Rough Homotopy and the Fundamental Group of Pawlak Approximation Spaces

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Abstract: In classical topology, homotopy theory depends fundamentally on continuous paths. Applying these standard concepts to discrete or granular spaces, such as rough sets, presents obvious mathematical challenges. Specifically, strict continuity in discrete environments often restricts admissible paths to constant maps, making the classical fundamental group trivial and largely uninformative for structural analysis. To address this issue within Pawlak approximation spaces (X, R), we propose a new algebraic invariant: the Rough Fundamental Group, denoted by $\pi_1^R(X, x_0^R)$. Our approach replaces standard continuous paths with "rough paths," defined as finite sequences of transitions between adjacent granules, or equivalence classes. By equipping the time index with a discrete (granular) structure, we ensure both temporal and spatial stability, creating a functional discrete homotopy framework. We show that the resulting rough path homotopy classes form a well-defined group under the operation of rough concatenation. Additionally, we prove that rough continuous maps induce group homomorphisms, establishing the functorial nature of our construction. To test this invariant, we compute the rough fundamental group for a discrete model of the Rough Circle, proving it is isomorphic to the group of integers, $\mathbb{Z}$. Conversely, the group is trivial for a contractible rough disk. These results confirm that $\pi_1^R(X, x_0^R)$ successfully detects topological holes and analyzes connectivity in granular spaces.

1 INTRODUCTION

Rough set theory, originally established in the early 1980s by Z. Pawlak [1], offers a formal framework for modeling uncertainty and granularity. Its significance is highlighted by broad applications in various domains such as data mining, AI, and pattern recognition [2], [3].Topological interpretations of rough sets have been extensively studied, spanning from the foundational works of Pawlak [4] to recent advancements in granular topological structures [5], Various generalized topological structures on rough sets have been investigated, including the study of connectivity, separation axioms, and continuity in approximation spaces [6], [7]. However, while the set-theoretic and general topological properties of rough sets have been widely studied, the algebraic topology aspect specifically the theory of homotopy and fundamental groups remains largely unexplored. Classical homotopy theory relies heavily on the concept of continuous paths with the standard Euclidean topology [8], [9]. Applying this classical definition directly to rough topological spaces, which are inherently discrete or granular, leads to significant difficulties. Strict continuity in discrete spaces often reduces admissible paths to constant maps, rendering the fundamental group trivial and uninformative. It is worth noting the earlier contributions of Abd El-Latif [10], who investigated some homotopic properties within the context of rough sets. However, the approach in [10] relied heavily on classical definitions of continuity and paths. As discussed in literature regarding discrete spaces, imposing classical continuous structures on rough (granular) spaces often leads to a restrictive theory where the only admissible loops are constant maps, or where the fundamental group becomes trivial. Thus, a strictly classical approach may not fully capture the "jumping" nature of rough paths across granules. This necessitates the construction of a more flexible framework to overcome these limitations. It is also crucial to distinguish the framework proposed in this paper from other discrete homotopy theories, specifically Digital Homotopy and Coarse

Homotopy. While Digital Homotopy [11] deals primarily with grid structures (typically $\mathbb{Z}^n$) and adjacency relations derived from pixel connectivity, our approach is based on general approximation spaces (X, R) where R can be any equivalence relation, not necessarily induced by a lattice structure. Similarly, Coarse Homotopy in the sense of Coarse Geometry [12] investigates the large-scale properties of metric spaces "from infinity". In contrast, our Rough Homotopy focuses on the local granularity and indiscernibility arising from limited information in finite or infinite sets, following Pawlak's semantics. Therefore, the rough fundamental group constructed here is an independent invariant tailored specifically for the rough set environment. We introduce the concept of a rough path, which replaces the classical continuous map with a sequence of transitions between adjacent granules. We define the time domain not as a continuous continuum, but as a partitioned space equipped with a granular topology. This approach allows us to capture the "jumps" inherent in rough spaces without violating the continuity required for group structures. We formally define the Rough Fundamental Group, denoted by $\pi_1^R(X, [x_0]_R)$, and prove that it forms a group under rough concatenation. We demonstrate that this group satisfies the standard group axioms - closure, associativity, identity, and inverse - and provides a non-trivial algebraic invariant for rough spaces.

2 PRELIMINARIES

In this section, we briefly review the basic concepts of rough sets and rough topological spaces that are required for the subsequent discussion. More comprehensive treatments of these concepts can be found in [1], [4], and [13].

Definition 2.1 Consider a finite set X with at least one element, which serves as the underlying universe of discourse. Suppose that a relation $R \subseteq X \times X$ is defined on X such that R satisfies the usual axioms of an equivalent relation. the order pair (X, R) is referred to as a Pawlak approximation space, for any element x ∈ X, the collection $[x]_R$={y ∈ X :(x, y) ∈ R} represents the equivalence class of x induced by the relation R. each such class forms a basic information unit, often called an elementary set or granule. the family of all equivalent classes generated by R constitutes a partition of R and this partition denoted by X/R.

Definition 2.2 For any subset A⊆ X, the definition of R-lower approximation and R-upper approximation of A are specified, respectively, by:

$$\underline{R}(A) = \{x \in X \mid [x]_R \subseteq A\} = \bigcup\{[x]_R \in X/R \mid [x]_R \subseteq A\},$$

$$\overline{R}(A) = \{x \in X \mid [x]_R \cap A \neq \emptyset\} = \bigcup\{[x]_R \in X/R \mid [x]_R \cap A \neq \emptyset\}.$$

A subset A is said to be R-exact (or definable) $if\ \underline{R}(A)=\overline{R}(A)$, and R-rough if $\underline{R}(A) \neq \overline{R}$ (A).

