Temimi-Ansari Method for a Class of Fifth-Order Korteweg-De Vries Equations

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Abstract: In this study examines the application of the Tamimi-Ansari method (TAM) as a semi-analytical iterative approach to solving three different models of nonlinear PDEs derived from fifth-order generalized Korteweg-de Vries (KdV) equation. The method relies on generating approximate solutions in the form of polynomial series and, in some cases, can yield the exact solution depending on the formula of the equation to be solved and the initial condition chosen for this purpose. The method's efficiency and accuracy were evaluated using numerical criteria to measure error and to assess the convergence of approximate and exact solutions. The numerical results demonstrated that the TAM possesses excellent convergence characteristics and high numerical stability and proved its ability to produce accurate approximate solutions for higher-order nonlinear equations. Furthermore, graphical comparisons showed a high degree of agreement between the approximate and exact solutions, confirming the method's effectiveness and reliability in handling this type of PDE.

1 INTRODUCTION

Nonlinear equations are fundamental to explaining a wide range of phenomena in biology, chemistry, and physics, where they are used in electromagnetic radiation, solid-state physics, fiber optics, plasma physics, and fluid dynamics. A wide range of numerical and analytical techniques are available for addressing complex scientific models [1]-[7]. Advancements in scientific and engineering research often require the precise formulation and conceptualization of mathematical models that represent real-world phenomena. Whether they are ODEs or PDEs with initial or boundary conditions, higher-order DEs are essential for modeling and simulating a wide range of physical and engineering problems. For example, higher-order DEs may be used to study the dynamics of an accelerating mass-spring system [8]. The consequences of imprecise interactions with the Nipah virus (NiV) are also studied [9]. Furthermore, it is employed to investigate bifurcations and chaos and has numerous uses in circuit design, signal processing, heat distribution, secure communications, plasma physics, describing the behavior of surface water waves in fluid dynamics, and the propagation of electrostatic waves in plasmas [10]-[14].

The KdV equation is an important nonlinear PDE that helps in simulating traveling waves and harmonic crystals in shallow water. It was introduced by the French mathematician Joseph Boussinesq in 1877 [15] and established in 1895 by the Dutch mathematicians Korteweg and Gustav de Vries [16]. In addition, there is a class of fifth-order KdV equations with three coefficients called the generalized fifth-order KdV equation (KdV5), which is written as follows [17]:

$$\mathcal{U}_t + a\mathcal{U}^2\mathcal{U}_x + b\mathcal{U}_x\mathcal{U}_{2x} + c\mathcal{U}\mathcal{U}_{3x} + \mathcal{U}_{5x} = 0, \quad \mathcal{U} = \mathcal{U}(x,t), \tag{1}$$

In this context, x and t denote the real spatial and temporal independent variables, respectively, while $\mathcal{U}(x,t)$ represents the dependent function. The parameters a, b and c are non-zero real-valued coefficients with randomly assigned values. The spatial domain extends over $-\infty < x < \infty$, and the temporal domain is defined for $0 \le t < \infty$. The KdV5 equation incorporates two dispersive terms, namely $\mathcal{U}_{3x}$ and $\mathcal{U}_{5x}$. The randomness and variability in the values of the coefficients substantially influence the behavior and characteristics of (1), allowing for the formulation of multiple variants of the KdV5 equation by altering these real-valued coefficients [18]. A variety of physical phenomena, such as wave propagation in nonlinear LC circuits with mutual inductance between nearby inductors and shallow water waves close to critical surface tension conditions, are fundamentally modeled by the KdV5 equation [19]. The exact solution for the single-wave case of KdV5 equation has been derived in [20]. Additionally, various numerical techniques such as finite difference method, method of collocation, and the Galerkin method have been employed to obtain approximate solutions. Kawahara [21] conducted numerical investigations into the stability of solutions to this equation. Further, Boyd [22] and Haupt-Boyd [23] reduced the equation to an ODE, enabling the application of both analytical and numerical techniques to explore its solutions in more depth.

The TAM is an innovative semi-analytical iterative technique for solving nonlinear DEs. It was first introduced in 2011 by Helmi Al-Tamimi and Abdulrazzaq Al-Ansari in a research paper [24]. It has gained widespread attention due to its simple steps and high accuracy. The basic idea of TAM is to transform the nonlinear equation into a series of simple linear problems that are solved sequentially without the need for small coefficients or prior linear assumptions, allowing for rapid convergence and high accuracy of the results [24]. The technique has subsequently been applied to multipoint boundary problems, and accurate and highly convergent results have been obtained, gaining it wider credibility in the academic community. Compared to various methods and approaches such as HPM and VIM [25], TAM often shows faster convergence and lower errors in various applications. Applications of this method have extended to physics, chemistry, and biology, from the Korteweg-DeVries KdV equations to mass transfer problems and epidemic propagation models, reflecting its flexibility and universality [26], [27].

The TAM was applied in a study that included a comparison with three semi-analytical methods: VIM, DTM, and ADM. The TAM achieved the highest convergence speed and best agreement compared to analytical solutions [28]. The TAM also succeeded in solving differential algebraic equations with higher accuracy and better convergence speed compared to other methods [25]. The TAM was applied to carbon dioxide (CO_2) mixing problems and complex chemical reactions, achieving fast convergence and high accuracy [26]. Additionally, the TAM achieved remarkable efficiency in solving linear and nonlinear Fokker-Planck equations in one, two, and three dimensions [29]. The TAM has demonstrated its effectiveness in solving large-dimensional problems with remarkable success in applied sciences [30]. In addition, the fractional version of the TAM was used to solve fractional Fisher equations with significant improvement in accuracy and stability [31].

