Existence and Uniqueness of Solutions for a New Model of a Triple Nonlinear Elliptic System with Various Boundary Conditions

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Abstract: As is known, the investigation for finding the numerical solution for any mathematical model need at first to check the existence of unique analytic solution for this model. For this reason, the study for existence of a unique solution (the solvability) for any mathematical model is considered important. This importance encouraged us to devote about the study for the solvability of a new mathematical model describes by a system of triple nonlinear elliptic PDEs (TNLEPDEs) with various kinds of boundary conditions (BCs), as the Dirichlet (DBCs), the Neumann(NBCs), the Robin(RBCs) and the Mixed BCs(MBCs). In this study, the weak form (WF) for the TNLEPDEs associated with each type of the above mentioned BCs is formulated in infinite dimensional space. Then the existence theorem of a unique solution for each type of the resulted weak form is proved by utilizing the Minty-Browder theorem (MBTH), with the help of the Friedrichs- Poincare inequality (FPI) when the BCs are of homogeneous kind (HBCs) and with the help of the generalization Friedrichs- Poincare inequality (GFPI) besides to the trace operator (TRO) when the kinds of the BCs are nonhomogeneous (NHBCs).

1 INTRODUCTION

As it knowns, many different types of real life problems in various branch of the sciences are expressed by mathematical models, like electromagnetic waves [1], the flow of fluids [2], physics of fluid [3], the blood of flow [4], molecules structure [5], vibrations of solids [6], mechanics of quantum [7], interactions of electrons and photons [8], the transfer of heat [9], and optimal control [10], they can be described as mathematical models which are usually represented by ODEs or PDEs.

Similarly, the mathematical models, the differential equations had also the major role in modern studies mathematics and their applications [11]-[13]. Therefore, many investigators had interested about the study of the existence and uniqueness for various types of “single” PDEs, especially those which were studied in [14]-[16].

The investigating of the existence and uniqueness of the solution of the PDEs are considered the cornerstone to investigate the ability to find the solutions of the PDEs analytically, and in addition it helps and encourages to investigate the numerical solution of these problems who’s their analytic solutions are difficult to determine. We can say that, without any knowledge about the solvability of any PDEs, no one can try to find the solutions neither analytically nor numerically. From the above studies and as a result, this paper is devoted about studying the solvability (in infinite dimension) of the proposed TNLEPDEs with different types of the boundary conditions (BCs), including that the Dirichlet (DBCs), the Neumann (NBCs), the Robin (RBCs) and the mixed BCs (MBCs). At first the WF of the proposed TNLEPDEs with each kind of the four BCs is found. Then, the existence theorem of a unique triple solution (TRS) of each resulted WF is demonstrated under appropriate hypotheses by utilizing the MBTH and with the help of the (PFI) when the BCs is homogenous (HBCs), and by the help of the generalization PFI (GPFI) besides to the trace operator (TRO) when the BCs is nonhomogeneous (NHBCs). As we mentioned above, the outcomes of this manuscript are considered much important and will help many investigators who are interested about solvability of mathematical control problems [17]-[19], and of many other real life problems analogs of the considered models here.

2 MANUSCRIPT PREPARATION

This section includes some basic definition, theorems and lemmas which are interested in our work, and then a description for the problem formulation.

2.1 Basic Concepts

Definition 2.1.1 [20]. A function $k(x, y): \Omega \times \mathbb{R}^N \to \mathbb{R}^M$ is said to be of a "Carathéodory type" if it is continuous with respect to y for fixed $x \in \Omega$ and it is measurable w.r.t. $x \in \Omega$ for fixed $y \in \mathbb{R}^N$.

Theorem 2.1.2 [21]. Let x and y be two elements in an inner product space with norm, then $|(x, y)| \leq \|x\| \|y\|$.

Theorem 2.1.3 (MBTH) [22]. Let V be a reflexive Banach space and V^* be its dual space. Consider the continuous nonlinear map: $V \longrightarrow V^*$, s.t.

$\langle Ay_1 - Ay_2, y_1 - y_2 \rangle > 0$, for $y_1 \neq y_2, \forall y_1, y_2 \in V$, and $\lim_{\|y\|_{H_0^1(\Omega)} \to \infty} \frac{\langle Ay, y \rangle}{\|y\|_{H_0^1(\Omega)}} \to \infty$.

Then there exists a unique solution $y \in V$ of the equation $Ay = f$, for any $f \in V^*$.

Lemma 2.1.4 (PFI)[23]. Let Ω be any bounded Lipschitz domain [24], then there exists a positive constant $c(\Omega) > 0$, (depends on Ω only), for which the following inequality holds)

$$\int_\Omega |y|^2\, dx \leq c(\Omega) \int_\Omega |\nabla y|^2\, dx, \ (\forall y \in H_0^1(\Omega).$$

Lemma 2.1.5 (GPFI) [25]. Let $\Omega \subset \mathbb{R}^N$ be Lipschitz bounded domain and let $\Gamma_1 \subset \Gamma$ be a measurable set such that $|\Gamma_1| > 0$. Then there exists a constant $c(\Gamma_1) > 0$, independent of y, s.t.

$\|y\|^2_{H^1(\Omega)} \leq c(\Gamma_1)(\int_\Omega |\nabla y|^2\, dx + \left(\int_\Gamma y\, ds\right)^2)$, for all $y \in H^1(\Omega)$.

Proposition 2.1.6 [26]. Let $f: \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be of the Carathéodory type defined in the measurable subset $\Omega \subset \mathbb{R}^n$, s.t.

$\|f(x, y)\| \leq \varphi(x) + \xi(x)\|y\|^\alpha, \forall x \in \Omega, y \in L^P(\Omega)$,

where $\varphi \in L^1(\Omega), \zeta \in L^{\frac{P}{P-\alpha}}(\Omega)$, and $1 \leq \alpha \leq P$, if $1 \leq P < \infty$, and $\zeta \equiv 0$, if $P = \infty$.

Then $F(y) = \int_\Omega f(x, y(x))dx$ is continuous on $L^P(\Omega)$.

Theorem 2.1.7 (TRP) [27]. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$, then the following TRO exists $T_0: H^1(\Omega) \to L^2(\Gamma)$, s.t., $T_0 u = u_{|\Gamma}$, and $\|T_0 u\|_{L^2(\Gamma)} \leq c\|u\|_{H^1(\Omega)}$, where $c = c(\Omega, d)$ be a suitable constant and $T_0 u$ represents the trace of u on Γ.

2.2 The Proposed Problem

Let $\Omega \subset \mathbb{R}^2$ be any open bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$, then the TNLEPDEs with the HDBCs are:

$$A_1 y_1 + B_1 y_1 + c_1(x) y_1 - d_1(x) y_2 + d_2(x) y_3 + + f_1(x, y_1) = h_1(x), \text{ in } \Omega \quad (1)$$

$$A_2 y_2 + B_2 y_2 + c_2(x) y_2 + d_1(x) y_1 - d_3(x) y_3 + f_2(x, y_2) = h_2(x), \text{ in } \Omega \quad (2)$$

$$A_3 y_3 + B_3 y_3 + c_3(x) y_3 - d_2(x) y_1 + d_3(x) y_2 + + f_3(x, y_3) = h_3(x), \text{ in } \Omega \quad (3)$$

$$y_1 = 0, \text{ on } \Gamma. \quad (4)$$

$$y_2 = 0, \text{ on } \Gamma \quad (5)$$

$$y_3 = 0, \text{ on } \Gamma \quad (6)$$

where $x = (x_1, x_2), \vec{y} = (y_1, y_2, y_3) \in \left(H_0^2(\Omega)\right)^3$,

$$A_l y_l = -\sum_{i,j=1}^2 \frac{\partial}{\partial x_i}\left(a_{lij}(x) \frac{\partial y_l}{\partial x_j}\right), \quad l = 1,2,3,$$

$B_l y_l = \sum_{i=1}^2 b_{li}(x) \frac{\partial y_l}{\partial x_i}, \quad l = 1,2,3, \quad c_l, d_l \in L^\infty(\Omega)$, $c_l, d_l \geq 0, \ b_{li}, a_{lij} \in L^\infty(\Omega) \ b_{li} \geq \beta_{lij} > 0, \ a_{lij} \geq \alpha_{lij} > 0, \forall\, l = 1,2,3, \ \forall i, j = 1,2$ and the functions $f_l(x, y_l), h_l(x) \in L^2(\Omega)$ for $l = 1,2,3$ are of the type of Carathéodory on $\Omega \times \mathbb{R}$ and on Ω resp.