It is well known that the approximation operators induce a natural topology on X. This topology is crucial for our construction of the rough fundamental group.

Definition 2.3 Consider (X, R) is an approximation space. The topology generated by the equivalence classes of R as a basis is named the rough topology on X and is denoted by $\mathcal{T}_R$. Explicitly,

$$\mathcal{T}_R = \{\bigcup\mathcal{B} \mid \mathcal{B} \subseteq X/R\}.$$

The pair (X, $\mathcal{T}_R$) is known as a rough topological space (RTS).

In (X, $\mathcal{T}_R$), the closure and interior operators correspond exactly to the upper and lower approximations defined by Pawlak:

$$\mathrm{int}(A) = \underline{R}(A) \quad \text{and} \quad \mathrm{cl}(A) = \overline{R}(A).$$

Definition 2.4 Suppose (X, $\mathcal{T}_R$) and (Y, $\mathcal{T}_S$) are two RTS, A map $\phi: X \to Y$ is said to be rough continuous if for every $V \in \mathcal{T}_S$, the inverse image $\phi^{-1}(V)$ belongs to $\mathcal{T}_R$.

3 THE MAIN RESULT

Definition 3.1 Let (X, τ_R) and (Y, τ_S) be two RTS and let $f, g: X \to Y$ be rough continuous maps. Consider $I = [0,1]$ equipped with an equivalence relation $\sim_I$ whose equivalence classes form a finite ordered partition $I = I_1 \cup I_2 \cup \ldots \cup I_n$, where for a sequential order $0 = t_0 < t_1 < \cdots < t_n = 1$ the classes are given by $I_k = [t_{k-1}, t_k) \quad where\ k = 1, \ldots \ldots, n-1 \quad I_n = [t_{n-1}, t_n]$.

This partition is intrinsic into I and independent of the domain X or the maps g and f. A rough homotopy between f and g is a map $H: X \times I \to Y$ Satisfy the following:

1) **Boundary Conditions:** $H(x, 0) = f(x)$, $H(x, 1) = g(x), \quad \forall x \in X.$
(This ensures that the homotopy connects the initial and terminal maps.)
2) **Spatial Rough Continuity:** For every fixed $t \in I$,
$H_t: X \to Y,, \quad H_t(x) = H(x, t),$

is rough continuous with respect to τ_R and τ_S. *(This ensures that each "slice in time" is rough continuous across X.)*

3) **Temporal Rough Stability:** For each $x \in X$ and each equivalence class $I_k \subseteq I$, there exists an equivalence class $[y]_S \subseteq Y$ such that

$$H(\{x\} \times I_k) \subseteq [y]_S.$$

(This ensures that along each "rough time interval," the image stays within a single equivalence class in Y, respecting the rough structure of the codomain.)

The existence of such a map implies that f and g are rough homotopic, expressed by $f \simeq_R g$.

Example 3.1 Let $X = S^1 = \{(\cos\theta, \sin\theta) \mid \theta \in [0,2\pi)\}$ be the unit circle and let $Y = S^1 \times [0,1]$ denote the cylinder.

Definition of the relation on S^1. We define a relation R on X by partitioning the circle into four subsets according to the angle θ:

$$[x]_R = \begin{cases} A_1, & \text{if } 0 \leq \theta < \frac{\pi}{2}, \\ A_2, & \text{if } \frac{\pi}{2} \leq \theta < \pi, \\ A_3, & \text{if } \pi \leq \theta < \frac{3\pi}{2}, \\ A_4, & \text{if } \frac{3\pi}{2} \leq \theta < 2\pi. \end{cases}.$$

Each set A_i represents one equivalence class of R.

Rough topology on S^1. The rough topology τ_R induced by R on X can be given by

$$\tau_R = \left\{ \bigcup_{i=1}^{n} [x_i]_R \,\middle|\, n \in \mathbb{N} \right\} \cup \{\varnothing\}.$$

Thus, the finite unions of all equivalence classes $\{A_1, A_2, A_3, A_4\}$ given the open sets.

Granulating the Cylinder: we define the rough structure on Y by partitioning the height $[0,1]$ into three bands (levels):

$$\begin{aligned} L_1 &= [0, 1/3), \\ L_2 &= [1/3, 2/3), \\ L_3 &= [2/3, 1]. \end{aligned}$$

The equivalence relation R_Y on Y is: $(x_1, t_1) \sim (x_2, t_2)$ iff $x_1, x_2 \in A_i$, $t_1, t_2 \in L_j$.

Thus, the equivalence classes (granules) of Y are "curved blocks" of the form $Q_{ij} = A_i \times L_j$.

The Maps: we define the map $f, g: X \to Y$ as the inclusion of the circle into the base and top of the cylinder:

$$f(x) = (x, 0) \in A_i \times L_1 \text{ and } \quad g(x) = (x, 1) \in A_i \times L_3.$$

Note that f maps into the bottom band, and g maps into the top band.

Rough Homotopy: We define the homotopy $H: X \times [0,1] \to Y$ by the natural vertical movement: $H(x,t) = (x,t)$

To verify this is a rough homotopy, we consider the partition of time corresponding to the bands: $0 < 1/3 < 2/3 < 1$.

- Interval $(0,1/3)$: For any fixed $x \in A_k$, and any t in this interval, $H(x,t) = (x,t)$ has height in L_1. Thus, the image stays entirely within the single equivalence class $A_k \times L_1$.
- Interval $(1/3,2/3)$: As t increases, the image moves to the band L_2. For t in this range, the image stays within the class $A_k \times L_2$.
- Interval $(2/3,1)$: Similarly, the image resides in $A_k \times L_3$.