In this paper, we will study the TAM for solving three different models of the generalized KdV5 equation. Although much research has been conducted to solve this equation, to the author's knowledge, no studies have been conducted using this method to date. This method is a semi-analytical numerical method that yields approximate solutions in the form of polynomials. This approximate solution sometimes leads to an exact solution, depending on the formula of the equation to be solved and the initial condition chosen for this purpose. The polynomials in the iterations generated by the TAM exhibit excellent convergence properties, resulting in high accuracy. The generalized KdV5 equation inherently contains higher-order derivatives, which is why this attribute makes the technique useful. The method also exhibits high numerical stability, an important feature when dealing with nonlinear equations. This stability ensures reliable results even in cases of high-order approximations.

The following is how this research is structured: The basic idea of TAM is described in Section 2, and two techniques for error analysis and measurement are covered in Section 3. The approximate solutions of three distinct models obtained from the generalized KdV5 equation are examined in Section 4. The approximate solutions' graphs and numerical results are shown. It is determined that the answers derived from the TAM are near and extremely accurate after comparing them with the precise solutions utilizing error measuring techniques. The study's most significant findings are covered in Section 5.

2 BASIC IDEA OF TAM

The principle of the TAM can be explained as follows [32]. Let us have the general form of the following nonlinear PDE:

$$\mathcal{L}\left(\mathcal{U}(x,t)\right) + \mathcal{N}\left(\mathcal{U}(x,t)\right) + \mathcal{F}(x,t) = 0, \qquad (2)$$

With boundary-conditions:

$$\mathcal{B}\left(\mathcal{U}, \frac{\partial \mathcal{U}}{\partial x}\right) = 0, \qquad (3)$$

Let x and t be the independent variables, where $\mathcal{U}(x,t)$ denotes unknown function and $\mathcal{F}(x,t)$ is defined function. The operator $\mathcal{L}$ represents linear part of the DE, $\mathcal{N}$ corresponds to the nonlinear part, and $\mathcal{B}$ denotes the boundary operator. A fundamental assumption in this method is that $\mathcal{L}$ captures linear behavior of equation, although certain linear terms may be incorporated into $\mathcal{N}$ if necessary to facilitate the analysis. The method proceeds through a series of well-defined steps, beginning with the assumption that $\mathcal{U}_0$ serves as an initial approximation of the exact solution $\mathcal{U}$, obtained by solving the following initial problem:

$$\mathcal{L}\left(\mathcal{U}_0\right)+\mathcal{F}=0, \qquad \mathcal{B}\left(\mathcal{U}_0, \frac{\partial \mathcal{U}_0}{\partial x}\right)=0. \tag{4}$$

To get the next iteration $\mathcal{U}_1$ of solution $\mathcal{U}$, we solve following equation:

$$\begin{gathered}\mathcal{L}\left(\mathcal{U}_1\right)+\mathcal{N}\left(\mathcal{U}_0\right)+\mathcal{F}=0, \\ \mathcal{B}\left(\mathcal{U}_1, \frac{\partial \mathcal{U}_1}{\partial x}\right)=0.\end{gathered} \tag{5}$$

Following the procedure outlined above, the TAM provides a framework for expressing the solution of general nonlinear (2) in following iterative form:

$$\begin{gathered}\mathcal{L}\left(\mathcal{U}_{n+1}\right)+\mathcal{N}\left(\mathcal{U}_n\right)+\mathcal{F}=0, \\ \mathcal{B}\left(\mathcal{U}_{n+1}, \frac{\partial \mathcal{U}_{n+1}}{\partial x}\right)=0.\end{gathered} \tag{6}$$

Now, it can be seen that every solution of $\mathcal{U}_n(x,t)$ is a solution of (2). Therefore, approximate bounds estimate provides greater accuracy, and has the advantage that each iteration is better than previous one, and as number of iterations increases, the solution converges to exact solution of the equation. Thus, a closed-form solution to (2) can be obtained by:

$$\mathcal{U}_{Exact}=\lim_{n \rightarrow \infty} \mathcal{U}_n. \tag{7}$$

In this research, which included a numerical study of three different models of the generalized KdV5 equation using the TAM, approximate solutions were obtained, and numerical results were studied after only three iterations, because higher iterations are difficult to write down and extract their numerical results.

3 ERROR MEASUREMENT

If we have an exact solution $q(x,t)$ and an approximate numerical solution $Q(x,t)$, two solutions to a PDE that depends on continuous real variables x and t, and is defined in the spatial domain $[x_{min}, x_{max}]$ and the time domain $[t_{min}, t_{max}]$. Now, we can measure the error of the numerical method using several methods, including [33]-[36].

3.1 MAE Method

The following formula may be used to determine the MAE of the numerical solution relative to the exact solution:

$$\mathrm{MAE}=\frac{1}{|\Omega|} \int_{t_{min}}^{t_{max}} \int_{x_{min}}^{x_{max}}|Q(x,t)-q(x,t)|\, dx\, dt. \tag{8}$$

3.2 RMSE Method

The RMSE may be computed in this formula:

$$\mathrm{RMSE}=\sqrt{\frac{1}{|\Omega|} \int_{t_{min}}^{t_{max}} \int_{x_{min}}^{x_{max}}|Q(x,t)-q(x,t)|^2\, dx\, dt}\,. \tag{9}$$

It is worth noting that $|\Omega|$ in (8) and (9) represents the total area of both the spatial and temporal domains of the PDE, which can be easily calculated by multiplying the length of the spatial interval by the length of the temporal interval as follows:

$$|\Omega|=\left(x_{max}-x_{min}\right)\left(t_{max}-t_{min}\right). \tag{10}$$

4 NUMERICAL APPLICATIONS

In this section, three different fifth-order models of the generalized KdV5 equation are solved using the TAM.

