Fuzzy Completely Equiprime Graph of Nearring via Fuzzy Ideals

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Abstract: This paper investigates a class of fuzzy graphs associated with nearrings through the concept of c-equiprime fuzzy ideals. The relationships between the structural properties of these fuzzy graphs and different classes of fuzzy ideals are examined, and conditions under which the resulting graphs become identical are established. In addition, the study integrates concepts from fuzzy graph theory and algebraic structures to analyze the interaction between fuzzy ideals and closed-loop structures, providing a unified framework for their investigation. The applicability of the proposed approach is discussed in terms of improving the representation and analysis of algebraic structures. Furthermore, a new concept, referred to as the annihilator ideal, is introduced and its fundamental properties are investigated. The study also identifies the conditions required for the existence of fuzzy cliques within the proposed graph model. It is shown that when the underlying fuzzy ideal satisfies c-equiprime conditions in simple or integral nearrings, the corresponding fuzzy graph exhibits a star-graph structure. Moreover, sufficient conditions are established under which an induced subgraph forms a fuzzy clique. These results provide new theoretical insights into the interplay between fuzzy graph structures and nearring theory while extending the applicability of fuzzy algebraic models.

1 INTRODUCTION

After the introduction of fuzzy set theory researchers applied it in different fields. Rosenfeld [1] generalized algebraic group structure by introducing fuzzy groups. Davvaz [2] introduced $\mathcal{F}$ Is of rings and near rings. Pilz [3], a mathematical system near ring $\mathbb{N}$, which is broader than ring theory and define $\mathbb{N} = \mathbb{N}_0$ if $x_0 = 0$ for all $x \in \mathbb{N}$, was presented. Kedukodi et al. [4], [5] examined distinct $\mathcal{F}$Is with respect to near rings. Davvaz [6], he studied the concept interval valued L- $\mathcal{F}$Is of nearrings. Jagadeesha et al. [7], [8] investigated properties of several interval valued L-FIs of nearrings. Beck [9] connected a ring to a graph by introducing the zero-divisor graph of a ring. Anderson and Livingston [10] generalized the definition of a zero-divisor graph. Wu [11] further studied zero divisor graphs. Bhavanari et al. [12] studied graphs of nearrings with respect to different ideals. Mordeson [13] studied different graphs and $\mathcal{F}$ Gs. Samanta, Akram and Pal [14] introduced m-step fuzzy competition graphs. And can generalized this paper in different topic such as ciphers and biology [15]-[17] and other ideals [18].

2 PRELIMINARIES

Definition 2.1 [13]: An $\mathcal{F}\mathrm{G}H = (\mathcal{F},\ P_{cv})$ of (V, E) is described by a fuzzy sub set $\mathcal{F}$ of V and fuzzy sub set P_{cv} of E whereby $P_{cv}(w, y) \leq \mathcal{F}(w) \wedge \mathcal{F}(y)\ \forall\, w, y \in V$.

Definition 2.2 [2]: For a set with at least one element X the mapping $\mathcal{F}: X \to [0,1]$ is labeled a fuzzy of X.

Definition 2.3 [3]: The annihilator of element $w \in \mathbb{N}$ is a set $Ann(x) = \{r \in \ \mathbb{N}\ | rw = 0\}$.

Definition 2.4 [4]: A right permutable $\mathbb{N}$ is $awy = ayw\ \forall\, w, y, z \in \mathbb{N}$ and the ideal I is totally reflexive if $wny \in$ I then $ynx \in$ I for $w, y \in$ I and $n \in\ \mathbb{N}$.

Definition 2.5 [4]: Make I is ideal of $\mathbb{N}$. Then I is labeled c-equiprime ideal $(c - eq - prime\ ideal)$ of $\mathbb{N}$ if $aw - ay \in$ I then $w - y \in$ I for all $w, y \in \mathbb{N}$ and $a \in \mathbb{N}\backslash$I and weakly completely prime ideal if $0 \neq aj \in$ I $\Leftrightarrow\ a \in$ I $\vee\, j \in$ I.

Definition 2.6 [2]: Let $\alpha, \beta \in (0,1]$ and $\alpha < \beta$. Let $\mathcal{F}$ be $\mathcal{F}$I with thresholds α, β if to supply all $w, y, i \in \mathbb{N}$:

- $\alpha \vee \mathcal{F}(w - y) \geq \beta \wedge \mathcal{F}(w) \wedge \mathcal{F}(y)$;
- $\alpha \vee \mathcal{F}(y - w + y) \geq \beta \wedge \mathcal{F}(w)$;
- $\alpha \vee \mathcal{F}(wy) \geq \beta \wedge \mathcal{F}(w)$;

- $\alpha \vee \mathcal{F}(w(y+i) - wy) \geq \beta \wedge \mathcal{F}(i)$.

We call α as the lower threshold of $\mathbb{N}$ and β as the upper threshold of $\mathbb{N}$. In this paper $\mathcal{F}$ means $\mathcal{F}$I with lower threshold α and threshold β.

Theorem 2.1 [2]: Let $\mathcal{F}$ be a fuzzy subset of $\mathbb{N}$ then $\mathcal{F}$ be a $\mathcal{F}$I of $\mathbb{N}$ if and only if to supply $\sigma \in (\alpha, \beta]$ the level subset $\mathcal{F}_\sigma$ is ideal of $\mathbb{N}$ whenever $\mathcal{F}_\sigma = \{w | \mathcal{F}(w) \geq \sigma\}$.