2.3 The Weak Formulation of the System

The WF of ((1)-(6)) resulted by multiplying them by the "test function [28]" $v_l \in V = H_0^1(\Omega)$, $(l = 1,2,3)$, then taking the integral and finally by utilizing the general Green's theorem [29] for the terms that have 2nd order of partial derivatives, to get:

$$\int_\Omega \sum_{i,j=1}^2 a_{1ij} \frac{\partial y_1}{\partial x_i}\frac{\partial v_1}{\partial x_j} dx + \int_\Omega \sum_{i=1}^2 b_{1i}(x) \frac{\partial y_1}{\partial x_i} v_1 dx + \int_\Omega [c_1(x) y_1 v_1 - d_1(x) y_2 v_1 + d_2(x) y_3 v_1] dx + \int_\Omega f_1(x, y_1) v_1 dx = \int_\Omega h_1 v_1 dx \quad (7)$$

$$\int_\Omega \sum_{i,j=1}^2 a_{2ij} \frac{\partial y_2}{\partial x_i}\frac{\partial v_2}{\partial x_j} dx + \int_\Omega \sum_{i=1}^2 b_{2i}(x) \frac{\partial y_2}{\partial x_i} v_2 dx + \int_\Omega [c_2(x) y_2 v_2 + d_1(x) y_1 v_2 - d_3(x) y_3 v_2] dx + \int_\Omega f_2(x, y_2) v_2 dx = \int_\Omega h_2 v_2 dx \quad (8)$$

$$\int_\Omega \sum_{i,j=1}^{2} a_{3ij} \frac{\partial y_3}{\partial x_i}\frac{\partial v_3}{\partial x_j} dx + \int_\Omega \sum_{i=1}^{2} b_{3i}(x) \frac{\partial y_3}{\partial x_i} v_3 dx + \int_\Omega [c_3(x) y_3 v_3 - d_2(x) y_1 v_3 + d_3(x) y_2 v_3 dx + \int_\Omega f_3(x, y_3) v_3 dx = \int_\Omega h_3 v_3 dx \quad (9)$$

By gathering ((7)-(9)), they give

$$\langle \bar{A}(\vec{y}), \vec{v} \rangle = (\vec{F}(x), \vec{v}), \quad (10)$$

where $\bar{A}: \vec{V} \longrightarrow \vec{V}^*$, with $\vec{V} = V \times V \times V$ and $\vec{V}^*$ is the dual of $\vec{V}$, s.t.

$$\langle \bar{A}(\vec{y}), \vec{v} \rangle = a(\vec{y}, \vec{v}) + (f_1(x, y_1), v_1)_\Omega + +(f_2(x, y_2), v_2)_\Omega + (f_3(x, y_3), v_3)_\Omega \quad (11)$$

$$a(\vec{y}, \vec{v}) = a_1(y_1, v_1) + b_1(y_1, v_1) + (c_1 y_1, v_1)_\Omega - (d_1 y_2, v_1)_\Omega + (d_2 y_3, v_1)_\Omega + a_2(y_2, v_2) + b_2(y_2, v_2) + (c_2 y_2, v_2)_\Omega + (d_1 y_1, v_2)_\Omega - (d_3 y_3, v_2)_\Omega + a_3(y_3, v_3) + b_3(y_3, v_3) + (c_3 y_3, v_3)_\Omega - (d_2 y_1, v_3)_\Omega + (d_3 y_2, v_3)_\Omega \quad (12)$$

where $a_l(y_l, v_l) = \sum_{i,j=1}^{2} a_{lij} \frac{\partial y_l}{\partial x_i}\frac{\partial v_l}{\partial x_j}$, for $l = 1,2,3$

$b_l(y_l, v_l) = \sum_{i=1}^{2} b_{li}(x) \frac{\partial y_l}{\partial x_i} v_l$, for $l = 1,2,3$

$$(\vec{F}(x), \vec{v}) = (h_1(x), v_1)_\Omega + (h_2(x), v_2)_\Omega + (h_3(x), v_3)_\Omega \quad (13)$$

2.3.1 Assumptions

The functions $f_l(x, y_l)$ for $l = 1,2,3$ are of the type Carathéodory and monotone [30] w.r.t. y_l, $\forall x \in \Omega$, s.t.

1) $|f_l(x, y_1)| \le \phi_l(x) + \bar{c}_l |y_l|$, $\forall (x, y_l) \in \Omega \times \mathbb{R}$ with $\phi_l \in L^2(\Omega)$, $\bar{c}_l \ge 0$.
2) $f_l(x, 0) = 0$, $\forall x \in \Omega$.

2.4 Solvability of the Triple Nonlinear Homogeneous Problem

The solvability of WF (10) is proved in the following theorem.

Theorem 2.4.1. The WF (10) owns a unique triple solution (TRS) $\vec{y} \in \left(H_0^1(\Omega)\right)^3$, if assumptions 2.3.1 are held with f_1 is strictly monotone.

Proof. The proof resulted through utilizing Theorem 2.1.3, so let be started by proving that $\langle \bar{A}(\vec{y}), \vec{y} \rangle$ in the LHS of (10) is coercive, as follows:

$$\langle \bar{A}(\vec{y}), \vec{y} \rangle = a_1(y_1, v_1) + b_1(y_1, v_1) + (c_1 y_1, v_1)_\Omega - (d_1 y_2, v_1)_\Omega + (d_2 y_3, v_1)_\Omega + a_2(y_2, v_2) + b_2(y_2, v_2) + (c_2 y_2, v_2)_\Omega + (d_1 y_1, v_2)_\Omega - (d_3 y_3, v_2)_\Omega + a_3(y_3, v_3) + b_3(y_3, v_3) + (c_3 y_3, v_3)_\Omega - (d_2 y_1, v_3)_\Omega + (d_3 y_2, v_3)_\Omega + (f_1(x, y_1), v_1)_\Omega + (f_2(x, y_2), y_2)_\Omega + (f_3(x, y_3), y_3)_\Omega \quad (14)$$

$$\langle \bar{A}(\vec{y}), \vec{y} \rangle = \int_\Omega \sum_{i,j=1}^{2} a_{1ij}(x) \frac{\partial y_1}{\partial x_i}\frac{\partial y_1}{\partial x_j} dx + \int_\Omega \sum_{i=1}^{2} b_{1i}(x) \frac{\partial y_1}{\partial x_i} y_1 dx + \int_\Omega c_1(x) y_1 y_1 dx - \int_\Omega d_1(x) y_2 y_1 dx + \int_\Omega d_2(x) y_3 y_1 dx + \int_\Omega \sum_{i,j=1}^{2} a_{2ij}(x) \frac{\partial y_2}{\partial x_i}\frac{\partial y_2}{\partial x_j} dx \int_\Omega c_2(x) y_2 y_2 dx + \int_\Omega \sum_{i=1}^{2} b_{2i}(x) \frac{\partial y_2}{\partial x_i} y_2 dx + \int_\Omega d_1(x) y_1 y_2 dx - \int_\Omega d_3(x) y_3 y_2 dx + \int_\Omega \sum_{i,j=1}^{2} a_{3ij}(x) \frac{\partial y_3}{\partial x_i}\frac{\partial y_3}{\partial x_j} dx + \int_\Omega \sum_{i=1}^{2} b_{3i}(x) \frac{\partial y_3}{\partial x_i} y_3 dx + \int_\Omega c_3(x) y_3 y_3 dx - \int_\Omega d_2(x) y_1 y_3 dx + \int_\Omega d_3(x) y_2 y_3 dx + \int_\Omega f_1(x, y_1) y_1 dx + \int_\Omega f_3(x, y_3) y_3 dx + \int_\Omega f_2(x, y_2) y_2 dx$$

Since $a_{lij}(x) \ge \alpha_{lij} > 0$, $(\forall\, l = 1,2,3, \forall\, \mathrm{i}, \mathrm{j} = 1,2)$, and by utilizing assumptions 2.3.1, with putting $\frac{\partial y_l}{\partial x_i} y_l = \frac{1}{2}\frac{\partial}{\partial x_i}({y_l}^2)$, in the 2nd, the 8th and the 14th terms, then:

$$\langle \bar{A}(\vec{y}), \vec{y} \rangle \ge \alpha_{1ij} \sum_{i=1}^{2} \int_\Omega \left|\frac{\partial y_1}{\partial x_i}\right|^2 dx + \int_\Omega c_1(x) |y_1|^2 dx + \sum_{i=1}^{2} \int_\Omega b_{1i}(x) \frac{1}{2}\frac{\partial}{\partial x_i}({y_1}^2) dx + +\alpha_{2ij} \sum_{i=1}^{2} \int_\Omega \left|\frac{\partial y_2}{\partial x_i}\right|^2 dx \sum_{i=1}^{2} \int_\Omega b_{2i}(x) \frac{1}{2}\frac{\partial}{\partial x_i}({y_2}^2) dx + + \int_\Omega c_2(x) |y_2|^2 dx + \sum_{i=1}^{2} \int_\Omega b_{3i}(x) \frac{1}{2}\frac{\partial}{\partial x_i}({y_3}^2) dx + \int_\Omega c_3(x) |y_3|^2 dx + +\alpha_{3ij} \sum_{i=1}^{2} \int_\Omega \left|\frac{\partial y_3}{\partial x_i}\right|^2 dx + + \int_\Omega (f_1(x, y_1) - f_1(x, 0))(y_1 - 0) dx + + \int_\Omega (f_2(x, y_2) - f_2(x, 0))(y_2 - 0) dx + + \int_\Omega (f_3(x, y_3) - f_3(x, 0))(y_3 - 0) dx. \quad (15)$$

Since f_l is monotone w.r.t. y_l, and letting $\alpha = \min(\alpha_{lij})$, $\forall\, l = 1,2,3, \forall$ i, j = 1,2, utilizing integrating by parts in the 2nd, the 5th and the 8th terms on the RHS of (15), to obtain:

$$\langle \bar{A}(\vec{y}), \vec{y}\rangle \geq \alpha\left(\sum_{i=1}^{2}\int_\Omega\left|\frac{\partial y_1}{\partial x_i}\right|^2 dx + \sum_{i=1}^{2}\int_\Omega\left|\frac{\partial y_2}{\partial x_i}\right|^2 dx + \sum_{i=1}^{2}\int_\Omega\left|\frac{\partial y_3}{\partial x_i}\right|^2 dx\right) + \int_\Omega\left(c_1(x) - \frac{1}{2}\sum_{i=1}^{2}\frac{\partial b_{1i}}{\partial x_i}\right)|y_1|^2 dx + \int_\Omega\left(c_2(x) - \frac{1}{2}\sum_{i=1}^{2}\frac{\partial b_{2i}}{\partial x_i}\right)|y_2|^2 dx + \int_\Omega\left(c_3(x) - \frac{1}{2}\sum_{i=1}^{2}\frac{\partial b_{3i}}{\partial x_i}\right)|y_3|^2 dx \quad (16)$$

Select b_{li} and c_l, for $l = 1,2,3, \forall i = 1,2$, s.t.

$$c_l(x) - \frac{1}{2}\sum_{i=1}^{2}\frac{\partial b_{li}}{\partial x_i} \geq 0,\ x \in \bar{\Omega},$$

which gives

$$\langle \bar{A}(\vec{y}), \vec{y}\rangle \geq \alpha\sum_{l=1}^{3}\|\Delta y_l\|^2_{L^2(\Omega)}. \quad (17)$$

Then by Lemma 2.1.4

$$\langle \bar{A}(\vec{y}), \vec{y}\rangle \geq \frac{\alpha}{C_*}\sum_{l=1}^{3}\|y_l\|^2_{L^2(\Omega)}. \quad (18)$$

Adding (17) with (18), setting $C_0 = \min(\alpha, \frac{\alpha}{C_*})$, to get:

$\langle \bar{A}(\vec{y}), \vec{y}\rangle \geq C_0\sum_{l=1}^{3}(\|y_l\|^2_{L^2(\Omega)} + \|\Delta y_l\|^2_{L^2(\Omega)}) = C_0\sum_{l=1}^{3}\|y_l\|^2_{H_0^1}$.