Conclusion**:** Since there exists a time partition such that the mapping is constant with respect to the equivalence classes of Y on each sub-interval, and the transitions occur between adjacent bands ($L_1 \to L_2 \to L_3$), g and f are roughly homotopic.

Theorem 3.1 The rough homotopy relation $\simeq_R$ is an equivalence relation.

Proof. Consider the functions $f, g, h: (X, \tau_{R_1}) \to (Y, \tau_{R_2})$ are rough continuous.

Reflexive: Define a homotopy $H_f(x,t) := f(x)$. That implies $H_f(x,0) = f(x)$, $H_f(x,1) = f(x)$, so $f \simeq_R f$.

Symmetric: If H is a homotopy between f and g, define $\widetilde{H}(x,t) := H(x, 1-t)$. Then $\widetilde{H}(x,0) = g(x)$, $\widetilde{H}(x,1) = f(x)$, so $g \simeq_R f$.

Transitive: If H is a homotopy between f and g, and K is between g and h, then before constructing L we refine the time partition associated with H and K to a common finite partition of I, ensuring that the temporal rough stability condition is preserved under rescaling and concatenation.

Define

$$L(x,t) := \begin{cases} H(x, 2t), & 0 \leq t \leq 1/2, \\ K(x, 2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

Then $L(x,0) = f(x)$, $L(x,1) = h(x)$ and L is a rough homotopy, so $f \simeq_R h$.

Hence $\simeq_R$ is an equivalence relation. ∎

Definition 3.2 If $(X, \mathcal{T}_R)$ is RTS. A rough path in X is a map $\alpha: ([0,1], \mathcal{T}_\Delta) \to (X, \mathcal{T}_R)$,

with Δ that is a finite partition of the interval $I = [0,1]$ into sub-intervals (granules) $J_0, J_1, \dots, J_n$. The map α is a rough path if it maps each time granule J_k into a single granule of the space X. That is, for each $k \in \{0, \dots, n\}$, there exists $x_k \in X$ such that: $\alpha(J_k) \subseteq [x_k]_R$ Consequently, the path can be represented as a finite sequence of granules: $[x_0]_R, [x_1]_R, \dots \dots, [x_n]_R$.

Remark 3.1 Although Definition **3.2** describes a rough path as a step function, an essential topological constraint must be imposed on the transitions between consecutive steps. We assume that for any two consecutive time granules J_k and J_{k+1}, the corresponding image granules $[x_k]_R$ and $[x_{k+1}]_R$ must be **adjacent** (or identical) in X. A transition between non-adjacent granules implies a lack of connectivity in the rough structure. Therefore, such "jumps" are inadmissible in our construction of the rough fundamental group, as they would violate the discrete continuity required for path connectedness.

Definition 3.3 Let $(X, \mathcal{T}_R)$ be RTS, and let $\alpha, \beta: I \to X$ be two rough paths defined on partitions Δ_α and Δ_β respectively.

A rough path homotopy between α and β is a map $H: (I \times I, \mathcal{T}_\Gamma) \to (X, \mathcal{T}_R)$,

where Γ is a grid partition on $I \times I$ formed by refining the time partitions. H is a rough path homotopy if it fulfills the following:

1) Boundary Conditions: For all $s \in I$, $H(s, 0) = \alpha(s)$ and $H(s, 1) = \beta(s)$.
2) Rough Endpoint Stability: At every stage of the homotopy, each endpoint remains within its original equivalence class. For all $t \in I$:
   a. $H(0, t) \in [\alpha(0)]_R$, $H(1, t) \in [\alpha(1)]_R$.
3) Adjacency Preservation (Rough Continuity): For any two adjacent cells C_1, C_2 in the grid partition of $I \times I$, their images $H(C_1)$ and $H(C_2)$ must be adjacent (or identical) equivalence classes in X.

We call α and β rough homotopic provided that such a map can be found, denoted by $\alpha \simeq_R \beta$.

Example 3.2 Let RTS $X = \{A, B, C\}$ be equipped with an adjacency relation R forming a complete triangle graph, i.e., every pair of distinct points is adjacent ($A \sim B, B \sim C, A \sim C$). The topology $\mathcal{T}_R$ on X consists of equivalence classes which are disjoint clopen sets.

Let $\alpha, \beta: [0,1] \to X$ be two rough paths from A to C:

Path α (via B): Moves from A to B and then to C.

$$\alpha(s) = \begin{cases} A & 0 \le s < 1/3 \\ B & 1/3 \le s < 2/3 \\ C & 2/3 \le s \le 1 \end{cases}$$

Path β (Direct): Moves directly from A to C.

$$\beta(s) = \begin{cases} A & 0 \le s < 1/2 \\ C & 1/2 \le s \le 1 \end{cases}$$

We analyze the homotopy at the critical mid-section, specifically at $s_0 = 0.5$. Note that:

$$\alpha(0.5) = B \quad \text{and} \quad \beta(0.5) = C.$$

1) Failure of Standard (Continuous) Homotopy; Suppose we attempt to construct a standard continuous homotopy $H: [0,1] \times [0,1] \to X$ where $H(s, 0) = \alpha(s)$ and $H(s, 1) = \beta(s)$. Consider the path of the point $s_0 = 0.5$ over time t:

$$\gamma(t) = H(0.5, t).$$

- At $t = 0$: $\gamma(0) = \alpha(0.5) = B$.
- At $t = 1$: $\gamma(1) = \beta(0.5) = C$.