Definition 2.7: Let $\mathcal{F}$ be a $\mathcal{F}$I of $\mathbb{N}$ then is labeled $c - eq - prime$ $\mathcal{F}$I if to supply $\sigma \in (\alpha, \beta]$ with $\alpha \vee \mathcal{F}(\mathrm{a}) \vee \mathcal{F}(w-y) \geq \beta \wedge \mathcal{F}(aw - ay)$ for all $w, y \in \mathbb{N}$ and $a \in \mathbb{N} \backslash \mathcal{F}_\sigma$.

Remark 2.1:

- In general, $0 \in \mathcal{F}_\sigma$, $\sigma \in (\alpha, \beta]$ whenever is an ideal;
- $\mathcal{F}(0)$ is the maximum value in$\mathcal{F}(w)$ for all $w \in \mathbb{N}$.

Also introduced some concept and definitions of graph [19], [20]. The number of edges incident to a vertex v is labeled the degree of v and is labeled by $\mathrm{dg}(v)$. A graph of order n without any edges, that means $E = \emptyset$ while cycle is a closed walk without repetition of vertices. Thus, the degree of each vertices of a cycle graph is two and if not is a cycle that is not circuit and a subgraph H is labeled an induced subgraph of G if the vertices $u, v \in V(H)$ are adjacent in G then they are adjacent in H, The set $W \subseteq V$ is labeled a clique if every different pair of vertices in W are adjacent confirmed by $< w >$ while a girth(gr) is the length of the shortest cycle contained in the graph. A proper coloring of a graph G is an assignment of colors to its vertices such that no two adjacent vertices receive the same color. Equivalently, an $n-$coloring of G is a mapping $\psi: V(G) \rightarrow \{1, 2, \cdots, n\}$. The adjacent vertices have different colors, that means $\psi(a) \neq \psi(b)$ assigns different colors whenever $\overline{ab} \in E(G)$ $\forall a, b \in V(G)$, and the minimum number of a proper colors required for G is labeled chromatic number of G and confirmed by $\chi(G)$.

3 C-EQUIPRIME $\mathcal{F}$G OF NEARRING WITH RESPECT TO LEVEL IDEAL

Definition 3.1: Let $\mathcal{F}: \mathbb{N} \rightarrow (0.1]$ be a $\mathcal{F}$I of $\mathbb{N}$. Let $\sigma \in (\alpha, \beta]$. Let v be fixed. Define $\mathcal{P}_{cv}: \mathbb{N} \times \mathbb{N} \rightarrow [0,1]$. Then the fuzzy graph with respect to the level ideal $\mathcal{F}_\sigma$ is defined as follows:

$$\mathcal{P}_{cv}(p, w) = \begin{cases} \mathcal{F}(p) \wedge \mathcal{F}(w) & if\ p \neq w\ and\ (pw - p0) \in \mathcal{F}_\sigma \\ & or\ (wp - wo) \in \mathcal{F}_\sigma \\ 0 & other\ wise \end{cases}$$

Example 3.1: Let $\mathbb{N} = \{0,1.2.3. a, j, v, d\}$ be the nearring equipped with addition and multiplication as described as Table 1.

Table 1: Nearring for $\mathbb{N} = \{0,1.2.3. a, j, v. d\}$.

$+$	0	1	2	3	a	j	v	d		$*$	0	1	2	3	a	j	v	d
0	0	1	2	3	a	j	v	d		0	0	0	0	0	0	0	0	0
1	1	2	3	0	d	v	a	j		1	0	1	2	3	a	j	v	d
2	2	3	0	1	j	a	d	v		2	0	2	0	2	0	2	0	2
3	3	0	1	2	v	d	j	a		3	0	3	2	1	j	a	v	d
a	a	d	j	v	2	0	1	3		a	0	a	2	j	a	j	v	d
j	j	v	a	d	0	2	3	1		j	0	j	2	a	j	a	v	d
v	v	a	d	j	1	3	0	2		v	0	v	0	v	0	v	0	v
d	d	j	v	a	3	1	2	0		d	0	d	0	d	2	2	0	0

Table 2: $\mathcal{P}_{cv}(p, w)$ when $\mathcal{F}_\sigma = \{0,2, v, d\}$.

$\mathcal{P}_{cv}(p, y)$	0	1	2	3	a	b	c	D
0	0	0.4	0.7	0.3	0.5	0.6	0.7	0.7
1	0.4	0	0.4	0	0	0	0.4	0.4
2	0.7	0.4	0	0.3	0.5	0.6	0.7	0.7
3	0.3	0	0.3	0	0	0	0.3	0.3
A	0.5	0	0.5	0	0	0	0.5	0.5
j	0.6	0	0.6	0	0	0	0.6	0.6
v	0.7	0.4	0.7	0.3	0.5	0.6	0	0.7
d	0.7	0.4	0.7	0.3	0.5	0.6	0.7	0

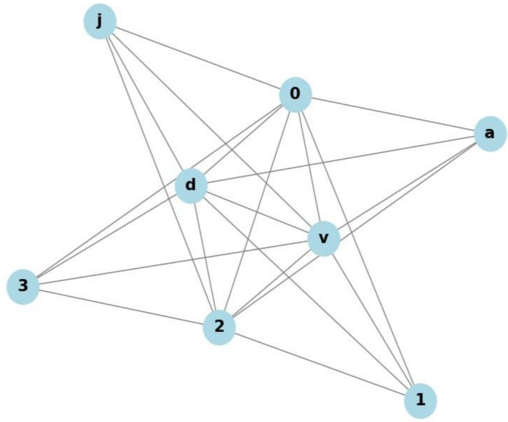

Figure 1: $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)$ when $\mathcal{F}_\sigma = \{0,2,v,d\}$.