4.1 Fifth Order Sawada-Kotera (SK5) Equation

By generalized KdV5 (1) and after choosing non-zero real values for the random coefficients $a=45, b=c=15$, the SK5 equation can be written as [37]:

$$\mathcal{U}_t+45 \mathcal{U}^2 \mathcal{U}_x+15 \mathcal{U}_x \mathcal{U}_{2x}+15 \mathcal{U} \mathcal{U}_{3x}+\mathcal{U}_{5x}=0, \tag{11}$$

Here, x and t denote the independent variables, $\mathcal{U}=\mathcal{U}(x,t)$ represents the dependent function. The exact solitary wavelet solution of (11) is given:

$$\mathcal{U}_{Exact}(x,t)=4k^2 \operatorname{sech}^2\left(kx-16k^5 t\right)-\frac{4k^2}{3}, \tag{12}$$

Where k is a non-zero random constant, we substitute zero for t in solution (12) to obtain the following initial condition:

$$\mathcal{U}(x,0) = 4k^2\,\mathrm{sech}^2(kx) - \frac{4k^2}{3}. \tag{13}$$

Equation 11 can be rewritten as:

$$\mathcal{U}_t = -45\mathcal{U}^2\mathcal{U}_x - 15\mathcal{U}_x\mathcal{U}_{2x} - 15\mathcal{U}\mathcal{U}_{3x} - \mathcal{U}_{5x}. \tag{14}$$

Following the analysis conducted using the TAM, the coefficients in (14) can be determined:

$$\begin{gathered}\mathcal{L}(\mathcal{U}) = \mathcal{U}_t, \qquad \mathcal{F}(x,t) = 0,\\ \mathcal{N}(\mathcal{U}) = -45\mathcal{U}^2\mathcal{U}_x - 15\mathcal{U}_x\mathcal{U}_{2x} - 15\mathcal{U}\mathcal{U}_{3x}\\ - \mathcal{U}_{5x}.\end{gathered} \tag{15}$$

We start by assuming $\mathcal{U}_0$ is an initial approximation of $\mathcal{U}$, obtained by solving following initial equation:

$$\begin{gathered}\mathcal{L}_t(\mathcal{U}_0) = 0, \qquad \mathcal{L}_t = \frac{\partial}{\partial x}\\ \mathcal{U}_0(x,0) = 4k^2\,\mathrm{sech}^2(kx) - \frac{4k^2}{3},\end{gathered} \tag{16}$$

After solving (16), we obtain the following initial approximation:

$$\mathcal{U}_0(x,t) = 4k^2\,\mathrm{sech}^2(kx) - \frac{4k^2}{3}. \tag{17}$$

Also, to get the next repetition, you must solve the following equation:

$$\begin{gathered}\mathcal{L}_t(\mathcal{U}_1) + \mathcal{N}(\mathcal{U}_0) = 0,\\ \mathcal{U}_1(x,0) = 4k^2\,\mathrm{sech}^2(kx) - \frac{4k^2}{3}.\end{gathered} \tag{18}$$

By integrating both sides of (18) over the interval from 0 to t, we obtain the first iteration as follows:

$$\begin{aligned}\mathcal{U}_1(x,t) = 4k^2\,\mathrm{sech}^2(kx) - \frac{4k^2}{3}\\ + \frac{512k^7t\,(e^{4kx} - e^{2kx})}{(e^{2kx}+1)^3}.\end{aligned} \tag{19}$$

Solving the following problem will yield the second iteration:

$$\begin{gathered}\mathcal{L}_t(\mathcal{U}_2) + \mathcal{N}(\mathcal{U}_1) = 0,\\ \mathcal{U}_2(x,0) = 4k^2\,\mathrm{sech}^2(kx) - \frac{4k^2}{3}.\end{gathered} \tag{20}$$

Similarly, by integrating of (20), the second iteration is obtained as follows:

$$\begin{aligned}&\mathcal{U}_2(x,t)\\ &= 4k^2\,\mathrm{sech}^2(kx) - \frac{4k^2}{3}\\ &+ 128k^7t\sinh(kx)\,\mathrm{sech}^3(kx)\\ &+ 2048k^{12}t^2\,\mathrm{sech}^2(kx)\\ &- 3072k^{12}t^2\,\mathrm{sech}^4(kx) + O(t^3).\end{aligned} \tag{21}$$

Further iterations may be gained by solving the following problem using the same procedure:

$$\begin{gathered}\mathcal{L}_t(\mathcal{U}_3) + \mathcal{N}(\mathcal{U}_2) = 0,\\ \mathcal{U}_3(x,0) = 4k^2\,\mathrm{sech}^2(kx) - \frac{4k^2}{3}.\end{gathered} \tag{22}$$

After integrating both sides of (22) over the interval from 0 to t, the third iteration is obtained as follows:

$$\begin{aligned}&\mathcal{U}_3(x,t)\\ &= 4k^2\,\mathrm{sech}^2(kx) - \frac{4k^2}{3}\\ &+ 128k^7t\sinh(kx)\,\mathrm{sech}^3(kx)\\ &+ 2048k^{12}t^2\,\mathrm{sech}^2(kx)\\ &- 3072k^{12}t^2\,\mathrm{sech}^4(kx)\\ &+ \frac{65536}{3}k^{17}t^3\sinh(kx)\,\mathrm{sech}^3(kx)\\ &- 65536k^{17}t^3\sinh(kx)\,\mathrm{sech}^5(kx) + O(t^4).\end{aligned} \tag{23}$$

$$\vdots$$

Furthermore, by continuing in this way, higher iterations can also be calculated, but for brevity, they are not mentioned here, since each iteration represents an approximate solution to (11). Thus, we obtain a series of the form:

$$\begin{aligned}&\mathcal{U}_n(x,t)\\ &= 4k^2\,\mathrm{sech}^2(kx) - \frac{4k^2}{3}\\ &+ 128k^7t\sinh(kx)\,\mathrm{sech}^3(kx)\\ &+ 2048k^{12}t^2\,\mathrm{sech}^2(kx)\\ &- 3072k^{12}t^2\,\mathrm{sech}^4(kx)\\ &+ \frac{65536}{3}k^{17}t^3\sinh(kx)\,\mathrm{sech}^3(kx)\\ &- 65536k^{17}t^3\sinh(kx)\,\mathrm{sech}^5(kx) + \cdots\\ &+ O(t^{n+1}), n \geq 0,\end{aligned} \tag{24}$$