Since the time interval $[0,1]$ is connected and H is continuous with the standard topology, the image $\gamma([0,1])$ must be a connected subset of X. However, in the rough topological space X, the sets $\{B\}$ and $\{C\}$ are disjoint and clopen (totally disconnected). Thus, no continuous path exists between B and C. The map $\gamma(t)$ must be constant.

$$\Longrightarrow H(0.5,1) = B \neq C = \beta(0.5).$$

This contradicts the boundary condition. Therefore, standard homotopy fails.

2) Success of Rough Homotopy (Partitioned Time) Now, we apply our definition of Rough Homotopy by partitioning the deformation time I_t into two intervals: $[0,0.5)$ and $[0.5,1]$. Define $H: I \times I \to X$ as:

$$H(s, t) = \begin{cases} \alpha(s) & 0 \le t < 0.5 \\ \beta(s) & 0.5 \le t \le 1 \end{cases}.$$

Let us verify the conditions:

1) Boundary Conditions (Substitution):
   - At $t = 0$ (Start): $H(s, 0) = \alpha(s)$. (Since $0 < 0.5$):
   - At $t = 1$ (End): $H(s, 1) = \beta(s)$. (Since $1 \ge 0.5$).
2) Admissibility of Transition (The Jump): The discontinuity occurs at $t = 0.5$. We must ensure that for every s, the jump is between adjacent granules.

$$\text{Jump}(s): \alpha(s) \to \beta(s).$$

Checking specific values:

- For $s \in [0,1/3)$: Jump is $A \to A$ (Identity, valid).
- For $s \in [1/3,1/2)$: Jump is $B \to A$. (Valid since $B \sim A$).

- For $s \in [1/2,2/3)$: Jump is $B \to C$. (Valid since $B \sim C$).
- For $s \in [2/3,1]$: Jump is $C \to C$ (Identity, valid).

Since all transitions are between adjacent (or identical) classes, H is a valid Rough Homotopy.

$$\therefore \alpha \simeq_R \beta.$$

Remark 3.2 The definition of rough path homotopy presented above is consistent with the general theory of rough homotopy between arbitrary rough continuous maps. Indeed, if we regard the unit interval $I = [0,1]$ as a rough space $(I, \mathcal{T}_\Delta)$ equipped with the partition topology, then the rough paths α and β are simply rough continuous maps. Classical path homotopy demands strict continuity in the standard topological sense, where distinct points are theoretically separable (Hausdorff property). In contrast, rough path homotopy relies on granular continuity governed by the approximation space (X, R). While the classical approach resolves individual points, the rough approach treats all points within a single equivalence class $[x]_R$ as topologically indistinguishable. Consequently, rough homotopy describes motion not as a flow of points, but as a sequence of transitions between granules.

Remark 3.3 Since rough homotopy between rough continuous maps in general is an equivalence relation, it follows immediately that rough path homotopy $\simeq_R$ on the set of rough paths from [0,1] to X is also an equivalence relation. Hence, The equivalence class of a rough path α with respect to the equivalence relation $\simeq_R$ will be denoted $[\alpha]$. Thus

$$\alpha \simeq_R \beta \text{ iff } [\alpha] = [\beta].$$

Definition 3.4 Let $\alpha, \beta: ([0,1], \mathcal{T}_\Delta) \to (X, \mathcal{T}_R)$ be two rough paths such that

$$\alpha(1) \in [x_1]_R = [\beta(0)]_R.$$

The rough concatenation of α and β, denoted $\alpha * \beta$, is the map

$$(\alpha * \beta)(s) = \begin{cases} \alpha(2s), & 0 \le s \le 1/2, \\ \beta(2s-1), & 1/2 \le s \le 1. \end{cases}$$

Proposition 3.1 Rough concatenation induces a well-defined operation on rough path homotopy classes. Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be rough paths such that $[\alpha_1(1)]_R = [\beta_1(0)]_R$. If $\alpha_1 \simeq_R \alpha_2$ and $\beta_1 \simeq_R \beta_2$, then: $\alpha_1 * \beta_1 \simeq_R \alpha_2 * \beta_2$.

Proof. Suppose that $H: I \times I \to X$ is a rough homotopy between α_1 and α_2, and $K: I \times I \to X$ be a rough homotopy between β_1 and β_2.By the definition of rough homotopy (Endpoint Stability condition), we have for all t belong to I: $H(1,t) \in [\alpha_1(1)]_R$ and $K(0,t) \in [\beta_1(0)]_R$.

Since the concatenation condition ensures $[\alpha_1(1)]_R = [\beta_1(0)]_R$, it follows that for every t, the "end" of H and the "start" of K lie in the same equivalence class: $[H(1,t)]_R = [K(0,t)]_R$.

We construct a combined homotopy $L: I \times I \to X$ by concatenating H and K along the time axis s. Define L as:

$$L(s,t) = \begin{cases} H(2s,t), & 0 \le s \le 1/2, \\ K(2s-1,t), & 1/2 \le s \le 1. \end{cases}$$

We must verify that L is a valid rough homotopy:

1) Boundary Conditions**:** At $t = 0$, $L(s,0)$ traces $\alpha_1 * \beta_1$. At $t = 1$, $L(s,1)$ traces $\alpha_2 * \beta_2$. Thus, L connects the product paths.
2) Endpoint Stability: $L(0,t) = H(0,t)$ and $L(1,t) = K(1,t)$. Since H and K are homotopies, these remain in the fixed start/end classes for all t.
3) Adjacency Preservation (Rough Continuity): The map L is clearly rough continuous on the sub-rectangles $[0,1/2] \times I$ and $[1/2,1] \times I$ (inherited from H and K). The critical verification is at the "seam" $s = 1/2$. For any fixed t, the transition from the left side ($s \to 1/2^-$) to the right side ($s \to 1/2^+$) corresponds to a transition from a granule in $H(1,t)$ to a granule in $K(0,t)$. As shown above, these granules are identical. Since a granule is adjacent to itself, the transition is admissible.