We define $\mathcal{F}: \mathbb{N} \to (0,1]$ by:

$$\mathcal{F}(w) = \begin{cases} 0.9 \ \ if\ w = 0 \\ 0.3\ if\ w = 3 \\ 0.4\ if\ w = 1 \\ 0.5\ if\ w = a \\ 0.6\ if\ w = j \\ 0.7\ if\ w \in \{2.v.d\} \end{cases}.$$

If we take threshold $\alpha = 0.6$ $\beta = 0.7$, then $\mathcal{F}$ is $\mathcal{F}$I of $\mathbb{N}$. Let $\sigma = \beta$ then $\mathcal{F}_\sigma = \{0,2,v.d\}$. Then the values of $\mathcal{P}_{cv}$ are as in Table 2. The graph is given in Figure 1.

Proposition 3.1: Let $\mathcal{F}$ be a $\mathcal{F}$I of $\mathbb{N}$ then $\mathcal{F}$ be a $c-eq-prime\ \mathcal{F}I$ of $\mathbb{N}$ iff $\forall \sigma\epsilon(\alpha,\beta]$ the level sub set $\mathcal{F}_\sigma$ *is* $c-eq-prime\ ideal$ of $\mathbb{N}$.

Proof: By Theorem 2.1, $\mathcal{F}_\sigma$ is ideal of $\mathbb{N}$ as $\mathcal{F}$ is $\mathcal{F}$I. Suppose $\mathcal{F}$ is c-equiprime $\mathcal{F}$I with $\alpha \vee \mathcal{F}(a) \vee \mathcal{F}(w-y) \geq \beta \wedge \mathcal{F}(aw-ay)$ for $w,y \in \mathbb{N}$, $a \in \mathbb{N}\backslash\mathcal{F}_\sigma$ and $\sigma\epsilon(\alpha,\beta]$. Now, if $aw-ay \in \mathcal{F}_\sigma$ then $\mathcal{F}(aw-ay) \geq \sigma$, so $\alpha \vee \mathcal{F}(a) \vee \mathcal{F}(w-y) \geq \beta \wedge \mathcal{F}(aw-ay) \geq \beta \wedge \sigma = \sigma$, that is $\alpha \vee \mathcal{F}(a) \vee \mathcal{F}(w-y) \geq \sigma$. So $\mathcal{F}(a) \geq \sigma$ or $\mathcal{F}(w-y) \geq \sigma$ then $a \in \mathcal{F}_\sigma$, a contradiction, therefore $w-y \in \mathcal{F}_\sigma$ then $\mathcal{F}_\sigma$ is $c-eq-prime\ ideal\ \mathbb{N}$. Conversely, we have to prove it by contrary, assume that $\alpha \vee \mathcal{F}(a) \vee \mathcal{F}(w-y) < \sigma < \beta \wedge \mathcal{F}(aw-ay)$ then $\mathcal{F}(a) < \sigma$ or $\mathcal{F}(w-y) < \sigma$ and $\mathcal{F}(ax-ay) > \sigma$, so $a \notin \mathcal{F}_\sigma$, $w-y \notin \mathcal{F}_\sigma$ and $aw-ay \in \mathcal{F}_\sigma$, a contradiction as $\mathcal{F}_\sigma$ is $c-eq-prime\ ideal$of $\mathbb{N}$. Then $\alpha \vee \mathcal{F}(a) \vee \mathcal{F}(w-y) \geq \beta \wedge \mathcal{F}(aw-ay)$. Therfore $\mathcal{F}$is c-equiprime $\mathcal{F}$I of $\mathbb{N}$ $\forall\ \sigma\epsilon(\alpha.\beta]$.

Proposition 3.2: Let $\mathcal{F}$ be a $c-eq-prime\ \mathcal{F}I$ of $\mathbb{N}$ then $\mathcal{F}_\sigma$ is weakly complete $\mathcal{F}$I of $\mathbb{N}$.

Proof: Let $\sigma\epsilon(\alpha,\beta]$ and $a,b \in \mathbb{N}$ such that $aj \in \mathcal{F}_\sigma$ with $aj \neq 0$ by Proposition 3.1 $\mathcal{F}_\sigma$ is a $c-eq-prime\ ideal$ of $\mathbb{N}$ we receive $aj-a0 \in \mathcal{F}_\sigma$ this implies $j-0 \in \mathcal{F}_\sigma$. Then $b \in \mathcal{F}_\sigma$.

Proposition 3.3: Let $\mathcal{F}$ be a $c-eq-prime\ \mathcal{F}$I of $\mathbb{N} = \mathbb{N}_0$ then $(\mathcal{F}_\sigma,\mathcal{F},\mathcal{P}_{cv},\sigma)$ is clique in $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)$.

Proof: Let $\sigma\epsilon(\alpha.\beta]$ and $p \in \mathcal{F}_\sigma$, suppose $p = 0$ then $pw - p0 \in \mathcal{F}_\sigma$ and $wp - w0 \in \mathcal{F}_\sigma\ \forall\ w \in \mathbb{N}$ since $\mathcal{F}$ is $\mathcal{F}$I of $\mathbb{N}$ we receive $\mathcal{F}_\sigma$ is an ideal of $\mathbb{N}$ then $\mathcal{P}_{cv}(p,w) > 0$.

Hence p is connected to all elements of $\mathbb{N}$. If $p \neq 0$, Let's now assume that there a vertex $w \in \mathbb{N}$ whereby $\mathcal{P}_{cv}(p,w) = 0$ this implies $pw \in \mathcal{F}_\sigma$ and $p0 \in \mathcal{F}_\sigma$ as $\mathcal{F}_\sigma$ is a right ideal that is $pw - p0 \in \mathcal{F}_\sigma$. Hence $\mathcal{P}_{cv}(p,w) > 0$ and every element of $\mathcal{F}_\sigma$ is connected, so $(\mathcal{F}_\sigma,\mathcal{F},\mathcal{P}_{c\,v},\sigma)$ is a clique.