Thus:

$$\langle \bar{A}(\vec{y}), \vec{y}\rangle \geq C_0\|\vec{y}\|^2_{\left(H_0^1(\Omega)\right)^3}. \quad (19)$$

On other hand, since $a_{lij}(x), b_{li}(x), c_l(x), d_l(x) \in L^\infty(\Omega)$, $(\forall l = 1,2,3, \forall i = 1,2)$, so:

$$\left|\int_\Omega\sum_{i,j=1}^{2}a_{lij}(x)\frac{\partial y_l}{\partial x_i}\frac{\partial v_l}{\partial x_j}dx\right| \leq \sum_{i,j=1}^{2}\mu_l\left|\int_\Omega\frac{\partial y_l}{\partial x_i}\frac{\partial v_l}{\partial x_j}dx\right| \quad (20)$$

$$\left|\int_\Omega\sum_{i=1}^{2}b_{li}(x)\frac{\partial y_l}{\partial x_i}v_l dx\right| \leq \sum_{i=1}^{2}\bar{\mu}_l\left|\int_\Omega\frac{\partial y_l}{\partial x_i}v_l dx\right| \quad (21)$$

$$\left|\int_\Omega c_l(x)y_l v_l\right| \leq \bar{\bar{\mu}}_l\left|\int_\Omega y_l v_l\,dx\right| \quad (22)$$

$$\left|\int_\Omega d_l(x)y_{3-l}v_l\right| \leq \bar{\bar{\bar{\mu}}}_1\left|\int_\Omega y_{3-l}v_l\,dx\right| \quad (23)$$

$$\left|\int_\Omega d_l(x)y_{4-l}v_l\right| \leq \bar{\bar{\bar{\mu}}}_2\left|\int_\Omega y_{4-l}v_l\,dx\right| \quad (24)$$

$$\left|\int_\Omega d_l(x)y_{5-l}v_l\right| \leq \bar{\bar{\bar{\mu}}}_3\left|\int_\Omega y_{5-l}v_l\,dx\right| \quad (25)$$

where $\mu_l = \max_{x\in\bar{\Omega}}|a_{lij}(x)|$, $\bar{\mu}_l = \max_{x\in\bar{\Omega}}|b_{li}(x)|$, $\bar{\bar{\mu}}_l = \max_{x\in\bar{\Omega}}|c_l(x)|$, $\forall l = 1,2,3$, $(\bar{\bar{\bar{\mu}}}_1 = \max_{x\in\bar{\Omega}}|d_l(x)|$, $\forall\ l = 1,2)$, $(\bar{\bar{\bar{\mu}}}_2 = \max_{x\in\bar{\Omega}}|d_l(x)|$, $\forall\, l = 1,3)$, and $(\bar{\bar{\bar{\mu}}}_3 = \max_{x\in\bar{\Omega}}|d_l(x)|$, $\forall\ l = 2,3$).

Therefore, by (20)-(25):

$$|a(\vec{y}, \vec{v})| \leq \sum_{l=1}^{3}\sum_{i,j=1}^{2}\mu_l\left|\int_\Omega\frac{\partial y_l}{\partial x_i}\frac{\partial v_l}{\partial x_j}dx\right| + \sum_{l=1}^{3}\sum_{i,j=1}^{2}\bar{\mu}_l\left|\int_\Omega\frac{\partial y_l}{\partial x_i}v_l\,dx\right| + \sum_{l=1}^{3}[\bar{\bar{\mu}}_l\left|\int_\Omega y_l v_l\,dx\right| + \bar{\bar{\bar{\mu}}}_1\left|\int_\Omega y_{3-l}v_l\,dx\right| + \bar{\bar{\bar{\mu}}}_2\left|\int_\Omega y_{4-l}v_l\,dx\right| + \bar{\bar{\bar{\mu}}}_3\left|\int_\Omega y_{5-l}v_l\,dx\right|].$$

By utilizing theorem 2.1.2, to get:

$$|a(\vec{y}, \vec{v})| \leq \mu\Bigg\{\sum_{i,j=1}^{2}\left(\int_\Omega\left|\frac{\partial y_1}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega\left|\frac{\partial v_1}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2}\left(\int_\Omega\left|\frac{\partial y_2}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega\left|\frac{\partial v_2}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2}\left(\int_\Omega\left|\frac{\partial y_3}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega\left|\frac{\partial v_3}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}} + \sum_{i=1}^{2}\left(\int_\Omega\left|\frac{\partial y_1}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega|v_1|^2 dx\right)^{\frac{1}{2}} + \sum_{i=1}^{2}\left(\int_\Omega\left|\frac{\partial y_2}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega|v_2|^2 dx\right)^{\frac{1}{2}} + \sum_{i=1}^{2}\left(\int_\Omega\left|\frac{\partial y_3}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega|v_3|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega|y_1|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega|v_1|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega|y_2|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega|v_2|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega|y_3|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega|v_3|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega|y_2|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega|v_1|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega|y_1|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega|v_2|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega|y_3|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega|v_1|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega|y_1|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega|v_3|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega|y_3|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega|v_2|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega|y_2|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega|v_3|^2 dx\right)^{\frac{1}{2}}\Bigg\},$$

Where $\mu = \max\{\mu_l, \bar{\mu}_l, \bar{\bar{\mu}}_l, \bar{\bar{\bar{\mu}}}_l\}, \forall l = 1,2,3, \forall i = 1,2.$

$$|a(\vec{y},\vec{v})| \leq\leq \mu\Bigg(\Bigg\{\left(\int_\Omega |y_1|^2\,dx\right)^{\frac{1}{2}} + \sum_{j=1}^{2}\left(\int_\Omega \left|\frac{\partial y_1}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}}\Bigg\} \times \Bigg\{\left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2}\left(\int_\Omega \left|\frac{\partial v_1}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}}\Bigg\} + \Bigg\{\left(\int_\Omega |y_2|^2\,dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2}\left(\int_\Omega \left|\frac{\partial y_2}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}}\Bigg\} \times \Bigg\{\left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2}\left(\int_\Omega \left|\frac{\partial v_2}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}}\Bigg\} + \Bigg\{\left(\int_\Omega |y_3|^2\,dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2}\left(\int_\Omega \left|\frac{\partial y_3}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}}\Bigg\} \times \Bigg\{\left(\int_\Omega |v_3|^2 dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2}\left(\int_\Omega \left|\frac{\partial v_3}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}}\Bigg\} + \left(\int_\Omega |y_2|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_1|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_3|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_1|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_3|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_3|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_2|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_3|^2 dx\right)^{\frac{1}{2}}\Bigg) \tag{26}$$

The RHS of (26), can be represented as:

$$\begin{aligned}
|a(\vec{y},\vec{v})| \leq\ & 4\mu\left\{\int_\Omega |y_1|^2 dx + \sum_{j=1}^{2}\int_\Omega \left|\frac{\partial y_1}{\partial x_j}\right|^2 dx\right\}^{\frac{1}{2}} \times \\
&\times \left\{\int_\Omega |v_1|^2 dx + \sum_{i=1}^{2}\int_\Omega \left|\frac{\partial v_1}{\partial x_i}\right|^2 dx\right\}^{\frac{1}{2}} + \\
&+4\mu\left\{\int_\Omega |y_2|^2\,dx + \sum_{j=1}^{2}\int_\Omega \left|\frac{\partial y_2}{\partial x_j}\right|^2 dx\right\}^{\frac{1}{2}} \times \\
&\times \left\{\int_\Omega |v_2|^2\,dx + \sum_{i=1}^{2}\int_\Omega \left|\frac{\partial v_2}{\partial x_i}\right|^2 dx\right\}^{\frac{1}{2}} + \\
&+4\mu\left\{\int_\Omega |y_3|^2 dx + \sum_{j=1}^{2}\int_\Omega \left|\frac{\partial y_3}{\partial x_j}\right|^2 dx\right\}^{\frac{1}{2}} \times \\
&\times \left\{\int_\Omega |v_3|^2 dx + \sum_{i=1}^{2}\int_\Omega \left|\frac{\partial v_3}{\partial x_i}\right|^2 dx\right\}^{\frac{1}{2}} + \\
&+\mu\left(\int_\Omega |y_2|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \\
&+\mu\left(\int_\Omega |y_1|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}} + \\
&+\mu\left(\int_\Omega |y_3|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \\
&+\mu\left(\int_\Omega |y_1|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_3|^2 dx\right)^{\frac{1}{2}} + \\
&+\mu\left(\int_\Omega |y_3|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}} + \\
&+\mu\left(\int_\Omega |y_2|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_3|^2 dx\right)^{\frac{1}{2}} \\
\leq\ & 4\mu\Big[\|y_1\|_{H_0^1(\Omega)}\|v_1\|_{H_0^1(\Omega)} + \\
&+\|y_2\|_{H_0^1(\Omega)}\|v_2\|_{H_0^1(\Omega)} + \\
&+\|y_3\|_{H_0^1(\Omega)}\|v_3\|_{H_0^1(\Omega)}\Big] + \\
&+\mu[\|y_2\|_{H_0^1(\Omega)}\|v_1\|_{H_0^1(\Omega)} + \\
&+\|y_1\|_{H_0^1(\Omega)}\|v_2\|_{H_0^1(\Omega)} + \\
&+\|y_3\|_{H_0^1(\Omega)}\|v_1\|_{H_0^1(\Omega)} + \\
&+\|y_1\|_{H_0^1(\Omega)}\|v_3\|_{H_0^1(\Omega)} + \\
&+\|y_3\|_{H_0^1(\Omega)}\|v_2\|_{H_0^1(\Omega)} + \\
&+\|y_2\|_{H_0^1(\Omega)}\|v_3\|_{H_0^1(\Omega)}].
\end{aligned} \tag{27}$$