Here, the symbol $O(t^{n+1})$ indicates that the omitted noise terms contain terms of degree $(n+1)$ or more for the time variable t. According to (7), and by using the Taylor Series, we obtain a closed form of exact solution:

$$\begin{gathered}\mathcal{U}_{Exact}(x,t) = \lim_{n\to\infty} \mathcal{U}_n(x,t) \\ = 4k^2 \operatorname{sech}^2(kx - 16k^5 t) - \frac{4k^2}{3}.\end{gathered} \tag{25}$$

A numerical study was conducted on the approximate solutions of the SK5 equation using the TAM after giving different non-zero values for the constant k. Table 1 shows a comparison of the numerical results obtained through MAE and RMSE to measure the error resulting from the approximate solutions at the spatial interval $-20 \le x \le 20$ and at the two-time intervals $t \in [0,1]$ and $t \in [0,5]$. Note that the value of n in the table represents the number of iterations resulting from the method used. The estimated errors in the table show that the approximate solutions obtained using the TAM are effective and have high accuracy at the specified time intervals. It also shows that the error decreases as the value of k decreases and number of iterations increases. In addition, the shapes of exact solution of SK5 equation and the approximate solutions resulting from using the TAM were studied after giving a value for the constant $k = 0.3$. Figure 1 shows the approximate solutions resulting from the used numerical method and compares them with the exact solution at the interval $x \in [-20, 20]$ and at time $t = 5$. Also, the errors arising from the approximate solutions were explained using the absolute error $|\mathcal{U}_n(x,t) - \mathcal{U}_{Exact}(x,t)|$. The findings show that the approximate solution graphs closely resemble the precise solution, demonstrating the precision and effectiveness of the employed numerical technique.

4.2 Fifth Order Kaup-Kupershmidt (KK5) Equation

After choosing non-zero real values for the coefficients $a = 20, b = 25, c = 10$ in the generalized KdV5 (1), the KK5 equation can be written as follows [37]:

$$\mathcal{U}_t + 20\mathcal{U}^2\mathcal{U}_x + 25\mathcal{U}_x\mathcal{U}_{2x} + 10\mathcal{U}\mathcal{U}_{3x} + \mathcal{U}_{5x} = 0, \tag{26}$$

Here, x and t are independent variables, $\mathcal{U} = \mathcal{U}(x,t)$ represents dependent function. The exact solution of (26) is:

$$\mathcal{U}_{Exact}(x,t) = 12k^2 \operatorname{sech}^2(kx - 176k^5 t) - 4k^2, \tag{27}$$

Where k is a non-zero random constant, we substitute zero for t in solution (27) to obtain the initial condition:

$$\mathcal{U}(x,0) = 12k^2 \operatorname{sech}^2(kx) - 4k^2, \tag{28}$$

Equation (26) can be rewritten as:

$$\mathcal{U}_t = -20\mathcal{U}^2\mathcal{U}_x - 25\mathcal{U}_x\mathcal{U}_{2x} - 10\mathcal{U}\mathcal{U}_{3x} - \mathcal{U}_{5x}. \tag{29}$$

Based on the analysis conducted using the TAM, the coefficients of (29) can be determined as:

$$\begin{gathered}\mathcal{L}(\mathcal{U}) = \mathcal{U}_t, \quad \mathcal{F}(x,t) = 0, \\ \mathcal{N}(\mathcal{U}) = -20\mathcal{U}^2\mathcal{U}_x - 25\mathcal{U}_x\mathcal{U}_{2x} - 10\mathcal{U}\mathcal{U}_{3x} \\ - \mathcal{U}_{5x},\end{gathered} \tag{30}$$

We start by assuming $\mathcal{U}_0$ is an initial approximation of solution $\mathcal{U}$ by solving initial equation:

$$\begin{gathered}\mathcal{L}_t(\mathcal{U}_0) = 0, \quad \mathcal{L}_t = \frac{\partial}{\partial x} \\ \mathcal{U}_0(x,0) = 12k^2 \operatorname{sech}^2(kx) - 4k^2.\end{gathered} \tag{31}$$

After solving (31), we get the initial approximation:

$$\mathcal{U}_0(x,t) = 12k^2 \operatorname{sech}^2(kx) - 4k^2 . \tag{32}$$

Also, to get the following iteration, you must solve the equation:

$$\begin{gathered}\mathcal{L}_t(\mathcal{U}_1) + \mathcal{N}(\mathcal{U}_0) = 0, \\ \mathcal{U}_1(x,0) = 12k^2 \operatorname{sech}^2(kx) - 4k^2 .\end{gathered} \tag{33}$$

By integrating of (33), the first iteration is obtained as follows:

$$\begin{aligned}\mathcal{U}_1(x,t) = {} & 12k^2 \operatorname{sech}^2(kx) - 4k^2 \\ & + \frac{16896k^7 t\,(e^{4kx} - e^{2kx})}{(e^{2kx}+1)^3}.\end{aligned} \tag{34}$$