Proposition 3.4: Let $\mathcal{F}$ be $c-eq-prime\ \mathcal{F}$I of $\mathbb{N} = \mathbb{N}_0$ and $\sigma\epsilon(\alpha,\beta]$ if $w \in \mathcal{F}_\sigma$ and $w \neq 0$ then $Annw$ is a clique in $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)$.

Proof: Let $a,j\ \in Ann(w)$ then $aw = 0$ and $jw = 0$, so that $aw - a0 \in \mathcal{F}_\sigma$ and $jw - j0\ \in \mathcal{F}_\sigma$ that is $\mathcal{P}_{cv}(a,w) > 0$ and $\mathcal{P}_{cv}(j,w) > 0$ so there exist an edge between a and w with j and w. Since $(aj)w = a(jw) = a0 = 0w$ by cancelation law then $aj = 0$ this implies $aj - a0 \in \mathcal{F}_\sigma$, we receive $\mathcal{P}_{cv}(p,w) > 0$, $\forall\ a,j \in Ann(w)$ then $Ann(w)$ is a clique in $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)$.

Proposition 3.5: Let $\mathcal{F}$ be a $c-eq-prime\ \mathcal{F}$I of $\mathbb{N} = \mathbb{N}_0$ then $(\mathbb{N}\backslash\mathcal{F}_\sigma,\mathcal{F},\mathcal{P}_{cv},\sigma)$ is null graph.

Proof: From Proposition 3.3, $\mathcal{F}_\sigma$ is a clique in $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)$ for $\sigma\epsilon(\alpha,\beta]$ that is a maximum complete subgraph. Now, for $w,y \in \mathbb{N}\backslash\mathcal{F}_\sigma$,if $\mathcal{P}_{cv}(w,y) > 0$ that is there exist an edge joining w and y and $wy - w0 \in \mathcal{F}_\sigma$ or $yw - y0 \in \mathcal{F}_\sigma$, without loss of generality if $wy - wo \in \mathcal{F}_\sigma$ by Proposition 3.1 then $\mathcal{F}_\sigma$ is a $c-eq-prime\ ideal$ that is $y - 0 \in \mathcal{F}_\sigma$, a contradiction. Then $\mathcal{P}_{c\,v}(w,y) = 0$ for all $w,y \in \mathbb{N}\backslash\mathcal{F}_\sigma$. Therefore $(\mathbb{N}\backslash\mathcal{F}_\sigma,\mathcal{F},\mathcal{P}_{cv},\sigma)$ is null graph.

Proposition 3.6: Let $\mathcal{F}$ be non $c-eq-prime\ \mathcal{F}$I of $\mathbb{N} = \mathbb{N}_0$, then $(\mathbb{N}\backslash\mathcal{F}_\sigma,\mathcal{F},\mathcal{P}_{cv},\sigma)$ is not necessarily null graph.

Proof: Directly from Proposition 3.5 that is some elements in $\mathbb{N}\backslash\mathcal{F}_\sigma$ may be is connected by an edge.

Example 3.2: Let $\mathbb{N} = \{0.1.2.3.4.5.6.7\}$ be the nearring equipped with addition and multiplication described as Table 3.

We define $\mathcal{F}: \mathbb{N} \to (0,1]$ by:

$$\mathcal{F}(w) = \begin{cases} 0.9\ if\ w = 0 \\ 0.2\ if\ w \in \{1.3.7\} \\ 0.4\ if\ w \in \{2.5.6\} \\ 0.7\ if\ w = 4 \end{cases}.$$

If we take threshold $\alpha = 0.2$ and $\beta = 0.7$ then $\mathcal{F}$is $\mathcal{F}$I of $\mathbb{N}$. Let $\sigma = \beta$ then $\mathcal{F}_{\sigma} = \{0,4\}$ is non $c-eq-prime\ ideal$ of $\mathbb{N}$. Then the values of $\mathcal{P}_{cv}$ are as in Table 4. The graph is given in Figures 2 and 3.

Note that $\mathcal{F}$is not c−equiprime $\mathcal{F}$I of $\mathbb{N}$ $as\ \alpha \vee \mathcal{F}\ (a)\ \vee \mathcal{F}(w-y) = \alpha \vee \mathcal{F}\ (2)\ \vee \mathcal{F}(7-1) = 0.2 \vee 0.4 \vee 0.4 = 0.4 \ngeq 0.7 = 0.7 \wedge 0.9 = \beta \wedge \mathcal{F}(aw-ay)$.

Proposition 3.7: Let $Ann(a)$ and $Ann(j)$ be a $c-eq-prime\ ideals$ of $\mathbb{N}$ with $a \neq 0$ and $j \neq 0 \in \mathbb{N}$then $aj = 0$.

Proof: Suppose $aj \neq 0$ so that a $\notin$ Ann(j) and j$\notin$ $Ann(a)$ that is $a-0 \notin Ann(j)$ and $j-0 \notin Ann(a)$, then $ja-j0 \notin Ann(b)$ and $aj-a0 \notin Ann(a)$, a contradiction, as $Ann(a)$ and $Ann(j)$ are $c-equiprime\ ideals$ of $\mathbb{N}$.

Proposition 3.8: Let $\mathcal{F}$ be a $\mathcal{F}$I of $\mathbb{N}$ and $\sigma\epsilon(\alpha,\beta]$ if $Ann(a)$ and $nn(j)$ are $c-eq-prime\ ideals\ of\ \mathbb{N}$ with $a \neq 0$ and $j \neq 0 \in \mathcal{F}_{\sigma}$ then $\mathcal{P}_{cv}(a,j) > 0$.