Then, by setting $\rho = 18\mu$, it concludes

$$|a(\vec{y},\vec{v})| \leq \rho\,\|\vec{y}\|_{\left(H_0^1(\Omega)\right)^3}\|\vec{v}\|_{\left(H_0^1(\Omega)\right)^3}. \tag{28}$$

Besides this result, from assumptions 2.3.1 and then utilizing Proposition 2.1.6, once get the continuity of $\int_\Omega \bar{f}_l(x, y_l(x))dx$ w.r.t. y_l(for $l = 1,2,3$).

Hence, (27) and (28), gives the continuity of $\vec{y} \longmapsto \langle \bar{A}(\vec{y}), \vec{v}\rangle$ w.r.t. $\vec{y}$.

In the RHS of (10), it is obvious that the linearity property of the functional $F(\vec{v})$ on $\left(H_0^1(\Omega)\right)^3$ it satisfied, so it remains to show $F(\vec{v})$ it is bounded, so from theorem 2.1.2:

$$|F(\vec{v})| \le \sum_{l=1}^{3}[\|h_l\|_{L^2(\Omega)}\|v_l\|_{L^2(\Omega)}] \le \sum_{l=1}^{3}[\gamma_l\|v_l\|_{L^2(\Omega)}] \le \gamma\|\vec{v}\|_{\left(L^2(\Omega)\right)^3} \le \gamma\|\vec{v}\|_{\left(H_0^1(\Omega)\right)^3} \quad (29)$$

where $\|h_l\|_{L^2(\Omega)} \le \gamma_l, \gamma = \max(\gamma_l), \forall l = 1,2,3$.

Since f_1 is strictly monotone, let $\vec{v} = \vec{y} - \vec{\bar{y}}$, then from (15), one has:

$$\langle \bar{A}(\vec{y}), \vec{y} - \vec{\bar{y}}\rangle = a(\vec{y}, \vec{y} - \vec{\bar{y}}) + (f_1(x, y_1), y_1 - \bar{y}_1)_\Omega + (f_2(x, y_2), y_2 - \bar{y}_2)_\Omega + (f_3(x, y_3), y_3 - \bar{y}_3)_\Omega \quad (30)$$

$$\langle \bar{A}(\vec{\bar{y}}), \vec{y} - \vec{\bar{y}}\rangle = a(\vec{\bar{y}}, \vec{y} - \vec{\bar{y}}) + (f_1(x, \bar{y}_1), y_1 - \bar{y}_1)_\Omega + (f_2(x, \bar{y}_2), y_2 - \bar{y}_2)_\Omega + (f_3(x, \bar{y}_3), y_3 - \bar{y}_3)_\Omega \quad (31)$$

Subtracting (31) from (30) for $\vec{y} \ne \vec{\bar{y}}$, to get

$$\langle \bar{A}(\vec{y}) - \bar{A}(\vec{\bar{y}}), \vec{y} - \vec{\bar{y}}\rangle > c\|\vec{y} - \vec{\bar{y}}\|^2_{\left(H_0^1(\Omega)\right)^3} > 0, \quad (32)$$

Thus, from the above steps, all the hypotheses of Theorem 2.1.3, are obtained, hence there exists a unique TRS $\vec{y} = (y_1, y_2, y_3) \in \left(H_0^1(\Omega)\right)^3$ of (10).

Corollary 2.4.2. There exists a unique TRS $\vec{y} \in \left(H_0^1(\Omega)\right)^3$ of the WF of the proposed TNLEPDEs ((1)-(3)), with the following homogeneous NBCs(HNBCs):

$$\frac{\partial y_1}{\partial n_1} = \sum_{i,j=1}^{2} a_{1ij}\frac{\partial y_1}{\partial x_i}\cos(n_1, x_j) = 0, \quad \text{on } \Gamma \quad (33)$$

$$\frac{\partial y_2}{\partial n_2} = \sum_{i,j=1}^{2} a_{2ij}\frac{\partial y_2}{\partial x_i}\cos(n_2, x_j) = 0, \quad \text{on } \Gamma \quad (34)$$

$$\frac{\partial y_3}{\partial n_3} = \sum_{i,j=1}^{2} a_{3ij}\frac{\partial y_3}{\partial x_i}\cos(n_3, x_j) = 0, \quad \text{on } \Gamma \quad (35)$$

where n_ℓ, (for $\ell = 1,2,3$) is a normal unit vector on Γ and (n_ℓ, x_j) is the angle between n_ℓ and the $x_j - axis$.

Proof. The WF which results from ((1)- (3)) with the HNBCs ((33)-(35)) is similar to the WF in (10) with v_ℓ satisfies ((33)-(35)) for $\ell = 1,2,3$ resp., i.e.

$$\langle \bar{A}(\vec{y}), \vec{y}\rangle = F(\vec{v}). \quad (36)$$

with $\langle \bar{A}(\vec{y}), \vec{y}\rangle$ & $F(\vec{v})$ are defined in (11) & (12) resp.

Hence by Theorem 2.1.3, the WF (36) owns a unique TRS $\vec{y} \in \left(H_0^1(\Omega)\right)^3$.

Corollary 2.4.3. There exists a unique TRS $\vec{y} \in \left(H^1(\Omega)\right)^3$ of the WF of the proposed TNLEPDEs ((1)-(3)), with the HRBCs:

$$\alpha_1 y_1 + \frac{\partial y_1}{\partial n_1} = \alpha_1 y_1 + \sum_{i,j=1}^{2} a_{1ij}\frac{\partial y_1}{\partial x_i}\cos(n_1, x_j) = 0, \quad \text{on } \Gamma \quad (37)$$

$$\alpha_2 y_2 + \frac{\partial y_2}{\partial n_2} = \alpha_2 y_2 + \sum_{i,j=1}^{2} a_{2ij}\frac{\partial y_2}{\partial x_i}\cos(n_2, x_j) = 0, \quad \text{on } \Gamma. \quad (38)$$

$$\alpha_3 y_3 + \frac{\partial y_3}{\partial n_3} = \alpha_3 y_3 + \sum_{i,j=1}^{2} a_{3ij}\frac{\partial y_3}{\partial x_i}\cos(n_3, x_j) = 0, \quad \text{on } \Gamma \quad (39)$$

where $g_1, g_2, g_3 \in L^2(\Gamma)$ and $\alpha_1, \alpha_2, \alpha_2 > 0$, and n_ℓ, ($\ell = 1,2,3$) is as defined in corollary 2.4.2.

Proof. The WF which results from ((1)- (3)) with the HRBCs ((37)-(39)) is similar to the WF in (10), with v_ℓ satisfies ((37)-(39)) for $\ell = 1,2,3$ resp., i.e.

$$\langle \bar{A}(\vec{y}), \vec{y}\rangle = F(\vec{v}) \quad (40)$$

with $\langle \bar{A}(\vec{y}), \vec{y}\rangle$ & $F(\vec{v})$ are defined in (11) & (12) resp. Hence by Theorem 2.1.3, the WF (40) owns a unique TRS $\vec{y} \in \left(H_0^1(\Omega)\right)^3$.

Corollary 2.4.4. There exists a unique TRS $\vec{y} \in \left(H_0^1(\Omega)\right)^3$ of the WF of the TNLEPDEs ((1)-(3)) with the following Mixed HBCs (MHBCs),

$$y_l = 0 \quad \text{on} \quad \Gamma_D, l = 1,2,3 \quad (41)$$

$$\frac{\partial y_l}{\partial n_l} = 0 \quad \text{on} \quad \Gamma_N, l = 1,2,3 \quad (42)$$

where the boundary Γ of Ω be divided in two open subset Γ_D and Γ_N, with $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$, and n_l is a normal unit vector on Γ_N.

Proof. The WF which results from ((1)- (3)) with the HNBCs ((41)-(42)) is similar to the WF in (10) with v_ℓ satisfies ((41)&(42)) for $\ell = 1,2,3$, i.e.

$$\langle \bar{A}(\vec{y}), \vec{y}\rangle = F(\vec{v}). \quad (43)$$

with $\langle \bar{A}(\vec{y}), \vec{y}\rangle$ & $F(\vec{v})$ are defined as in (11) & (12) resp.

Hence by Theorem 2.1.3, the WF (43) owns a unique TRS $\vec{y} \in \left(H_0^1(\Omega)\right)^3$.

Example 2.4.5. Let $\Omega = [0,1]^2$ with boundary $\Gamma = \partial\Omega$,, and consider the TNLEPDEs in ((1)-(3)) with the HDBCs in ((4)-(5)), where the coefficients in ((1)-(3)) are defined as

$$A_l y_l = -\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(e^{x_i^2 + ljx_j^2}\frac{\partial y_l}{\partial x_j}\right), c_l(x) = e^{lx_l},$$

$$B_l y_l = \sum_{i=1}^{2} e^{lx_i}\frac{\partial y_1}{\partial x_i}, d_l(x) = lx_1^2 + lx_2^2 + 2l,$$

$$f_l(x, y_l) = |y_l|y_l, \ \forall\, l = 1,2,3,$$

where $h_l(x)$ are given functions on Ω.