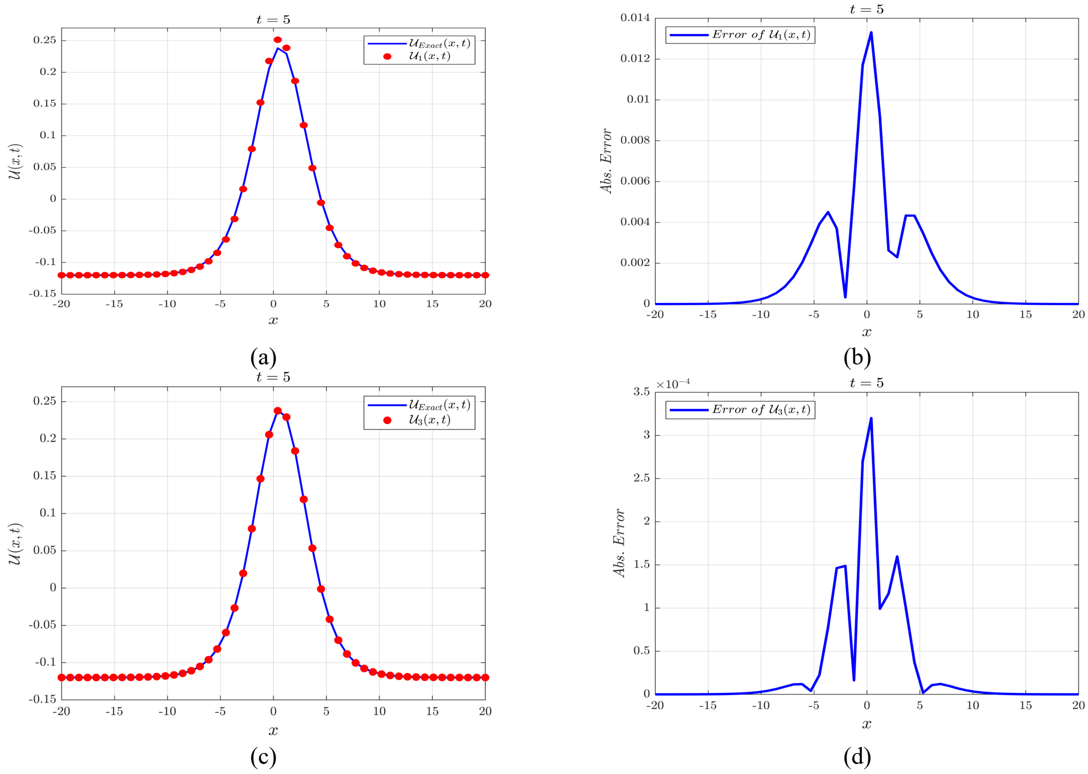


Figure 1: Comparison of the exact and approximate solutions of the SK5 equation and their corresponding approximation errors: a) first iteration, b) error after the first iteration, c) third iteration, and d) error after the third iteration.

Table 1: Numerical comparison of methods for calculating the error arising from the approximate solutions $\boldsymbol{\mathcal{U}_n(x,t)}$ for the SK5 equation.

$\boldsymbol{t}$	n	$k = 0.3$		$k = 0.1$		$k = 0.07$	
		MAE	RMSE	MAE	RMSE	MAE	RMSE
1	1	2.3270e-05	6.1314e-05	1.1980e-10	1.9943e-10	1.9301e-12	3.1683e-12
	2	4.8969e-07	1.5788e-06	1.0665e-14	2.1172e-14	3.1288e-17	5.7830e-17
	3	1.0729e-08	4.1685e-08	4.0087e-18	5.9249e-18	2.2354e-18	3.0037e-18
5	1	0.00058036	0.00152670	2.9940e-09	4.9858e-09	4.8920e-11	7.9207e-11
	2	6.1020e-05	0.00019770	1.3178e-12	2.6463e-12	3.9137e-15	7.2929e-15
	3	6.6810e-06	2.5536e-05	5.8467e-16	1.4003e-15	2.4755e-18	3.4765e-18

The following problem must be solved in order to receive the second iteration:

$$\begin{gathered} \mathcal{L}_t\,(\mathcal{U}_2) + \mathcal{N}(\mathcal{U}_1) = 0, \\ \mathcal{U}_2(x,0) = 12k^2\operatorname{sech}^2(kx) - 4k^2\,. \end{gathered} \tag{35}$$

Similarly, by integrating from 0 to t of (35) yields the following second iteration:

$$\begin{aligned} &\mathcal{U}_2(x,t) \\ &= 12k^2\operatorname{sech}^2(kx) - 4k^2 \\ &+ 4224k^7 t\sinh(kx)\operatorname{sech}^3(kx) \\ &+ 743424k^{12}t^2\operatorname{sech}^2(kx) \\ &- 1115136k^{12}t^2\operatorname{sech}^4(kx) + O(t^3)\,. \end{aligned} \tag{36}$$

The subsequent iteration can also be obtained by solving the following problem:

$$\begin{gathered} \mathcal{L}_t\,(\mathcal{U}_3) + \mathcal{N}(\mathcal{U}_2) = 0, \\ \mathcal{U}_3(x,0) = 12k^2\operatorname{sech}^2(kx) - 4k^2\,. \end{gathered} \tag{37}$$

After integrating both sides of (37) over the interval from 0 to t, the third iteration is obtained as follows:

$$\begin{aligned}
&\mathcal{U}_3(x,t)\\
&= 12k^2\,\mathrm{sech}^2(kx) - 4k^2\\
&+ 4224k^7 t\,\sinh(kx)\,\mathrm{sech}^3(kx)\\
&+ 743424k^{12}t^2\,\mathrm{sech}^2(kx)\\
&- 1115136k^{12}t^2\,\mathrm{sech}^4(kx)\\
&+ 87228416k^{17}t^3\sinh(kx)\,\mathrm{sech}^3(kx)\\
&- 261685248k^{17}t^3\sinh(kx)\,\mathrm{sech}^5(kx)\\
&+ O(t^4)\,.
\end{aligned} \tag{38}$$

