L^p(0,T; L^p(0,R))$, and weakly in $L^p(0,T; W^{1,p}(0,R))$. Passing to the limit in the nonlinear terms is justified by Minty-Browder monotonicity arguments [6]. The limit pair (f_a, f_s) satisfies the weak formulation of the original problem, including the boundary terms, justified by the trace theorem [25]. Assume two solutions $(f_a^{(1)}, f_s^{(1)})$ and $(f_a^{(2)}, f_s^{(2)})$. Subtracting the corresponding equations and applying monotonicity of the diffusion operators yields:

$$\int (D(u_1)\nabla u_1 - D(u_2)\nabla u_2) \cdot \nabla(u_1 - u_2) \ge 0.$$

Using Gronwall's inequality, one concludes that the difference vanishes, ensuring uniqueness [26].

Remark 1. This theorem ensures that the coupled nonlinear diffusion-reaction system admits a unique weak solution under reasonable structural conditions. It confirms that the model is mathematically well-posed, meaning the system's evolution is predictable and stable with respect to the initial and boundary data. Such a result provides the foundation for both analytical and numerical investigation of dust dispersion processes [27]. Following the local well-posedness of self-similar solutions, we now examine the qualitative spatial behavior of the profiles. Of particular interest is whether the dust concentration remains localized in space over time. The following result establishes that the self-similar profiles possess compact support, provided the nonlinear diffusion is sufficiently strong [28].

To illustrate the qualitative structure of the initial data and validate the consistency of the self-similar formulation, a numerical visualization is provided [29]. The parameters are selected to reflect physically relevant regimes, including nonlinear diffusion, interphase coupling, and heterogeneous spatial distributions:

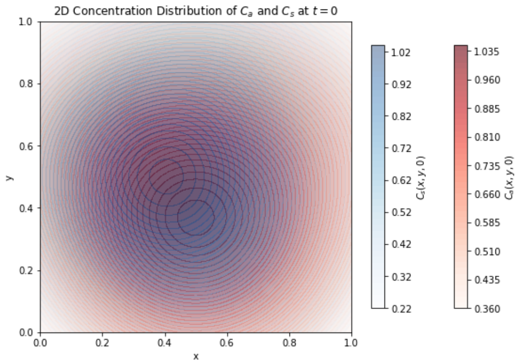

Figure 1: A 2D graph for the parameters with.

$$C_{a,\max} = 1.2, C_{s,\max} = 0.6, D_a = 0.03, D_s = 0.11, m = 2.3, k = 1.4, p = 2.9, l = 1.5, \lambda_{1,2} = 0.2, \alpha = 0.3.$$

Figure 1 presents the two-dimensional spatial distributions of the airborne concentration $C_a(x, y, 0)$ and the surface-bound concentration $C_s(x, y, 0)$ at the initial time. The spatial coordinates (x, y) define the computational domain, while the color intensity represents the magnitude of the concentrations [30]. The chosen parameter set ensures nontrivial profiles with localized peaks and oscillatory perturbations, capturing realistic heterogeneity and serving as a consistent basis for subsequent numerical simulations.

3 CONCLUSIONS

In this study, a nonlinear coupled reaction-diffusion model describing atmospheric dust transport has been formulated and examined. In contrast to classical approaches, the proposed framework incorporates degenerate nonlinear diffusion, explicitly accounts for interphase exchange, and allows for spatially heterogeneous coefficients. These extensions make it possible to represent more realistic transport mechanisms arising in nonuniform environments.

To analyze the qualitative behavior of the system, a self-similar reduction was employed. Within this framework, the existence of weak solutions was established under appropriate structural assumptions on the nonlinear terms and coefficients. This provides a rigorous mathematical basis for the model and ensures its well-posedness in the considered setting.

The representative numerical simulation shown in Figure 1 supports the analytical findings and illustrates several characteristic features of the model; a more extensive set of numerical experiments (varying parameters, time evolution, and comparison against the analytical self-similar profiles) is left for a follow-up study. In particular, the simulations reveal the formation of localized concentration profiles and

Taken together, the analytical results and computational evidence indicate that the proposed model provides a consistent and sufficiently flexible description of atmospheric dust dynamics. At the same time, the structure of the model allows for natural extensions. For instance, the inclusion of advective transport or more complex boundary interactions would enable the study of a broader class of environmental processes and may serve as a direction for further investigation in applied mathematical modeling.

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