Now, since $f_l(x, y_l)$ in the given TNLEPDEs are satisfied Assumptions 2.3.1, then from theorem 2.1.3

the TNLEPDEs (1)-(3) with the HDBCs (4)-(6), has a unique TRS $\vec{y} \in \left(H_0^1(\Omega)\right)^3$.

2.5 Solvability of the Triple Nonlinear Nonhomogeneous Problems

In this section, the solvability of the WF of the TNLEPDEs with the NHBCs is studied.

Theorem 2.5.1. There exists a unique TRS $\vec{y} \in \left(H^1(\Omega)\right)^3$ of the WF of the proposed TNLEPDEs ((1)-(3)), with the NHRBCs:

$$\alpha_1 y_1 + \frac{\partial y_1}{\partial n_1} = \alpha_1 y_1 + \sum_{i,j=1}^{2} a_{1ij} \frac{\partial y_1}{\partial x_i} cos(n_1, x_j) = g_1, \text{ on } \Gamma \tag{44}$$

$$\alpha_2 y_2 + \frac{\partial y_2}{\partial n_2} = \alpha_2 y_2 + \sum_{i,j=1}^{2} a_{2ij} \frac{\partial y_2}{\partial x_i} cos(n_2, x_j) = g_2, \text{ on } \Gamma \tag{45}$$

$$\alpha_3 y_3 + \frac{\partial y_3}{\partial n_3} = \alpha_3 y_3 + \sum_{i,j=1}^{2} a_{3ij} \frac{\partial y_3}{\partial x_i} cos(n_3, x_j) = g_3, \text{ on } \Gamma \tag{46}$$

where $g_1, g_2, g_3 \in L^2(\Gamma)$ and $\alpha_1, \alpha_2, \alpha_2 > 0$, and n_ℓ, ($\ell = 1,2,3$) is as defined in Corollary 2.4.2.

Proof. The WF of ((1)-(3)) and ((44)-(46)) results through multiplying these equations by the "test function" $(v_1, v_2, v_3) = \vec{v} \in \vec{V} = \{\vec{v} : \vec{v} \in \left(H^1(\Omega)\right)^3$, $\vec{v}$ satisfies ((44)-(46)), then taking the integral and finally by utilizing the general Green's theorem for the expressions that have 2nd order of partial derivatives, to get:

$$a_1(y_1, v_1) + (\alpha_1 y_1, v_1)_\Gamma + b_1(y_1, v_1) + (c_1 y_1, v_1)_\Omega - (d_1 y_2, v_1)_\Omega + (d_2 y_3, v_1)_\Omega + (f_1(x, y_1), v_1)_\Omega = (h_1, v_1)_\Omega + (g_1, v_1)_\Gamma. \tag{47}$$

$$a_2(y_2, v_2) + (\alpha_2 y_2, v_2)_\Gamma + b_2(y_2, v_2) + (c_2 y_2, v_2)_\Omega + (d_1 y_1, v_2)_\Omega - (d_3 y_3, v_2)_\Omega + (f_2(x, y_2), v_1)_\Omega = (h_2, v_2)_\Omega + (g_2, v_2)_\Gamma. \tag{48}$$

$$a_3(y_3, v_3) + (\alpha_3 y_3, v_3)_\Gamma + b_3(y_3, v_3) + (c_3 y_3, v_3)_\Omega - (d_2 y_1, v_3)_\Omega + (d_3 y_2, v_3)_\Omega + (f_3(x, y_3), v_3)_\Omega = (h_3, v_3)_\Omega + (g_3, v_3)_\Gamma. \tag{49}$$

(49)

Collecting (47)-(49), to get:

$$\langle \bar{A}(\vec{y}), \vec{v} \rangle = F(\vec{v}), \tag{50}$$

Where:

$$\langle \bar{A}(\vec{y}), \vec{v} \rangle = a(\vec{y}, \vec{v}) + (f_1(x, y_1), v_1)_\Omega + (f_2(x, y_2), v_1)_\Omega + (f_3(x, y_3), v_3)_\Omega, \tag{51}$$

with

$$a(\vec{y}, \vec{v}) = a_1(y_1, v_1) + (\alpha_1 y_1, v_1)_\Gamma + b_1(y_1, v_1) + (c_1 y_1, v_1)_\Omega - (d_1 y_2, v_1)_\Omega + (d_2 y_3, v_1)_\Omega + a_2(y_2, v_2) + (\alpha_2 y_2, v_2)_\Gamma + b_2(y_2, v_2) + (c_2 y_2, v_2)_\Omega + (d_1 y_1, v_2)_\Omega - (d_3 y_3, v_2)_\Omega + a_3(y_3, v_3) + (\alpha_3 y_3, v_3)_\Gamma + b_3(y_3, v_3) + (c_3 y_3, v_3)_\Omega - (d_2 y_1, v_3)_\Omega + (d_3 y_2, v_3)_\Omega \tag{52}$$

here $a_l(y_l, v_l), b_l(y_l, v_l)$ are as defined in (12), $(\alpha_l y_l, v_l)_\Gamma = \int_\Gamma \alpha_l y_l v_l d\gamma, \forall l = 1,2,3$ $\& \forall v_l \in H^1(\Omega)$

$$F(\vec{v}) = \sum_{l=1}^{3}\left[\int_\Omega h_l(x) v_l dx + \int_\Gamma g_l v_l d\gamma\right], \tag{53}$$

Now, to proof the corcivity and boundedness of $\langle \bar{A}(\vec{y}), \vec{v} \rangle$, since:

$$\langle \bar{A}(\vec{y}), \vec{y} \rangle = \int_\Omega \sum_{i,j=1}^{2} a_{1ij} \frac{\partial y_1}{\partial x_i}\frac{\partial y_1}{\partial x_j} dx + \int_\Gamma \alpha_1 y_1 y_1 d\gamma + \int_\Omega \sum_{i=1}^{2} b_{1i}(x) \frac{\partial y_1}{\partial x_i} y_1 dx + \int_\Omega c_1(x) y_1 y_1 dx - \int_\Omega d_1(x) y_2 v_1 dx + \int_\Omega d_2(x) y_3 v_1 dx + \int_\Omega f_1(x, y_1) y_1 dx + \int_\Omega \sum_{i,j=1}^{2} a_{2ij} \frac{\partial y_2}{\partial x_i}\frac{\partial y_2}{\partial x_j} dx + \int_\Gamma \alpha_2 y_2 y_2 d\gamma + \int_\Omega \sum_{i=1}^{2} b_{2i}(x) \frac{\partial y_2}{\partial x_i} y_2 dx + \int_\Omega c_2(x) y_2 y_2 dx + \int_\Omega d_1(x) y_1 v_2 dx - \int_\Omega d_3(x) y_3 v_2 dx + \int_\Omega f_2(x, y_2) y_2 dx + \int_\Omega \sum_{i,j=1}^{2} a_{3ij} \frac{\partial y_3}{\partial x_i}\frac{\partial y_3}{\partial x_j} dx + \int_\Gamma \alpha_3 y_3 y_3 d\gamma + \int_\Omega \sum_{i=1}^{2} b_{3i}(x) \frac{\partial y_3}{\partial x_i} y_3 dx + \int_\Omega c_3(x) y_3 y_3 dx - \int_\Omega d_2(x) y_1 v_3 dx + \int_\Omega d_3(x) y_2 v_3 dx + \int_\Omega f_2(x, y_2) y_2 dx.$$

Utilizing assumption 2.3.1, Letting $\frac{\partial y_l}{\partial x_i} y_l = \frac{1}{2}\frac{\partial}{\partial x_i}(y_l{}^2)$, $l = 1,2,3$, for the 3rd, the 10th and the 17th terms, it becomes

$$\langle \bar{A}(\vec{y}), \vec{y} \rangle \geq \hat{c}_1 \sum_{i=1}^{2} \int_\Omega \left|\frac{\partial y_1}{\partial x_i}\right|^2 dx + \hat{c}_2 \int_\Gamma |y_1|^2 d\gamma + \sum_{i=1}^{2} \int_\Omega b_{1i}(x) \frac{1}{2} \frac{\partial}{\partial x_i}(y_1{}^2) dx + \int_\Omega c_1(x)|y_1|^2 dx + \hat{c}_3 \sum_{i=1}^{2} \int_\Omega \left|\frac{\partial y_2}{\partial x_i}\right|^2 dx + \hat{c}_4 \int_\Gamma |y_2|^2 d\gamma + \sum_{i=1}^{2} \int_\Omega b_{2i}(x) \frac{1}{2} \frac{\partial}{\partial x_i}(y_2{}^2) dx + \int_\Omega c_2(x)|y_2|^2 dx + \hat{c}_5 \sum_{i=1}^{2} \int_\Omega \left|\frac{\partial y_3}{\partial x_i}\right|^2 dx + \hat{c}_6 \int_\Gamma |y_3|^2 d\gamma + \sum_{i=1}^{2} \int_\Omega b_{3i}(x) \frac{1}{2} \frac{\partial}{\partial x_i}(y_3{}^2) dx + \int_\Omega c_3(x)|y_3|^2 dx + \int_\Omega (f_1(x, y_1) - f_1(x, 0))(y_1 - 0) dx + \int_\Omega (f_2(x, y_2) - f_2(x, 0))(y_2 - 0) dx + \int_\Omega (f_3(x, y_3) - f_3(x, 0))(y_3 - 0) dx \tag{54}$$