Proof: From Proposition 3.7, then $aj = 0$, so $ab \in \mathcal{F}_{\sigma}$ as $0 \in \mathcal{F}_{\sigma}$ and $a0 \in \mathcal{F}_{\sigma}$ since $\mathcal{F}_{\sigma}$ is an ideal and $ab -$ ab $\in \mathcal{F}_{\sigma}$ or $ja - j0 \in \mathcal{F}_{\sigma}$ then $\mathcal{P}_{cv}(a,j) > 0$ and there exist an edge between a and j in $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)$.

Proposition 3.9: Let $\mathcal{F}$ be a $c-eq-prime\ \mathcal{F}\text{I}\ of\ right\ permutajle\ \mathbb{N} = \mathbb{N}_0$ and $\sigma\epsilon(\alpha,\beta]$ if $a \in Annw$ with $\mathcal{P}_{cv}(a,w) > 0$ then an $\in$ Annw $\forall$ n $\in$ $\mathbb{N}$.

Proof: As $a \in Annw$ then $aw = 0, a0 = 0$ and $\mathcal{P}_{cv}(a,w) > 0$ in $(\mathbb{N},\mathcal{F},\mathcal{P}_{c\,v},\sigma)$ then $aw - a0 \in \mathcal{F}_{\sigma}$ assume that $aw - a0 = \lambda$ for $\lambda \in \mathcal{F}_{\sigma}$ then $awn - a0n = \lambda n$, as $aw - a0 = \lambda$ that is $0 - 0 = \lambda$ therefore $awn - a0n = 0n = 0$. Hence, $awn = 0$ as right permutable, then $anw = 0$ that is $an \in Annw$.

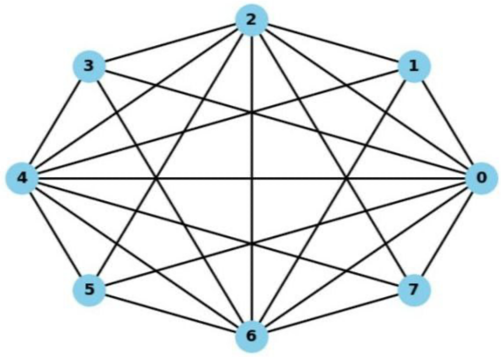

Figure 2: $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)$ when $\mathcal{F}_{\sigma} = \{0,4\}$.

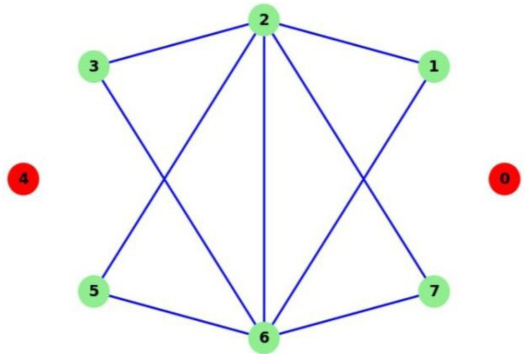

Figure 3: $(\mathbb{N}\backslash\mathcal{F}_{\sigma},\mathcal{F},\mathcal{P}_{cv},\sigma)$ when $\mathcal{F}_{\sigma} = \{0,4\}$.

Table 3: Nearring for example 3.2.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

*	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	2	4	2	0	2	4	6
2	0	4	0	4	0	4	0	4
3	0	6	4	6	0	6	4	2
4	0	0	0	0	0	0	0	0
5	0	2	4	2	0	2	4	6
6	0	4	0	4	0	4	0	4
7	0	6	4	6	0	6	4	2

Table 4: $\mathcal{P}_{cv}(p,w)$ when $\mathcal{F}_{\sigma} = \{0\}$.

$\mathcal{P}_{\text{cv}}(\text{w},\text{y})$	0	1	2	3	4
0	0	0.6	0.6	0.2	0.3
1	0.6	0	0	0	0
2	0,6	0	0	0	0
3	0.2	0	0	0	0
4	0.3	0	0	0	0

Proposition 3.10: Let $\mathcal{F}$ be $c-eq-prime\ \mathcal{F}$I of $\mathbb{N}=\mathbb{N}_0$ then $w\in\mathcal{F}_\sigma$ if and only if $dg(w)=dg(0)=n-1$ in $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)$ whenever $|\mathbb{N}|=n$.

Proof: Suppose that $dg(w)=dg(0)$ that is w and 0 is adjacent to all elements of $\mathbb{N}$ then $wy-w0\in\mathcal{F}_\sigma$ or $yw-y0\in\mathcal{F}_\sigma$ and $0y-00\in\mathcal{F}_\sigma$ or $y0-00\in\mathcal{F}_\sigma$ for an $y\in\mathbb{N}\backslash\mathcal{F}_\sigma$ whereby $y\neq w$. As $\mathcal{F}_\sigma$ is $c-eq-prime\ ideal$ of $\mathbb{N}$, $yx-y0\in\mathcal{F}_\sigma$ then $w-0\in\mathcal{F}_\sigma$ that is $w\in\mathcal{F}_\sigma$.

Conversely, easy see that $dg(0)=n-1$, $|\mathbb{N}|=n$, now let $w\in\mathcal{F}_\sigma$:

Case 1: If $w=0$ then the statement holds.

Case 2: If $w\neq 0$ if possible. Say that $dg(w)<dg(0)$ then it comes to vertex z whereby $\mathcal{P}_{cv}(w,z)=0$ this implies $wz-w0\notin\mathcal{F}_\sigma$ and $zw-z0\notin\mathcal{F}_\sigma$ then $w\notin\mathcal{F}_\sigma$, a contradiction.

Proposition 3.11: Let $\mathcal{F}$ be $c-eq-prime\ \mathcal{F}$I of $\mathbb{N}$ and $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)$ contain a cycle then $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)$ is isomorphic to star graph whenever $\mathcal{F}_\sigma=\{0\}$.