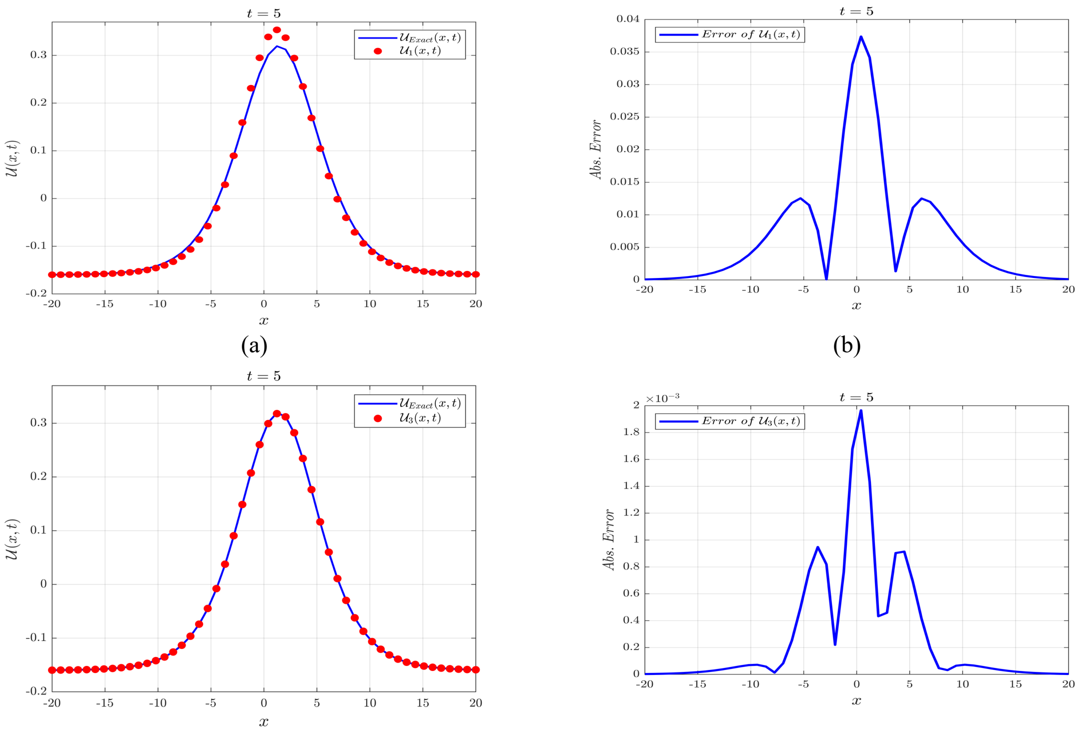
$$\vdots$$

Continuing in this way, we can also calculate higher iterations, which for brevity's sake are not mentioned here, where each iteration represents an approximate solution to (26). Thus, we obtain the series:

$$\begin{aligned}
&\mathcal{U}_n(x,t)\\
&= 12k^2\,\mathrm{sech}^2(kx) - 4k^2\\
&+ 4224k^7 t\,\sinh(kx)\,\mathrm{sech}^3(kx)\\
&+ 743424k^{12}t^2\,\mathrm{sech}^2(kx)\\
&- 1115136k^{12}t^2\,\mathrm{sech}^4(kx)\\
&+ 87228416k^{17}t^3\sinh(kx)\,\mathrm{sech}^3(kx)\\
&- 261685248k^{17}t^3\sinh(kx)\,\mathrm{sech}^5(kx) + \cdots\\
&+ O(t^{n+1}), n \geq 0,
\end{aligned} \tag{39}$$

Where $O(t^{n+1})$ indicates that the omitted noise terms contain terms of degree $(n+1)$ or more for the time variable t. According to (7), and by using the Taylor series, a closed form of exact solution is obtained:

$$\begin{aligned}
\mathcal{U}_{Exact}(x,t) &= \lim_{n\to\infty} \mathcal{U}_n(x,t)\\
&= 12k^2\,\mathrm{sech}^2(kx - 176k^5 t) - 4k^2.
\end{aligned} \tag{40}$$



(a) (b) (c) (d)

Figure 2: Comparison of the exact and approximate solutions of the KK5 equation and their corresponding approximation errors: a) first iteration, b) error after the first iteration, c) third iteration, and d) error after the third iteration.

Table 2: Numerical comparison of methods for calculating the error arising from the approximate solutions $\mathcal{U}_n(x,t)$ for the KK5 equation.

t	n	$k=0.2$		$k=0.1$		$k=0.05$	
		MAE	RMSE	MAE	RMSE	MAE	RMSE
1	1	9.7480e-05	0.00021006	4.3472e-08	7.2395e-08	1.3628e-11	2.2538e-11
	2	2.9718e-06	7.8873e-06	4.2032e-11	8.4533e-11	5.4562e-16	9.0307e-16
	3	9.4353e-08	3.2413e-07	4.1108e-14	9.8378e-14	4.0971e-18	5.4061e-18
5	1	0.0024246	0.0052074	1.0868e-06	1.8100e-06	3.4045e-10	5.6345e-10
	2	0.00036903	0.00097595	5.2484e-09	1.0565e-08	6.8153e-14	1.1283e-13
	3	5.8521e-05	0.00018251	2.6818e-11	6.1472e-11	1.1686e-17	2.0129e-17

A numerical study was conducted to find approximate solutions to the KK5 equation using the TAM after choosing different non-zero values for the constant k. Table 2 shows a numerical comparison for results of approximate solutions by using MAE and RMSE to measure the resulting error within the spatial domain $-20 \leq x \leq 20$ for the two time periods $t \in [0,1]$ and $t \in [0,5]$. The value n in the table represents the number of iterations found by the numerical method. The results show that the approximate solutions obtained using the TAM are characterized by a high degree of accuracy, as the error decreases with the decrease in the value of k and increase in number of iterations, which indicates the efficiency of the method within the specified time periods. Moreover, the form of the analytical solution of the KK5 equation was studied and compared with the approximate solutions obtained from the TAM at the value $k=0.2$. Figure 2 shows the approximate solutions within the period $x \in [-20,20]$ at $t=5$. In addition, the form of the resulting error was explained using the absolute error measure $|\mathcal{U}_n(x,t) - \mathcal{U}_{Exact}(x,t)|$. The results indicate that the approximate solutions closely match the analytical solution, thereby confirming the accuracy and effectiveness of the employed numerical method.