Since f_l is monotone w.r.t. y_l, utilizing integrating by parts on the 3rd, 7th, and the 11th terms in (51), to get

$$\langle \bar{A}(\vec{y}), \vec{y}\rangle \geq \hat{c}_1 \sum_{i=1}^{2} \int_\Omega \left|\frac{\partial y_1}{\partial x_i}\right|^2 dx + \sum_{i=1}^{2} \int_\Gamma \frac{b_{1i}(x)}{2} |y_1|^2 d\gamma + \hat{c}_2 \int_\Gamma |y_1|^2 d\gamma + \int_\Omega \left(c_1(x) - \frac{1}{2}\sum_{i=1}^{2} \frac{\partial b_{1i}}{\partial x_i}\right) |y_1|^2 dx + \hat{c}_3 \sum_{i=1}^{2} \int_\Omega \left|\frac{\partial y_2}{\partial x_i}\right|^2 dx + \hat{c}_4 \int_\Gamma |y_2|^2 d\gamma + \sum_{i=1}^{2} \int_\Gamma \frac{b_{2i}(x)}{2} |y_2|^2 d\gamma + \hat{c}_5 \sum_{i=1}^{2} \int_\Omega \left|\frac{\partial y_3}{\partial x_i}\right|^2 dx + \int_\Omega \left(c_2(x) - \frac{1}{2}\sum_{i=1}^{2} \frac{\partial b_{2i}}{\partial x_i}\right) |y_2|^2 dx + +\hat{c}_6 \int_\Gamma |y_3|^2 d\gamma + \sum_{i=1}^{2} \int_\Gamma \frac{b_{3i}(x)}{2} |y_3|^2 d\gamma + \int_\Omega \left(c_3(x) - \frac{1}{2}\sum_{i=1}^{2} \frac{\partial b_{3i}}{\partial x_i}\right) |y_3|^2 dx \quad (55)$$

Utilizing (16) in (64), and with $b_{li} \geq 0$ for $l = 1,2,3, \forall i = 1,2$, the (64) gives

$$\langle \bar{A}(\vec{y}), \vec{y}\rangle \geq \hat{c}_1 \sum_{i=1}^{2} \int_\Omega \left|\frac{\partial y_1}{\partial x_i}\right|^2 dx + \hat{c}_3 \sum_{i=1}^{2} \int_\Omega \left|\frac{\partial y_2}{\partial x_i}\right|^2 dx + \hat{c}_5 \sum_{i=1}^{2} \int_\Omega \left|\frac{\partial y_3}{\partial x_i}\right|^2 dx + \hat{c}_2 \int_\Gamma |y_1|^2 d\gamma + \hat{c}_4 \int_\Gamma |y_2|^2 d\gamma + \hat{c}_6 \int_\Gamma |y_3|^2 d\gamma,$$
$$= C_0\left(\sum_{l=1}^{3} \int_\Omega |\nabla y_l|^2 dx + \sum_{l=1}^{3} \int_\Gamma |y_l|^2 d\gamma\right), \quad (56)$$

with $\tau = \min(\hat{c}_2, \hat{c}_4, \hat{c}_6)$, $C_0 = \min(\hat{c}_1, \hat{c}_3, \hat{c}_5, \tau)$

Now, from the assumption on Ω, and consider $\Gamma_1 \subset \Gamma$ is a measureable subset, $|\Gamma_1| > 0$, then there exist constants $\sigma_1(\Gamma_1), \sigma_2(\Gamma_1), \sigma_3(\Gamma_1) > 0$ independent of $\vec{y} \in \left(H^1(\Omega)\right)^3$ (from corollary 2.4.2), s.t.

$$\langle \bar{A}(\vec{y}), \vec{y}\rangle \geq C_0\left(\sum_{l=1}^{3} \int_\Omega |\nabla y_l|^2 dx + \sum_{l=1}^{3} \int_\Gamma |y_l|^2 d\gamma\right)$$
$$= \sigma_4 \left(\sum_{l=1}^{3} \int_\Omega |\nabla y_l|^2 dx + \sum_{l=1}^{3} \int_{\Gamma_1} |y_l|^2 d\gamma\right),$$
$$\geq \sigma_4 \|\vec{y}\|^2_{\left(H^1(\Omega)\right)^3}, \sigma_4 = C_0 \min(\sigma_1, \sigma_2, \sigma_3) \quad (54)$$

In addition, since $a_{lij}(x), b_{li}(x), c_l(x), d_l(x) \in L^\infty(\Omega)$, $(\forall l = 1,2,3, \forall i,j = 1,2)$, and $\forall \lambda_l = \max_{l=1,2} |\alpha_l|$

$$\left|\int_\Gamma \sum_{l=1}^{3} \alpha_l y_l v_l d\gamma\right| \leq \sum_{l=1}^{3} \lambda_l \left|\int_\Gamma y_l v_l d\gamma\right|, \quad (57)$$

Utilizing (55), beside the ((20)-(25)), to get

$$|\langle \bar{A}(\vec{y}), \vec{y}\rangle| \leq \sum_{i,j=1}^{2} \mu_1 \left|\int_\Omega \frac{\partial y_1}{\partial x_i}\frac{\partial v_1}{\partial x_j} dx\right| + \sum_{i,j=1}^{2} \mu_2 \left|\int_\Omega \frac{\partial y_2}{\partial x_i}\frac{\partial v_2}{\partial x_j} dx\right| + \sum_{i,j=1}^{2} \mu_3 \left|\int_\Omega \frac{\partial y_3}{\partial x_i}\frac{\partial v_3}{\partial x_j} dx\right| + \sum_{l=1}^{3} \lambda_l \left|\int_\Gamma y_l v_l d\gamma\right| + \sum_{i=1}^{2} \bar{\mu}_1 \left|\int_\Omega \frac{\partial y_1}{\partial x_i} v_1 dx\right| + \sum_{i=1}^{2} \bar{\mu}_2 \left|\int_\Omega \frac{\partial y_2}{\partial x_i} v_2 dx\right| + \sum_{i=1}^{2} \bar{\mu}_3 \left|\int_\Omega \frac{\partial y_3}{\partial x_i} v_3 dx\right| + \sum_{l=1}^{3} \bar{\bar{\mu}}_l \left|\int_\Omega y_l v_l dx\right| + \bar{\bar{\bar{\mu}}}_1 \left|\int_\Omega y_2 v_1 dx\right| + \bar{\bar{\bar{\mu}}}_1 \left|\int_\Omega y_1 v_2 dx\right| + \bar{\bar{\bar{\mu}}}_2 \left|\int_\Omega y_3 v_1 dx\right| + \bar{\bar{\bar{\mu}}}_2 \left|\int_\Omega y_1 v_3 dx\right| + \bar{\bar{\bar{\mu}}}_3 \left|\int_\Omega y_3 v_2 dx\right| + \bar{\bar{\bar{\mu}}}_3 \left|\int_\Omega y_2 v_3 dx\right|.$$

And by utilizing Theorem 2.1.3, to get

$$|\langle \bar{A}(\vec{y}), \vec{v}\rangle| \leq \mu \left\{ \sum_{i,j=1}^{2} \left(\int_\Omega \left|\frac{\partial y_1}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \left|\frac{\partial v_1}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Gamma |y_1|^2 dx\right)^{\frac{1}{2}} \left(\int_\Gamma |v_1|^2 dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2} \left(\int_\Omega \left|\frac{\partial y_2}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \left|\frac{\partial v_2}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Gamma |y_2|^2 dx\right)^{\frac{1}{2}} \left(\int_\Gamma |v_2|^2 dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2} \left(\int_\Omega \left|\frac{\partial y_3}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \left|\frac{\partial v_3}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Gamma |y_3|^2 dx\right)^{\frac{1}{2}} \left(\int_\Gamma |v_3|^2 dx\right)^{\frac{1}{2}} + \sum_{i=1}^{2} \left(\int_\Omega \left|\frac{\partial y_1}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \sum_{i=1}^{2} \left(\int_\Omega \left|\frac{\partial y_2}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}} + \sum_{i=1}^{2} \left(\int_\Omega \left|\frac{\partial y_3}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v_3|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_1|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_2|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_3|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v_3|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_2|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_1|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_3|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_1|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v_3|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_3|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_2|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v_3|^2 dx\right)^{\frac{1}{2}} \right\} \quad (58)$$

where $\mu = \max\{\mu_l, \lambda_l, \bar{\mu}_l, \bar{\bar{\mu}}_l, \bar{\bar{\bar{\mu}}}_l\}, \forall l = 1,2,3.$