Therefore, L is a valid rough homotopy, and the operation is well-defined on the classes. ∎

Definition 3.5 Let $x_0 \in X$ be a point in RTS $(X, \mathcal{T}_R)$. The Rough constant path, denoted by e_{x_0} (or c_{x_0}), is the map $e_{x_0}: ([0,1], \mathcal{T}_\Delta) \to (X, \mathcal{T}_R)$ defined by $e_{x_0}(s) = x_0, \forall\, s \in [0,1]$. Note that this is trivially a rough path with respect to any time partition, as its image $\{x_0\}$ is contained in a single granule $[x_0]_R$.

Lemma 3.1 Let e_x denote the constant rough path at $x \in X$, and let $\alpha: ([0,1], \mathcal{T}_\Delta) \to (X, \mathcal{T}_R)$ be a rough path with $\alpha(0) \in [x]_R$ and $\alpha(1) \in [y]_R$. Then e_x acts as a two-sided identity up to rough homotopy:

$$e_x * \alpha \simeq_R \alpha \qquad \text{and} \qquad \alpha * e_y \simeq_R \alpha.$$

Proof. We give explicit rough homotopies.

Left Identity **($e_x * \alpha \simeq_R \alpha$):** Define the seam-position function for $t \in [0,1]$ by:

$$a(t) := \frac{1-t}{2} \in [0, 1/2].$$

Define the homotopy $L: [0,1] \times [0,1] \to X$ by:

$$L(s,t) = \begin{cases} x, & 0 \le s \le a(t), \\ \alpha\left(\dfrac{s - a(t)}{1 - a(t)}\right), & a(t) \le s \le 1. \end{cases}$$

For each fixed t, the second piece is a re-parameterization of α on the interval $[a(t), 1]$. Since the re-parameterization is continuous, $L(\cdot, t)$ preserves the adjacency structure and is a valid rough path.

- At $t = 0$: We have $a(0) = 1/2$, so:

$$L(s,0) = \begin{cases} x, & 0 \le s \le 1/2, \\ \alpha(2s - 1), & 1/2 \le s \le 1, \end{cases}$$

which is exactly $e_x * \alpha$.

- At $t = 1$: We have $a(1) = 0$, so $L(s,1) = \alpha(s)$ for all $s \in [0,1]$.

Moreover, $L(0,t) = x \in [x]_R$ and $L(1,t) = \alpha(1) \in [y]_R$ for all t. Thus, L is a rough homotopy from $e_x * \alpha$ to α.

Right Identity **($\alpha * e_y \simeq_R \alpha$):** Define for $t \in [0,1]$:

$$b(t) := \frac{1+t}{2} \in [1/2, 1].$$

Define the homotopy $M: [0,1] \times [0,1] \to X$ by:

$$M(s,t) = \begin{cases} \alpha\left(\dfrac{s}{b(t)}\right), & 0 \le s \le b(t), \\ y, & b(t) \le s \le 1. \end{cases}$$

For fixed t, the first piece is a re-parameterization of α on $[0, b(t)]$, hence $M(\cdot, t)$ is a valid rough path.

- At $t = 0$: We have $b(0) = 1/2$, so:

$$M(s,0) = \begin{cases} \alpha(2s), & 0 \le s \le 1/2, \\ y, & 1/2 \le s \le 1, \end{cases}$$

which equals $\alpha * e_y$.

- At $t = 1$: We have $b(1) = 1$, so $M(s,1) = \alpha(s)$.

Also, $M(0,t) = \alpha(0) \in [x]_R$ and $M(1,t) = y \in [y]_R$ for all t. Hence M is a rough homotopy from $\alpha * e_y$ to α.

Therefore, the constant path acts as the identity element for concatenation on rough path homotopy classes. ∎

Definition 3.6 Let $\alpha: ([0,1], \mathcal{T}_\Delta) \to (X, \mathcal{T}_R)$ be a rough path from x_0 to x_1. The inverse rough path of α, denoted by α^{-1}, is the path from x_1 to x_0 defined by

$$\alpha^{-1}(s) := \alpha(1 - s), \quad \forall s \in [0,1].$$

Lemma 3.2 Let $\alpha: ([0,1], \mathcal{T}_\Delta) \to (X, \mathcal{T}_R)$ be a rough path with endpoints $\alpha(0) \in [x_0]_R$ and $\alpha(1) \in [x_1]_R$. Then the inverse path α^{-1} defined by $\alpha^{-1}(s) = \alpha(1 - s)$ satisfies:

$$\alpha * \alpha^{-1} \simeq_R e_{\alpha(0)} \quad \text{and} \quad \alpha^{-1} * \alpha \simeq_R e_{\alpha(1)}.$$

Moreover, if $\alpha_1 \simeq_R \alpha_2$, then $\alpha_1^{-1} \simeq_R \alpha_2^{-1}$. Thus, the inversion operation is well-defined on rough path homotopy classes.

Proof. We provide the explicit rough homotopies.

1) Homotopy $\alpha * \alpha^{-1} \simeq_R e_{\alpha(0)}$. For $t \in [0,1]$, define the scaling factor:

$$a(t) := \frac{1-t}{2} \in [0, 1/2].$$

Define the homotopy map $L: [0,1] \times [0,1] \to X$ by:

$$L(s,t) = \begin{cases} \alpha\left(\dfrac{2s}{1-t}\right), & 0 \le s \le a(t), \\ \alpha\left(2 - \dfrac{2s}{1-t}\right), & a(t) \le s \le 1,\ t < 1, \\ \alpha(0), & t = 1. \end{cases}$$

For each fixed $t < 1$, the first piece parameterizes α on $[0,1]$ compressed into the interval $[0, a(t)]$, and the second piece parameterizes α^{-1} compressed into $[a(t), 1]$. Thus, $L(\cdot, t)$ is a valid rough path formed by concatenation.