Proof: As by Proposition 3.10, $dg(0)=n-1$, $|\mathbb{N}|=n$ and $\mathcal{P}_{cv}(0,w)>0$ for any elements of $w\in\mathcal{F}_\sigma$ whenever $w\neq 0$. As $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)$, $\sigma\epsilon(\alpha,\beta]$ contains acycle and $\mathcal{F}$ be $a\ c-eq-prime\ \mathcal{F}$I by Proposition 3.1 then $\mathcal{F}_\sigma$ is $c-eq-prime\ ideal\ of\ \mathbb{N}$. Also, by Proposition 3,5 then $\mathbb{N}\backslash\mathcal{F}_\sigma$ is null subgraph that is every element $w\in\mathbb{N}\backslash\mathcal{F}_\sigma$ are independent elements and there are no edges between every neighborhood of w. Now, suppose $y\neq z\in\mathbb{N}\backslash\mathcal{F}_\sigma$, if:

Case1: w is adjacent to y or z then the path $0-w-y-0$ is cycle of length 3, contradiction.

Case 2: w is adjacent to y and z then a path $0-y-w-z-0$ is cycle of length 4, contradiction.

Case 3: w is not adjacent to y and z the study every element y or z with connect 0 and each other and repeat same case 1 and case 2, then in all cases we receive $\mathbb{N}\backslash\mathcal{F}_\sigma$ are independent set and $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)$ is isomorphic to star graph.

Remark 3.1: If $\mathcal{F}_\sigma\neq\{0\}$ then $(\mathcal{F}_\sigma,\mathcal{F},\mathcal{P}_{cv},\sigma)$ is isomorphic to star graph since $deg(w)=deg(0)\ \forall\ w\in\mathcal{F}_\sigma$.

Proposition 3.12: Let $\mathcal{F}$ be a $c-eq-prime\ \mathcal{F}$I $of\ \mathbb{N}=\mathbb{N}_0$ and $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)$ contains cycle then $gr(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)=3$.

Proof: If $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)$ contains cycle of length 3 the proof is complete.

Let $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)$ contains cycle of length 4 thus $\mathbb{N}$ contains at least 4 elements since by proposition (3.13), $deg(0)=n-1$, $|\mathbb{N}|=$ n and $\mathcal{P}(0,w)>0\ \forall\ w\in\mathbb{N}$. So, there will be star graph $K_{1.3}$ with center vertex 0 and same steps of Proposition 3.11 we receive in all cases the length of cycle is 3 and $gr(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)=3$.

Proposition 3.13: Let $\mathcal{F}$ be a $c-eq-prime\ \mathcal{F}$I $of\ \mathbb{N}$ and $\mathcal{F}_s\subseteq\mathcal{F}_\sigma\forall s<\sigma$ and $s,\sigma\in(\alpha,\beta]$ then $(\mathbb{N},\mathcal{F}_s,\mathcal{P}_{sv},s)\subseteq(\mathbb{N},\mathcal{F}_\sigma,\mathcal{P}_{cv},\sigma)$.

Proof: $V(\mathbb{N},\mathcal{F}_s,\mathcal{P}_{sv},s)=(\mathbb{N},\mathcal{F}_\sigma,\mathcal{P}_{cv},\sigma)=\mathbb{N}$ for any level ideal of $\mathbb{N}$. Now, let $\mathcal{P}_{sv}(w,y)>0$ in $(\mathbb{N},\mathcal{F}_s,\mathcal{P}_{sv},s)$ then $wy-w0\in\mathcal{F}_s$ OR $yw-y0\in\mathcal{F}_s$, since $\mathcal{F}_s\subseteq\mathcal{F}_\sigma$ this implies $wy-w0\in\mathcal{F}_\sigma$ or $yw-y0\in\mathcal{F}_\sigma$, so we receive $\mathcal{P}_{cv}(w,y)>0$ in $(\mathbb{N},\mathcal{F}_\sigma,\mathcal{P}_v,\sigma)$ that is every element w and y are adjacent with respect to level set $\mathcal{F}_S$ then is an adjacent with respect to $\mathcal{F}_\sigma$ then $(\mathbb{N},\mathcal{F}_s,\mathcal{P}_{sv},s)\subseteq(\mathbb{N},\mathcal{F}_\sigma,\mathcal{P}_{cv},\sigma)$.

Remark 3.2:

- The convers of Proposition 3.13 is not work in general that is if $(\mathbb{N},\mathcal{F}_s,\mathcal{P}_{sv},s)\subseteq(\mathbb{N},\mathcal{F}_\sigma,\mathcal{P}_{cv},\sigma)$ then $\mathcal{F}_s\subseteq\mathcal{F}_\sigma$.
- If $\mathcal{F}_s\nsubseteq\mathcal{F}_\sigma$ then in general $(\mathbb{N},\mathcal{F}_s,\mathcal{P}_{sv},s)\subseteq(\mathbb{N},\mathcal{F}_\sigma,\mathcal{P}_{cv},\sigma)$.

Corollary 3.1: Let $\mathcal{F}$ be a $c-eq-prime\ \mathcal{F}$I $of\ \mathbb{N}$ and $\mathcal{F}_s\subseteq\mathcal{F}_\sigma\forall s<\sigma$ and $s,\sigma\in(\alpha,\beta]$ then $(\mathbb{N},\mathcal{F}_s,\mathcal{P}_{sv},s)$ is induced subgraph of $(\mathbb{N},\mathcal{F}_\sigma,\mathcal{P}_{cv},\sigma)$.