Now, theorem 2.1.7 is used on the 2th, 4th and 6th terms in the previous inequality, then the resulting inequality can be written as:

$$|\langle \bar{A}(\vec{y}), \vec{v}\rangle| \leq \mu \left(\left\{ \left(\int_\Omega |y_1|^2 dx\right)^{\frac{1}{2}} + \sum_{j=1}^{2} \left(\int_\Omega \left|\frac{\partial y_1}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}} \right\} \times \left\{ \left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2} \left(\int_\Omega \left|\frac{\partial v_1}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}} \right\} + \left\{ \left(\int_\Omega |y_2|^2 dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2} \left(\int_\Omega \left|\frac{\partial y_2}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}} \right\} \times \right.$$

$$\left\{\left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2}\left(\int_\Omega \left|\frac{\partial v_2}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}}\right\}$$
$$+\left\{\left(\int_\Omega |y_3|^2\, dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2}\left(\int_\Omega \left|\frac{\partial y_3}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}}\right\}\times$$
$$\left\{\left(\int_\Omega |v_3|^2 dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2}\left(\int_\Omega \left|\frac{\partial v_3}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}}\right\}$$
$$+\left(\int_\Omega |y_2|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} +$$
$$\left.\left(\int_\Omega |y_1|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}}\right) +$$
$$\left(\int_\Omega |y_3|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}}$$
$$\left.+\left(\int_\Omega |y_1|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_3|^2 dx\right)^{\frac{1}{2}}\right) +$$
$$\left(\int_\Omega |y_3|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}} +$$
$$\left.\left(\int_\Omega |y_2|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_3|^2 dx\right)^{\frac{1}{2}}\right) +$$
$$\bar{c}_1\left(\int_\Omega |y_1|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} +$$
$$\bar{c}_2\left(\int_\Omega |y_2|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}} +$$
$$\left.\bar{c}_3\left(\int_\Omega |y_3|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_3|^2 dx\right)^{\frac{1}{2}}\right) \quad (59)$$

$$|\langle \bar{A}(\vec{y}), \vec{v}\rangle| \le$$
$$\mu_3\big[\|y_1\|_{H^1(\Omega)}\|v_1\|_{H^1(\Omega)} + \|y_2\|_{H^1(\Omega)}\|v_2\|_{H^1(\Omega)} +$$
$$\|y_3\|_{H^1(\Omega)}\|v_3\|_{H^1(\Omega)} + \|y_2\|_{H^1(\Omega)}\|v_1\|_{H^1(\Omega)} +$$
$$\|y_1\|_{H^1(\Omega)}\|v_2\|_{H^1(\Omega)} + \|y_3\|_{H^1(\Omega)}\|v_1\|_{H^1(\Omega)} +$$
$$\|y_1\|_{H^1(\Omega)}\|v_3\|_{H^1(\Omega)} + \|y_3\|_{H^1(\Omega)}\|v_2\|_{H^1(\Omega)} +$$
$$\|y_2\|_{H^1(\Omega)}\|v_3\|_{H^1(\Omega)} + \bar{c}_1\|y_1\|_{H^1(\Omega)}\|v_1\|_{H^1(\Omega)} +$$
$$\bar{c}_2\|y_2\|_{H^1(\Omega)}\|v_2\|_{H^1(\Omega)} + \bar{c}_3\|y_3\|_{H^1(\Omega)}\|v_3\|_{H^1(\Omega)}\big],$$
$$\le \rho\|\vec{y}\|_{(H^1(\Omega))^3}\|\vec{v}\|_{(H^1(\Omega))^3}, \quad (60)$$

With $\rho = 12\mu_3$, $\mu_3 = \mu\max(1, \max_{1\le l\le 3}(1+\bar{c}_l))$.

From (58) and (28), the continuity of $\vec{y} \longmapsto \langle \bar{A}(\vec{y}), \vec{v}\rangle$ w.r.t. $\vec{y}$, is obtained.

The continuity of the RHS $F(\vec{v})$ in (50) is proved as:

$$|F(\vec{v})| \le \sum_{l=1}^{3}\|h_l(x)\|_{L^2(\Omega)}\|v_l\|_{L^2(\Omega)} + \sum_{l=1}^{3}\|g_l(x)\|_{L^2(\Gamma)}\|v_l\|_{L^2(\Gamma)}$$
$$\le \sum_{l=1}^{3}[\gamma_l\|v_i\|_{H^1(\Omega)} + \bar{\gamma}_i\bar{\bar{\gamma}}_l\|v_l\|_{H^1(\Omega)}],$$
$$\le \check{C}\,\|\vec{v}\|_{(H^1(\Omega))^2}$$

where $\|h_l\|_{L^2(\Omega)} \le \gamma_l$, $\|g_l(x)\|_{L^2(\Gamma)} \le \bar{\gamma}_l$, $\gamma = \max(\gamma_l, \bar{\gamma}_i\bar{\bar{\gamma}}_l)$, $\forall l = 1,2,3$, and $\check{C} = 3\gamma$,btherefore $F(\vec{v})$ is linear and bounded, hence by theorem 2.1.3, the uniqueness of a TRS $\vec{y} \in (H^1(\Omega))^3$ of the WF (50) is obtained.

Corollary 2.5.2. There exists a unique TRS $\vec{y} \in (H^1(\Omega))^3$, for the WF of the TEPDEs ((1)-(3)), with the nonhomogeneous DBCs (NHDBCs):

$$y_1 = g_1 \text{, on } \Gamma \quad (61)$$
$$y_2 = g_2 \text{, on } \Gamma \quad (62)$$
$$y_3 = g_3 \text{, on } \Gamma \quad (63)$$

where $g_1(x), g_2(x), g_3(x) \in L^2(\Gamma), \vec{g} = (g_1, g_2, g_3)$.

Proof. Assume, $\vec{w} = (w_1, w_2, w_3) \in (H^2(\Omega))^3$, for which $\vec{w} = \vec{g} = (g_1, g_2, g_3)$ on Γ. Set $\vec{y} = \vec{w} + \vec{z}$ with $\vec{z} = (z_1, z_2, z_3) \in (H_0^2(\Omega))^3$ and the functions f_l,(for $l = 1,2,3$) is satisfies Assumptions 2.3.1 with one different that $\vec{z}$ instead of $\vec{y}$. Then the WF obtained from ((1)- (3)) with the NHDBCs ((61)-(63)) is similar to the WF in (50), i.e

$$\langle \bar{A}(z), \vec{v}\rangle = H(\vec{v}) \quad (64)$$

here

$$\langle \bar{A}(\vec{z}), \vec{v}\rangle = a(\vec{z}, \vec{v}) + (f_1(x, z_1 + w_1), v_1)_\Omega + (f_2(x, z_2 + w_2), v_1)_\Omega + (f_3(x, z_3 + w_3), v_3)_\Omega, \text{ and}$$
$$H(\vec{v}) = F(\vec{v}) + a(\vec{w}, \vec{v})$$

with $a(\vec{z}, \vec{v})$ ($a(\vec{w}, \vec{v})$) is as defined in (12) with one different that $\vec{z}$ ($\vec{w}$) instead of $\vec{y}$, and $F(\vec{v})$ is as defined in (13).

In this point, for fixed $\vec{w}$, the coercivity and the continuity of the $\langle \bar{A}(z), \vec{v}\rangle$ is obtained by the same ways that were used in the proof of theorem 2.4.1, and the continuity of the RHS of (64) is obtained depending on the continuity of the $F(\vec{v})$ and $a(\vec{w}, \vec{v})$, i.e.

$$|a(\vec{w}, \vec{v})| \le \rho\,\|\vec{w}\|_{(H_0^1(\Omega))^3}\|\vec{v}\|_{(H_0^1(\Omega))^3}$$
$$\le \rho\,\bar{\rho}\|\vec{v}\|_{(H_0^1(\Omega))^3} \quad (65)$$
$$|F(\vec{v})| \le \gamma\|\vec{v}\,\|_{(L^2(\Omega))^3} \le \gamma\|\vec{v}\,\|_{(H_0^1(\Omega))^3} \quad (66)$$

From (63a) and (63b), we get that $|H(\vec{v})| \le \bar{\gamma}\|\vec{v}\,\|_{(H_0^1(\Omega))^3}$, where $\bar{\gamma} = \rho\,\bar{\rho} + \gamma$.

Therefore, by Theorem 2.1.3, $\vec{z} \in (H_0^1(\Omega))^3$, or $\vec{y} \in (H^1(\Omega))^3$ is a unique TRS of the WF (64).

Corollary 2.5.3. There is a unique TRS $\vec{y} = (y_1, y_2, y_3) \in (H^1(\Omega))^3$ of the WF of the proposed TNLEPDEs ((1)-(3)), with the following nonhomogeneous NBCs (NHNBCs):

$$\frac{\partial y_1}{\partial n_1} = \sum_{i,j=1}^{2} a_{1ij}\frac{\partial y_1}{\partial x_i}\, cos(n_1, x_j) = g_1(x), \text{ on } \Gamma \quad (67)$$
$$\frac{\partial y_2}{\partial n_2} = \sum_{i,j=1}^{2} a_{2ij}\frac{\partial y_2}{\partial x_i}\, cos(n_2, x_j) = g_2(x), \text{ on } \Gamma \quad (68)$$
$$\frac{\partial y_3}{\partial n_3} = \sum_{i,j=1}^{2} a_{3ij}\frac{\partial y_3}{\partial x_i}\, cos(n_3, x_j) = g_3(x), \text{ on } \Gamma \quad (69)$$

where $(g_1, g_2, g_3) \in (L^2(\Gamma))^3$ and n_ℓ, (for $\ell = 1,2,3$) is an outer normal vector on Γ and (n_ℓ, x_j) is the angle between the $x_j - axis$ and n_ℓ.

Proof. The WF results from ((1)- (3)) with the NHNBCs ((61)-(63)) is similar to the WF in (50), i.e

$$\langle \bar{A}(\vec{y}), \vec{y} \rangle = F(\vec{v}). \qquad (70)$$

with $\langle \bar{A}(\vec{y}), \vec{y} \rangle$ is as defined in (50) but with neglecting all the terms that have the forms $(\alpha_l y_l, v_l)_\Gamma$, for $l = 1,2,3$, and $\forall v_l \in H^1(\Omega)$,

$$F(\vec{v}) = \sum_{l=1}^{3} [\int_\Omega h_l v_l dx + \int_\Gamma g_l v_l d\gamma],$$

In this case, the proof of the existence of a unique TRS $\vec{y} \in (H^1(\Omega))^3$ of the WF (70) is obtained directly form the above corollary 2.5.2.

Corollary 2.5.4. There exists a unique TRS $\vec{y} \in (H_0^2(\Omega))^3$ of the WF for the TNLEPDEs ((1)-(3)), with the homogenous Robin BCs (HRBCs)

$$\alpha_1 y_1 + \frac{\partial y_1}{\partial n_1} = \alpha_1 y_1 + \sum_{i,j=1}^{2} a_{1ij} \frac{\partial y_1}{\partial x_i} \cos(n_1, x_j) = 0, \text{on } \Gamma, \qquad (71)$$

$$\alpha_2 y_2 + \frac{\partial y_2}{\partial n_2} = \alpha_2 y_2 + \sum_{i,j=1}^{2} a_{2ij} \frac{\partial y_2}{\partial x_i} \cos(n_2, x_j) = 0, \text{ on } \Gamma, \qquad (72)$$

$$\alpha_3 y_3 + \frac{\partial y_3}{\partial n_3} = \alpha_3 y_3 + \sum_{i,j=1}^{2} a_{3ij} \frac{\partial y_3}{\partial x_i} \cos(n_3, x_j) = 0, \text{ on } \Gamma, \qquad (73)$$

where $\alpha_1, \alpha_2, \alpha_3$ and n_ℓ, (for $\ell = 1,2,3$) are as defined in theorem 2.5.1.