- At $t = 0$: We have $a(0) = 1/2$, and:

$$L(s,0) = \begin{cases} \alpha(2s), & 0 \le s \le 1/2, \\ \alpha(2 - 2s), & 1/2 \le s \le 1, \end{cases}$$

which is exactly $\alpha * \alpha^{-1}$.

- At $t = 1$: The definition gives $L(s,1) = \alpha(0)$ for all s, which is the constant path $e_{\alpha(0)}$.

The seam condition at $s = a(t)$ is satisfied for every t because both pieces meet at $\alpha(1)$, ensuring they belong to the same R-class. The endpoints satisfy $L(0,t) = \alpha(0)$ and $L(1,t) = \alpha(0)$ (since $\alpha^{-1}(1) = \alpha(0)$). Hence, L is a rough homotopy from $\alpha * \alpha^{-1}$ to $e_{\alpha(0)}$.

2) Homotopy $\alpha^{-1} * \alpha \simeq_R e_{\alpha(1)}$. Similarly, define for $t \in [0,1]$:

$$b(t) := \frac{1+t}{2} \in [1/2, 1].$$

Define the map $M: [0,1] \times [0,1] \to X$ by:

$$M(s,t) = \begin{cases} \alpha\left(1 - \dfrac{2s}{1+t}\right), & 0 \le s \le b(t), \\ \alpha\left(\dfrac{2s - (1+t)}{1+t}\right), & b(t) \le s \le 1,\ t < 1, \\ \alpha(1), & t = 1. \end{cases}$$

One checks as before that $M(\cdot\,,0) = \alpha^{-1} * \alpha$ and $M(\cdot\,,1) = e_{\alpha(1)}$, with the required seam (at $\alpha(0)$) and endpoint-class compatibility. Therefore, M is a rough homotopy from $\alpha^{-1} * \alpha$ to $e_{\alpha(1)}$.

3) Inversion respects homotopy**.** Let $H: [0,1] \times [0,1] \to X$ be a rough homotopy from α_1 to α_2 (i.e., $H(s,0) = \alpha_1(s)$ and $H(s,1) = \alpha_2(s)$). Define the map: $\widetilde{H}(s,t) := H(1-s,t)$.

Then $\widetilde{H}$ is a rough homotopy from α_1^{-1} to α_2^{-1}. The endpoints are preserved roughly because $H(0,t) \in [\alpha(0)]_R$ and $H(1,t) \in [\alpha(1)]_R$. Thus: $\alpha_1 \simeq_R \alpha_2 \Longrightarrow \alpha_1^{-1} \simeq_R \alpha_2^{-1}$.

Combining these three parts proves that inverses exist and behave correctly in the rough homotopy group structure. ◻

Lemma 3.3 Concatenation is associative up to rough homotopy. Let α, β, γ be rough paths such that the endpoint compatibility conditions hold:

$$[\alpha(1)]_R = [\beta(0)]_R \quad \text{and} \quad [\beta(1)]_R = [\gamma(0)]_R.$$

Then:

$$(\alpha * \beta) * \gamma \simeq_R \alpha * (\beta * \gamma).$$

Proof. We construct an explicit rough homotopy $L: [0,1] \times [0,1] \to X$ from $(\alpha * \beta) * \gamma$ to $\alpha * (\beta * \gamma)$ by continuously sliding the interior partition points (seams).

For $t \in [0,1]$, define two moving breakpoints $s_1(t)$ and $s_2(t)$ as: $s_1(t) := \frac{1-t}{4} + \frac{t}{2} \in [1/4, 1/2]$, $s_2(t) := \frac{1-t}{2} + \frac{3t}{4} \in [1/2, 3/4]$.

Observe the boundary values:

- At $t = 0$: $s_1(0) = 1/4$ and $s_2(0) = 1/2$. These correspond to the partition for $(\alpha * \beta) * \gamma$.
- At $t = 1$: $s_1(1) = 1/2$ and $s_2(1) = 3/4$. These correspond to the partition for $\alpha * (\beta * \gamma)$.

Now, define the homotopy map $L(s,t)$ by:

$$L(s,t) = \begin{cases} \alpha\left(\dfrac{s}{s_1(t)}\right), & 0 \le s \le s_1(t), \\ \beta\left(\dfrac{s - s_1(t)}{s_2(t) - s_1(t)}\right), & s_1(t) \le s \le s_2(t), \\ \gamma\left(\dfrac{s - s_2(t)}{1 - s_2(t)}\right), & s_2(t) \le s \le 1. \end{cases}$$

Verification: For each fixed t, the three pieces are linear re-parameterizations of α, β, γ onto the sub-intervals $[0, s_1(t)]$, $[s_1(t), s_2(t)]$, and $[s_2(t), 1]$ respectively. Since α, β, γ are rough paths and the re-parameterization functions are continuous, $L(\cdot, t)$ preserves the adjacency structure (rough continuity).

1) At $t = 0$:

$$L(s,0) = \begin{cases} \alpha(4s), & 0 \le s \le 1/4, \\ \beta(4s-1), & 1/4 \le s \le 1/2, \\ \gamma(2s-1), & 1/2 \le s \le 1, \end{cases}$$

which is exactly the definition of $(\alpha * \beta) * \gamma$.