4.3 Fifth Order Lax (L5) Equation

In the generalized KdV5 (1) and after choosing real non-zero values for the coefficients $a=30, b=20, c=10$, the L5 equation can be written as follows [37]:

$$\mathcal{U}_t + 30\mathcal{U}^2\mathcal{U}_x + 20\mathcal{U}_x\mathcal{U}_{2x} + 10\mathcal{U}\mathcal{U}_{3x} + \mathcal{U}_{5x} = 0, \quad (41)$$

Here, x and t are independent variables, $\mathcal{U} = \mathcal{U}(x,t)$ denotes dependent function. The exact solution of (41) is:

$$\mathcal{U}_{Exact}(x,t) = 6k^2 \operatorname{sech}^2(kx - 56k^5 t) - 2k^2, \quad (42)$$

Where k is a non-zero constant, we substitute zero for t in solution (42) to obtain the following initial condition:

$$\mathcal{U}(x,0) = 6k^2 \operatorname{sech}^2(kx) - 2k^2, \quad (43)$$

Equation (41) can be rewritten as:

$$\mathcal{U}_t = -30\mathcal{U}^2\mathcal{U}_x - 20\mathcal{U}_x\mathcal{U}_{2x} - 10\mathcal{U}\mathcal{U}_{3x} - \mathcal{U}_{5x}. \quad (44)$$

After the analysis presented in the TAM, the factors of (44) as follows:

$$\mathcal{L}(\mathcal{U}) = \mathcal{U}_t, \quad \mathcal{F}(x,t) = 0,$$
$$\mathcal{N}(\mathcal{U}) = -30\mathcal{U}^2\mathcal{U}_x - 20\mathcal{U}_x\mathcal{U}_{2x} - 10\mathcal{U}\mathcal{U}_{3x} - \mathcal{U}_{5x}, \quad (45)$$

We begin by assuming $\mathcal{U}_0$ is initial approximation of $\mathcal{U}$, obtained by solving the following initial equation:

$$\mathcal{L}_t(\mathcal{U}_0) = 0, \quad \mathcal{L}_t = \frac{\partial}{\partial x}$$
$$\mathcal{U}_0(x,0) = 6k^2 \operatorname{sech}^2(kx) - 2k^2. \quad (46)$$

After solving (46), we obtain the following initial approximation:

$$\mathcal{U}_0(x,t) = 6k^2 \operatorname{sech}^2(kx) - 2k^2. \quad (47)$$

Additionally, you need to solve the following equation to obtain the next iteration:

$$\mathcal{L}_t(\mathcal{U}_1) + \mathcal{N}(\mathcal{U}_0) = 0,$$
$$\mathcal{U}_1(x,0) = 6k^2 \operatorname{sech}^2(kx) - 2k^2. \quad (48)$$

By integrating of (48) over the interval $[0,t]$, we obtain the first iteration:

$$\mathcal{U}_1(x,t) = 6k^2 \operatorname{sech}^2(kx) - 2k^2 + \frac{2688k^7 t\,(e^{4kx} - e^{2kx})}{(e^{2kx}+1)^3}. \quad (49)$$

To obtain the second iteration, the following problem must be solved:

$$\mathcal{L}_t(\mathcal{U}_2) + \mathcal{N}(\mathcal{U}_1) = 0,$$
$$\mathcal{U}_2(x,0) = 6k^2 \operatorname{sech}^2(kx) - 2k^2. \quad (50)$$

Furthermore, by integrating of (50), the second iteration is obtained:

$$\begin{aligned}\mathcal{U}_2(x,t) &\\ = 6k^2\operatorname{sech}^2(kx) - 2k^2 &\\ + 672k^7 t\sinh(kx)\operatorname{sech}^3(kx) &\\ + 37632k^{12}t^2\operatorname{sech}^2(kx) &\\ - 56448k^{12}t^2\operatorname{sech}^4(kx) + O(t^3)\,. &\end{aligned} \tag{51}$$

The next iteration follows from solving the problem below:

$$\begin{gathered}\mathcal{L}_t\,(\mathcal{U}_3) + \mathcal{N}(\mathcal{U}_2) = 0,\\ \mathcal{U}_3(x,0) = 6k^2\operatorname{sech}^2(kx) - 2k^2.\end{gathered} \tag{52}$$

Now, by integrating of (52) over the same interval, the third iteration is obtained:

$$\begin{aligned}\mathcal{U}_3(x,t) &\\ = 6k^2\operatorname{sech}^2(kx) - 2k^2 &\\ + 672k^7 t\sinh(kx)\operatorname{sech}^3(kx) &\\ + 37632k^{12}t^2\operatorname{sech}^2(kx) &\\ - 56448k^{12}t^2\operatorname{sech}^4(kx) &\\ + 1404928k^{17}t^3\sinh(kx)\operatorname{sech}^3(kx) &\\ - 4214784k^{17}t^3\sinh(kx)\operatorname{sech}^5(kx) + O(t^4)\,. &\end{aligned} \tag{53}$$

$$\vdots$$

By continuing in this way, we can also calculate higher iterations, which are difficult to write down and are not mentioned here, since each iteration represents an approximate solution to (41). Thus, we obtain the following series (54).

$$\begin{aligned}\mathcal{U}_n(x,t) &\\ = 6k^2\operatorname{sech}^2(kx) - 2k^2 &\\ + 672k^7 t\sinh(kx)\operatorname{sech}^3(kx) &\\ + 37632k^{12}t^2\operatorname{sech}^2(kx) &\\ - 56448k^{12}t^2\operatorname{sech}^4(kx) &\\ + 1404928k^{17}t^3\sinh(kx)\operatorname{sech}^3(kx) &\\ - 4214784k^{17}t^3\sinh(kx)\operatorname{sech}^5(kx) + \cdots &\\ + O(t^{n+1}), n \geq 0, &\end{aligned} \tag{54}$$

Where $O(t^{n+1})$ indicates that the omitted noise terms contain terms of degree $(n+1)$ or greater for the time variable t. By using the Taylor series and taking the limit of (54), we obtain the exact solution:

$$\begin{aligned}\mathcal{U}_{Exact}(x,t) &= \lim_{n\to\infty}\mathcal{U}_n(x,t)\\ &= 6k^2\operatorname{sech}^2(kx - 56k^5 t) - 2k^2.\end{aligned} \tag{55}$$