Proof: As by Proposition 3.3 that is $(\mathcal{F}_\sigma,\mathcal{F},\mathcal{P}_{cv},\sigma)$ is complete subgraph of $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)$ and from Proposition 3.5, then $(\mathbb{N}\backslash\mathcal{F}_s,\mathcal{F},\mathcal{P}_{cv},\sigma)$ with $(\mathbb{N}\backslash\mathcal{F}_\sigma,\mathcal{F},\mathcal{P}_{cv},\sigma)$ is null graphs then by Proposition 3.13 we receive $(\mathbb{N},\mathcal{F}_s,\mathcal{P}_{sv},s)$ is induced subgraph of $(\mathbb{N},\mathcal{F}_\sigma,\mathcal{P}_{cv},\sigma)$.

Proposition 3.14: Let $\mathcal{F}$ be a $c-eq-prime\ ideal\ \mathcal{F}$I $of\ \mathbb{N}$ and $\mathcal{F}_\sigma\neq 0$ then $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)\subseteq(\mathbb{N},rad(\mathcal{F}),\mathcal{P}_{cv},\sigma)$.

Proof: Since $\mathcal{F}_\sigma$ is an ideal of a nearring we receive $\mathcal{F}\subseteq rad(\mathcal{F})$ by Proposition 3.13 we receive $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)\subseteq(\mathbb{N},rad(\mathcal{F}),\mathcal{P}_{cv},\sigma)$.

Corollary 3.2: Let $\mathcal{F}$ be a $c-eq-prime\ \mathcal{F}$I $of\ \mathbb{N}$ then $(\mathbb{N},\mathcal{F}^2_\sigma,\mathcal{P}_{cv},\sigma)\subseteq(\mathbb{N},\mathcal{F}_\sigma,\mathcal{P}_{cv},\sigma)$.

Proof: Since $\mathcal{F}^2_\sigma\subseteq\mathcal{F}_\sigma$, so by Proposition 3.13 we receive the result.

Corollary 3.3: Let $\mathcal{F}$ be a $c-eq-prime\ ideal\ \mathcal{F}$I $of\ \mathbb{N}$ then $(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)$ is induced subgraph of $(\mathbb{N},rad(\mathcal{F}),\mathcal{P}_{cv},\sigma)$.

Proof: From Proposition 3.13 and Corollary 3.1 we receive the result.

Proposition 3.15: Let $\mathcal{F}$ be a $c-eq-prime\ \mathcal{F}$I $of\ \mathbb{N}$ then:

$$\chi(\mathbb{N},\mathcal{F},\mathcal{P}_{cv},\sigma)=$$

$$\begin{cases} 2 \quad if \mathcal{F}_\sigma = 0 \\ |\mathcal{F}_\sigma| + 1 \, if \mathcal{F}_\sigma \neq 0 \\ n-1 \, if \ \mathcal{F}_\sigma = \mathbb{N}, \lceil \mathbb{N} \rceil = n. \end{cases}$$

Proof :We have the cases:

Case 1: If $\mathcal{F}_\sigma = 0$ then by Proposition 3.5 we receive $(\mathbb{N}\backslash\mathcal{F}_\sigma, \mathcal{F}, \mathcal{P}_{cv}, \sigma)$ is null graph then $\big(\mathcal{F}_\sigma, \mathcal{F}, \mathcal{P}_{cv}, \sigma)\big)$ is star graph so $\mathcal{P}_{cv}(w,y) = 0$ $\forall\, w, y \in \mathbb{N}\backslash\mathcal{F}_\sigma$ and $\mathcal{P}_{cv}(0,w) > 0$ we receive vertices $w \neq y$ has same clors and only vertex has colors different and we receive two colors so $\chi(\mathbb{N}, \mathcal{F}, \mathcal{P}_{cv}, \sigma) = 2$.

Case 2: If $\mathcal{F}_\sigma \neq \mathbb{N}$ since $\mathcal{F}_\sigma$ is a clique then $\chi(\mathcal{F}_\sigma) = |\mathcal{F}_\sigma|$.

Now, if $w \in \mathbb{N}\backslash\mathcal{F}_\sigma$, since $\mathcal{F}_\sigma$ is clique and by Proposition 3.5 that is $(\mathbb{N}\backslash\mathcal{F}_\sigma, \mathcal{F}, \mathcal{P}, \sigma)$ is null subgraph that is induced subgraph then $\mathcal{P}_{cv}(w,y) > 0$ $\forall$y $\in \mathcal{F}_\sigma$ thus induced subgraph $\langle < \mathcal{F}_\sigma > \cup \{w\}\rangle$ is complete over $|\mathcal{F}_\sigma| + 1$, so $\chi(\mathbb{N}, \mathcal{F}, \mathcal{P}_{cv}, \sigma) = |\mathcal{F}_\sigma| + 1$.

Cases 3: If $\mathcal{F}_\sigma = \mathbb{N}$ then $(\mathbb{N}, \mathcal{F}, \mathcal{P}_\sigma, \sigma)$ is complete graph then $\chi(\mathbb{N}, \mathscr{F}, \mathcal{P}_\sigma, \sigma) = n - 1$, $|\mathbb{N}| = n$ ■.

Example 3.3 :Let $\mathbb{N} = z_5$, we define $\mathcal{F}: \mathbb{N} \to (0.1]$ by:

$$\mathcal{F}(w) = \begin{cases} 0.9 \, if \ w = 0 \\ 0.2 \, if \ w = 3 \\ 0.3 \, if \ w = 4 \\ 0.6 \, if \ w \in \{1.2\} \end{cases}.$$

If we take threshold $\alpha = 0{,}6$ and $\beta = 0{,}9$ then $\mathcal{F}$ is $\mathcal{F}$I of $\mathbb{N}$. Let $\sigma = \beta$ then $\mathcal{F}_\sigma = \{0\}$. Therefore, the values of $\mathcal{P}_{cv}$ are as in Table 4 and $\chi(\mathbb{N}, \mathscr{F}, \mathcal{P}_{cv}, \sigma) = 2$. The graph is given in Figure 4.