2) At $t = 1$:

$$L(s,1) = \begin{cases} \alpha(2s), & 0 \le s \le 1/2, \\ \beta(4s-2), & 1/2 \le s \le 3/4, \\ \gamma(4s-3), & 3/4 \le s \le 1, \end{cases}$$

which is exactly the definition of $\alpha * (\beta * \gamma)$.

The seam compatibility holds for every t because the values at the matching points $s = s_1(t)$ and $s = s_2(t)$ lie in the appropriate rough-equivalence classes (specifically $[\alpha(1)]_R$ and $[\beta(1)]_R$). Since L is continuous on each sub-rectangle and respects the adjacency relation at the boundaries, L is a valid rough homotopy. ◻

Definition 3.7 Let $(X, \mathcal{T}_R)$ RTS and $x_0 \in X$. A rough loop based at x_0 is a rough path $\alpha: ([0,1], \mathcal{T}_\Delta) \to (X, \mathcal{T}_R)$ such that both endpoints belong to the equivalence class of the base point:

$$\alpha(0) \in [x_0]_R \quad \text{and} \quad \alpha(1) \in [x_0]_R.$$

The equivalence class $[x_0]_R$ serves as the "rough base point" of the loop.

Definition 3.8 Let $(X, \mathcal{T}_R)$ be RTS and let $x_0 \in X$. The rough fundamental group of X at the rough base point $[x_0]_R$, denoted by $\pi_1^R(X, [x_0]_R)$, is the set of all rough path homotopy classes of rough loops based at $[x_0]_R$:

$$\pi_1^R(X, [x_0]_R) = \{[\alpha]_R \mid \alpha: ([0,1], \mathcal{T}_\Delta) \to (X, \mathcal{T}_R),\ \alpha(0), \alpha(1) \in [x_0]_R\}.$$

The group operation is defined by the rough concatenation of path classes:

$$[\alpha]_R \cdot [\beta]_R = [\alpha * \beta]_R.$$

Theorem 3.2 The set $\pi_1^R(X, [x_0]_R)$ forms a group together with the rough concatenation operation defined in Definition 3.4

Proof. Combining the results proved above yields the desired conclusion: Proposition 3.1 ensures the operation is well-defined on homotopy classes, Lemma **3.3** establishes associativity, Lemma 3.1 identifies the identity element, and Lemma 3.2 guarantees the existence of inverse elements.

Remark 3.5 It is instructive to consider the behavior of the rough fundamental group in two extreme cases of the approximation space (X, R):

1) Identity relation (fine limit): If R is the identity relation (i.e., $xRy \Leftrightarrow x = y$), the granules become singletons. In this limit, the rough structure approaches the classical discrete structure, and the rough fundamental group $\pi_1^R(X)$ naturally relates to the classical fundamental group $\pi_1(X)$.
2) Universal relation (coarse limit): If R is the universal relation (i.e., xRy for all $x, y \in X$), the entire space X forms a single granule. Consequently, any rough loop is roughly homotopic to the constant loop (since all points are indistinguishable), and the rough fundamental group collapses to the trivial group $\{e\}$.

This demonstrates that $\pi_1^R(X)$ provides a non-trivial topological invariant strictly between these extremes.

Example 3.3 Let $X = \{1,2,3,4,5,6,7,8\}$ be equipped with an equivalence relation R that partitions X into four distinct granules (equivalence classes):$E_1 = \{1,2\}$, $E_2 = \{3,4\}$, $E_3 = \{5,6\}$, $E_4 = \{7,8\}$.

We define the rough topology $\mathcal{T}_R$ such that the cyclic adjacency is preserved: E_1 is adjacent to E_2, E_2 to E_3, E_3 to E_4, and E_4 back to E_1. This structure models a discretized circle with a "hole" in the center. Let the base point be $[x_0]_R = E_1$. Consider the elementary rough loop α that traverses the granules in a counterclockwise sequence:

$$\alpha = (E_1 \to E_2 \to E_3 \to E_4 \to E_1).$$

Formally, as a map from the partitioned interval, α visits each granule sequentially.

Analysis of Homotopy Classes:

1) Non-Triviality. The loop α cannot be roughly homotopic to the constant loop c_{E_1}. In our rough path framework, a homotopy requires "sliding" the path through adjacent granules. Since there is no "central" granule connecting E_1, E_2, E_3, E_4 directly (the center is empty), the loop α cannot be contracted to a point without breaking the adjacency chain.
2) Winding Number. Any rough loop in this space can be characterized by counting the total number of oriented turns about the center.
   - $\alpha^0 \simeq_R c_{E_1}$ (Identity, 0 winds).
   - $\alpha^1 = \alpha$ (1 wind).
   - α^{-1} (traverse $E_1 \to E_4 \to E_3 \to E_2 \to E_1$, -1 wind).
   - α^n (n winds).
3) Isomorphism. It can be shown that $\alpha^n \simeq_R \alpha^m$ if and only if $n = m$. The composition of loops corresponds to the addition of their winding numbers.

Conclusion: This rough circle has a rough fundamental group that is algebraically equivalent to the infinite cyclic group of integers: $\pi_1^R(X, E_1) \cong \mathbb{Z}$. This result confirms that the rough fundamental group successfully captures the topological feature (the 1-dimensional hole) despite the granular nature of space.