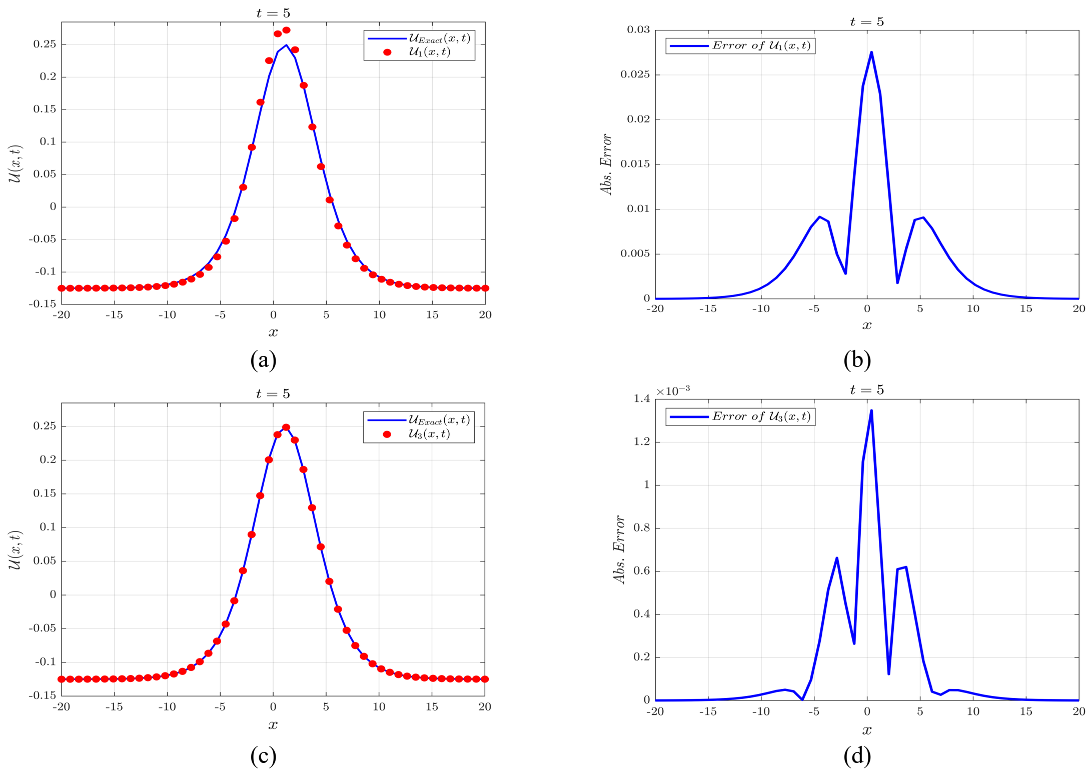


Figure 3: Comparison of the exact and approximate solutions of the L5 equation and their corresponding approximation errors: a) first iteration, b) error after the first iteration, c) third iteration, and d) error after the third iteration.

Table 3: Numerical comparison of methods for calculating the error arising from the approximate solutions $\mathcal{U}_n(x,t)$ for the L5 equation.

t	n	$k = 0.25$		$k = 0.1$		$k = 0.075$	
		MAE	RMSE	MAE	RMSE	MAE	RMSE
1	1	5.7531e-05	0.00013840	2.2009e-09	3.6646e-09	8.0530e-11	1.3017e-10
	2	1.7029e-06	5.0376e-06	6.8578e-13	1.3616e-12	6.1556e-15	1.1725e-14
	3	5.2480e-08	1.9634e-07	2.1083e-16	5.0398e-16	4.3443e-18	5.9057e-18
5	1	0.00143140	0.00343260	5.5013e-08	9.1616e-08	2.0078e-09	3.2543e-09
	2	0.00021155	0.00062474	8.4542e-11	1.7018e-10	7.6941e-13	1.4653e-12
	3	3.2563e-05	0.00011345	1.3171e-13	3.1509e-13	3.3588e-16	6.5941e-16

A numerical study was applied to find approximate solutions to the L5 equation using the TAM after giving different non-zero values for the constant k. Table 3 shows a comparison of numerical results obtained based on the MAE and RMSE criteria to evaluate the amount of error resulting from the approximate solutions within spatial domain $-20 \leq x \leq 20$ and for the two time periods $t \in [0,1]$ and $t \in [0,5]$. It is worth noting that value n in table represents the number of iterations resulting from applying the numerical method. The values listed in the table show that the approximate solutions calculated using the TAM are characterized by a high degree of accuracy and efficiency within the specified time periods. It was also shown that the amount of error decreases with the decrease in the value of k and the increase in number of iterations. In addition, the analytical solution form of L5 equation was analyzed and studied, and compared with the approximate solutions resulting from applying the TAM at a value of $k = 0.25$. Figure 3 shows the agreement of approximate solutions resulting from the numerical method with the analytical solution at the interval $x \in [-20, 20]$ and for time $t = 5$. The differences between the two solutions were also clarified by calculating the absolute of error $|\mathcal{U}_n(x,t) - \mathcal{U}_{Exact}(x,t)|$. The outcomes illustrate the precision and efficacy of the used numerical approach by showing that the graphs of the approximate solutions closely match the analytical solution.

5 CONCLUSIONS

In this work, several conclusions were reached. The TAM is a semi-analytical method that yields approximate solutions with an acceptable margin of error, and sometimes an exact solution, depending on the formula of the equation to be solved and the initial condition chosen for this purpose. An exact solution is obtained if the first terms of the TAM can be compared to the well-known series associated with the Taylor series. The TAM relies on the initial condition formula to obtain the initial problem and subsequent iterations of the approximate solutions. Extending the time period can increase the errors of the numerical method, due to the fact that approximate solutions obtained using TAM are polynomials in time variable. Therefore, to obtain good approximate solutions with high accuracy and very low errors, we need to perform more iterations. The TAM can be further developed in the future by adopting different initial conditions to obtain new approximate solutions and applying it to other types of nonlinear partial differential equations that are difficult to solve analytically. Its results can also be compared with those of other numerical methods to evaluate its accuracy and efficiency in finding approximate solutions.

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