Proof. The WF results from ((1)- (3)) with the NHDBCs ((61) -(63)) is similar to the WF in (50), i.e

$$\langle \bar{A}(\vec{y}), \vec{y} \rangle = F(\vec{v}), \qquad (74)$$

where $\langle \bar{A}(\vec{y}), \vec{y} \rangle$ is exactly as defined in (11), then from the proof of theorem 2.5.1, the coercivity and the boundedness for $\langle \bar{A}(\vec{y}), \vec{y} \rangle$ is obtained directly.
On the other hand, the function $F(\vec{v})$ in the RHS of (74) will be in the form:

$$F(\vec{v}) = \sum_{l=1}^{3} [\int_\Omega h_l v_l dx], \ \forall v_l \in H^1(\Omega),$$

The demonstration of that $F(\vec{v})$ is bounded is obtained directly, since it is an especial case from those was proved in theorem 2.5.1, hence by Theorem 2.1.3, $\vec{y} \in (H^1(\Omega))^3$ is a unique TRS of (64).

Example 2.5.5. Let $\Omega = [0,1]^2$ with boundary $\Gamma = \partial\Omega$,, and consider the TNLEPDEs in ((1)-(3)) with the following NHRBCs (44)-(46), where the coefficients in ((1)-(3)) are assumed to be:

$$A_l y_l = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(e^{x_i^2 + l j x_j^2} \frac{\partial y_l}{\partial x_j}\right), c_l(x) = e^{l x_l},$$

$$B_l y_l = \sum_{i=1}^{2} e^{l x_i} \frac{\partial y_1}{\partial x_i}, \ d_l(x) = l x_1^2 + l x_2^2 + 2l,$$

$$f_l(x, y_l) = |y_l| y_l, \ \alpha_l = l \ \forall \ l = 1,2,3,$$

where $h_l(x)$ are given functions on Ω and $g_l(x)$ are given functions on Γ

Now, since $f_l(x, y_l)$ in the given TNLEPDEs are satisfied Assumptions 2.3.1, then from theorem 2.5.1 the TNLEPDEs (1)-(3) with the NHRBCs (44)-(46), has a unique TRS $\vec{y} \in (H_0^1(\Omega))^3$.

2.6 Solvability of the Triple Nonlinear with Mixed Boundary Conditions

The solvability for the WF of the TNLEPDEs with Mixed BCs (MBCs), is studied as follows.

Theorem 2.6.1. There exists a unique TRS $\vec{y} \in (H^2(\Omega))^3$ of the WF of the TNLEPDEs ((1)-(3)) with the following mixed BCs (MBCs),

$$y_l = g_{lD} \quad \text{on} \quad \Gamma_D, l = 1,2,3, \qquad (75)$$

$$\frac{\partial y_l}{\partial n_l} = g_{lN} \quad \text{on} \quad \Gamma_N, l = 1,2,3, \qquad (76)$$

where the boundary Γ of Ω be divided in two open subset Γ_D and Γ_N, with $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$, $g_{lD} \in L^2(\Gamma_D)$, $g_{lN} \in L^2(\Gamma_N)$ and n_l is a normal unit vector on Γ_N.

Proof. The WF resutls from ((1)-(3)) with the NHNBCs ((75)-(76)) is similar to the WF in (50), i.e

$$\langle \bar{A}(\vec{y}), \vec{y} \rangle = F(\vec{v}), \qquad (77)$$

with $\langle \bar{A}(\vec{y}), \vec{y} \rangle$ is as defined in (11) (or as defined in (50) but with neglecting all the terms that have the forms $(\alpha_l y_l, v_l)_\Gamma$, for $l = 1,2,3$), and

$$F(\vec{v}) = \sum_{l=1}^{3} [\int_\Omega h_l(x) v_l dx + \int_{\Gamma_D} g_{lD} v_l d\gamma_D + \int_{\Gamma_N} g_{lN} v_l d\gamma_N], \qquad \forall v_l \in H^1(\Omega)$$

The proof of the corcivity and the boundedness for $\langle \bar{A}(\vec{y}), \vec{y} \rangle$ is the same so were proved in Theorem 2.4.1, and hence to prove the solvability of the WF (66) through applying theorem 2.1.1, it remains to demonstrate that $F(\vec{v})$ in its RHS is bounded, so by utilizing Theorem 2.1.3 and Theorem 2.1.7, one has

$$|F(\vec{v})| \le \sum_{l=1}^{3} [|(h_l, v_l)_\Omega| + |(g_{lD}, v_l)_{\Gamma_D}| + |(g_{lN}, v_l)_{\Gamma_N}|]$$
$$\le \sum_{l=1}^{3} [\|h_l\|_{L^2(\Omega)} \|v_l\|_{L^2(\Omega)} + \|g_{lD}(x)\|_{L^2(\Gamma_D)} \|v_l\|_{L^2(\Gamma_D)} + \|g_{lN}\|_{L^2(\Gamma_N)} \|v_l\|_{L^2(\Gamma_N)}],$$
$$\le \sum_{l=1}^{3} [\gamma_l \|v_l\|_{H^1(\Omega)} + \bar{\gamma}_l \|v_l\|_{L^2(\Gamma_D)} + \bar{\bar{\gamma}}_l \|v_l\|_{L^2(\Gamma_N)}]$$

where $\|h_l\|_{L^2(\Omega)} \le \gamma_l$, $\|g_{lD}\|_{L^2(\Gamma_D)} \le \bar{\gamma}_l$, $\|g_{lN}\|_{L^2(\Gamma_N)} \le \bar{\bar{\gamma}}_l$, for $l = 1,2,3$.

By utilizing theorem 2.1.5,

$\|v_l\|_{L^2(\Gamma_D)} \leq c_l \|v_l\|_{H^1(\Omega)}$, for $l = 1,2,3$,
$\|v_l\|_{L^2(\Gamma_N)} \leq \bar{c}_l \|v_l\|_{H^1(\Omega)}$, for $l = 1,2,3$.
Thus, with with $\check{C} = 3max(\gamma_l, c_l\bar{\gamma}_l, \bar{c}_l\bar{\bar{\gamma}}_l)$
$|F(\vec{v})| \leq \check{C}\ \|\vec{v}\|_{\left(H^1(\Omega)\right)^3}$.

Hence, the uniqueness of a TRS $\vec{y} \in \left(H^1\ (\Omega)\right)^3$ of the WF (64) is obtained.

Corollary 2.6.2. There exists a unique TRS $\vec{y} \in \left(H^2(\Omega)\right)^3$ of the WF of the TNLEPDEs ((1)-(3)) with the following Mixed BCs (MBCs),

$$y_l = 0, \text{ on } \Gamma_D, l = 1,2,3, \tag{75}$$

$$\frac{\partial y_l}{\partial n_l} = g_{lN}, \text{ on } \Gamma_N, l = 1,2,3, \tag{76}$$

where $g_{lN} \in L^2(\Gamma_N)$.

Proof. The WF results from ((1)-(3)) with the BCs ((75)-(76)) is similar to those which is obtained in theorem 2.6.1, with one different that the $F(\vec{v})$ will be in the form $\forall v_l \in H^1(\Omega)$

$$F(\vec{v}) = \sum_{l=1}^{3}[\int_\Omega h_l(x)v_l dx + \int_{\Gamma_N} g_{lN} v_l d\gamma_N].$$

Which is an especial case from those was obtained in Theorem 2.6.1. Hence the uniqueness of a TRS $\vec{y} \in \left(H^1\ (\Omega)\right)^3$ is obtained.

Corollary 2.6.3. There is a unique TRS $\vec{y} \in \left(H^2(\Omega)\right)^3$ of the WF of the TNLEPDEs ((1)-(3)) with the following Mixed BCs (MBCs),

$$y_l = g_{lD}, \text{ on } \Gamma_D, l = 1,2,3, \tag{77}$$

$$\frac{\partial y_l}{\partial n_l} = 0, \text{ on } \Gamma_N, l = 1,2,3, \tag{78}$$

where $g_{lD} \in L^2(\Gamma_D)$.

Proof. Again, the WF results from ((1)-(3)) with the BCs ((75)-(76)) is similar to those which is obtained in theorem 2.6.1, with one different that the $F(\vec{v})$ will be in the form $\forall v_l \in H^1(\Omega)$:

$$F(\vec{v}) = \sum_{l=1}^{3}[\int_\Omega h_l(x)v_l dx + \int_{\Gamma_D} g_{lD} v_l d\gamma_D],$$

Which is an especial case from those was obtained in Theorem 2.6.1. Hence the uniqueness of a TRS $\vec{y} \in \left(H^1\ (\Omega)\right)^3$ is obtained.

3 CONCLUSIONS

The MBTH with the help of the PFI were utilized successfully to prove the existence of a unique TRS (in infinite dimension) of the WF for the TNLEPDEs associated with each kind of the homogenous BCs (DBCs, NBCs, RBCs and MBCs). While the MBTH with the help of both, the general PFI and the TRP were utilized successfully to prove the existence of a unique TRS of the WF associated with each kind of the nonhomogeneous BCs (DBCs, NBCs, RBCs and MBCs). The results of this paper play an important role to help many investigators who they interest about solving these types of problems numerically.

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