Remark 3.4. It is instructive to observe what happens if we "fill" the hole in the previous example. Let us augment the space X by adding a central point $p = 9$, forming a new granule $E_{center} = \{9\}$. Suppose the adjacency relation is extended such that E_{center} is adjacent to all boundary granules E_1, E_2, E_3, E_4 (simulating a solid rough disk). In this new space X', the loop α is no longer essential. We can construct a rough homotopy that "pulls" the loop towards the center. Specifically, any transition $E_i \to E_{i+1}$ on the boundary can be deformed into $E_i \to E_{center} \to E_{i+1}$. By repeating this, the entire loop contracts to the constant loop at E_{center}. Consequently, the rough fundamental group becomes trivial:

$$\pi_1^R(X', E_1) \cong \{e\}.$$

This contrast between Example **3.3** ($\cong \mathbb{Z}$) and this remark ($\cong \{0\}$) highlights that π_1^R is a genuine topological invariant capable of distinguishing between rough spaces with and without holes.

Lemma 3.4. Consider $\phi: (X, R_X) \to (Y, R_Y)$ is a rough continuous map. If $\alpha: [0,1] \to X$ is a rough path, then $\phi \circ \alpha: [0,1] \to Y$ is also rough path.

Proof. Since rough paths are defined as rough continuous maps and the composition of rough continuous maps is again rough continuous, the result follows.

Because of the previous lemma, we now show that a rough continuous map induces a homomorphism between rough fundamental groups.

Theorem 3.3. Let (X, $\mathcal{T}_R$) and (Y, $\mathcal{T}_S$) be two RTS and let $\phi: (X, \mathcal{T}_R) \to (Y, \mathcal{T}_R)$ be a rough continuous map for which $\phi([x_0]_R) \subseteq [y_0]_\mathrm{S}$. Then ϕ induces a group homomorphism:

$$(\phi)_*: \pi_1^R(X, [x_0]_\mathrm{R}) \to \pi_1^R(Y, [y_0]_\mathrm{S}),$$

defined by $(\phi)_*([\alpha]_R) = [\,\phi \circ \alpha\,]_R$, where α is any rough loop in X based at $[x_0]_\mathrm{R}$.

Proof. The first step is to establish that the induced map $(\phi)_*$ is well defined. Let α and β be two rough loops in X based at $[x_0]_{\mathrm{R}}$ such that $\alpha \simeq_R \beta$ via a rough homotopy H. Since ϕ is rough continuous, the composition $\phi \circ H$ is again a rough homotopy, and hence $\phi \circ \alpha \simeq_R \phi \circ \beta$. Therefore,

$$(\phi)_*([\alpha]_R) = (\phi)_*([\beta]_R),$$

which shows that $(\phi)_*$ is well defined.

Next, we verify that $(\phi)_*$ is a group homomorphism. Let α and β be rough loops based at $[x_0]_{\mathrm{R}}$. Then $(\phi)_*([\alpha]_R \cdot [\beta]_R) = (\phi)_*([\alpha * \beta]_R) = [\,\phi \circ (\alpha * \beta)\,]_R$. By the definition of concatenation of rough paths, we have $\phi \circ (\alpha * \beta) = (\phi \circ \alpha) * (\phi \circ \beta)$.

Hence $[\,\phi \circ (\alpha * \beta)\,]_R = [\,\phi \circ \alpha\,]_R \cdot [\,\phi \circ \beta\,]_R = (\phi)_*([\alpha]_R) \cdot (\phi)_*([\beta]_R)$.

This proves that $(\phi)_*$ is a group homomorphism.

4 DISCUSSION

While recent advancements have successfully integrated algebraic structures with rough set theory, such as rough semigroups [14] and soft covering spaces [15]. rough homotopy theory has remained limited. Previous attempts to establish a fundamental group in this context, notably by Abd El-Latif [9], relied strictly on classical continuity. However, imposing classical continuous mappings on inherently discrete granular spaces restricts admissible paths to merely constant maps. To overcome this limitation, our proposed Rough Fundamental Group redefines paths as discrete transitions between adjacent granules. This novel approach preserves a non-trivial algebraic structure, perfectly aligning with modern algebraic-topological trends [14], [15] while effectively detecting the topological holes within Pawlak approximation spaces.

5 CONCLUSIONS

We have successfully constructed a novel algebraic invariant for rough topological spaces: the Rough Fundamental Group $\pi_1^R(X)$. By replacing the classical notion of continuous paths with sequences of transitions between adjacent granules, we established a rigorous homotopy theory that respects the discrete and granular nature of approximation spaces. Because Pawlak approximation spaces model information systems and relational databases, the rough fundamental group can serve as a diagnostic algorithm to detect structural anomalies, identify missing data "holes" and analyze connectivity breaches within discrete data networks.

We can sum up our primary contributions as follows:

1) We introduced formal definitions for rough paths, rough loops, and rough homotopy, overcoming the triviality issues associated with classical continuity in discrete settings.
2) We proved that the set of rough homotopy classes forms a group structure, satisfying the axioms of closure, associativity, identity, and inverse.
3) We established the functoriality of our construction by proving that rough continuous maps induce homomorphisms between rough fundamental groups.
4) We demonstrated the computational power of π_1^R through the example of the Rough Circle, showing that $\pi_1^R(S_R^1) \cong \mathbb{Z}$. Furthermore, we showed that the invariant correctly distinguishes between a rough circle (with a hole) and a rough disk (contractible), validating its topological significance.

6 FUTURE WORK

This work lays the foundation for a broader "Rough Algebraic Topology." Potential directions for future research include:

- Rough Homology: Investigating the relationship between π_1^R and a corresponding theory of rough homology groups.
- Higher Homotopy Groups: Extending the construction to define higher-dimensional rough homotopy groups $\pi_n^R(X)$ using rough spheres.
- Applications: Applying this invariant to digital image processing and sensor networks, specifically for identifying topological features and coverage holes in data with granular uncertainty.

We believe that this algebraic framework provides a robust tool for analyzing the topological structure of rough sets beyond standard set-theoretic properties.

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