Proposition 3.16: Let $\mathcal{F}$ be a $c-eq-prime \ \mathcal{F}I \ of \ \mathbb{N} = \mathbb{N}_o$ and $\mathcal{P}_{cv}(w,y) = 0$ and $w + \mathcal{F}_\sigma = y + \mathcal{F}_\sigma$ then $w^2, y^2 \notin \mathcal{F}_\sigma$ for $w, y \in \mathbb{N}$.

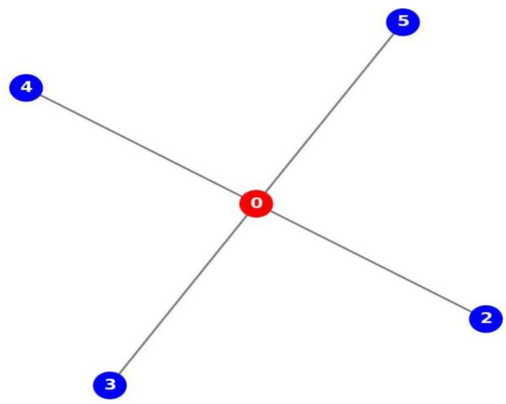

Figure 4: $(\mathbb{N}, \mathcal{F}, \mathcal{P}_{cv}, \sigma)$ when $\mathcal{F}_\sigma = \{0\}$.

Proof: We have the cases:

Case 1: Since $\mathcal{P}_{cv}(w,y) = 0$ then $wy - w0 \notin \mathcal{F}_\sigma$ since $\mathrm{w} + \mathcal{F}_\sigma = \mathrm{y} + \mathcal{F}_\sigma$ this implies $w - y \in \mathcal{F}_\sigma$, let $\sigma = w - y$ consider $w^2 - wy = w(w - y) = w\sigma \in \mathscr{F}_\sigma$.

If $w^2 \in \mathcal{F}_\sigma$ then from $w^2 - wy \in \mathcal{F}_\sigma$ if follows that $wy = w^2 - (w^2 - wy) \in \mathcal{F}_\sigma$.

Since $\mathcal{F}_\sigma$ is ideal sub near ring but this contradiction $wy - w0 \notin \mathcal{F}_\sigma$ since $\mathcal{P}_{cv}(\mathrm{w}, y) = 0$ therefore for $w^2 \notin \mathcal{F}_\sigma$.

Case 2: Since $w - y \in \mathcal{F}_\sigma$ we receive $w = y + \sigma$, consider $y^2 - wy = y(y - w) = y(-\sigma) \in \mathcal{F}_\sigma$.

If $y^2 \in \mathcal{F}_\sigma$ as $y^2 - wy \in \mathcal{F}_\sigma$ it follows that $wy \in \mathcal{F}_\sigma$ contradiction since $\mathcal{P}(w,y) = 0$ therefore $y^2 \notin\in \mathcal{F}_\sigma$.

Proposition 3.17: Let $\mathcal{F}_\sigma$ be totally reflexive ideal of a near ring if $a - w - j$ is path in $(\mathbb{N}, \mathcal{F}, \mathcal{P}_{c\,v}, \sigma)$ then either $\mathcal{F}_\sigma \cup \{w\}$ is ideal of $\mathbb{N}$ or the path is contained in cycle of length ≤ 4.

Proof: Let S_a is the set that represents the set of elements in $\mathbb{N}$ which, if we multiply element a by it from the right, the result belongs to that set $\mathcal{F}_\sigma$ and S_j is the set resulting from multiplying element b from right by elements $\mathbb{N}$ that belong to the set $\mathcal{F}_\sigma$. If $\mathcal{F}_\sigma \cup \{w\} = S_a \cap S_j$ then $\mathcal{F}_\sigma \cup \{w\}$ is an ideal. Otherwise, there is element $v \in S_a \cap S_j$ whereby $v \notin \mathcal{F}_\sigma \cup \{w\}$. From this we have $a - w - b - v - a$ is a cycle of length ≤ 4.

4 CONCLUSIONS

This study introduced a new class of fuzzy graphs, referred to as c-equiprime fuzzy graphs, associated with fuzzy ideals of nearrings, and established their fundamental algebraic and graph-theoretic properties. The relationships between c-equiprime fuzzy ideals, annihilator ideals, and the corresponding fuzzy graph structures were investigated, providing new links between nearring theory and fuzzy graph theory. Several structural properties of these graphs were derived, including conditions under which the associated fuzzy graph becomes a star graph or a clique, together with results concerning adjacency relations, induced subgraphs, cycles, graph radius, and chromatic number. Furthermore, the inclusion relationships between level fuzzy ideals and their corresponding graphs were established, showing how changes in fuzzy ideal levels influence the resulting graph structure.

The obtained results demonstrate that c-equiprime fuzzy graphs provide an effective framework for describing algebraic structures through graph-theoretic concepts. In particular, it was proved that vertices corresponding to elements of level fuzzy ideals generate complete subgraphs under suitable conditions, while graphs associated with simple or integral nearrings exhibit characteristic star-graph structures. The introduction of annihilator ideals further extends the theoretical framework and provides additional tools for analyzing fuzzy nearrings.

The proposed approach enriches the interaction between fuzzy algebra and graph theory and opens several directions for future research. These include extending the concept of c-equiprime fuzzy graphs to other algebraic structures, investigating weighted and intuitionistic fuzzy graph models, studying graph invariants under different classes of fuzzy ideals, and exploring potential applications in information systems, knowledge representation, and uncertainty modeling.

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