Graph Representations of Finite Groups via Conjugacy Relations and Their Induced Topologies

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Keywords: Group, Conjugacy Relation, Graph, Factorial Relation, Topological Space.

Abstract: In this work, we explore the construction of topological spaces related to finite sets through graph-theoretical representations derived from conjugation relations. This study establishes a conceptual link between three fundamental areas of mathematics: set theory, graph theory, and topology. We introduce a framework, referred to as the "bridge between groups, graphs, and topology," in which a finite set is represented by a graph whose vertices correspond to the elements of a group, and whose edges are defined by conjugation relations. Using the adjacency matrix of the resulting graph, we derive an adjacency rule that constructs a topology on the set of vertices. This methodology facilitates the exploration of various topological properties, including continuity, open and closed subsets, and possible covering spaces. Furthermore, we present algorithmic procedures applicable within the *GAP* (Group, Algorithm and Programming) framework for the systematic and accurate construction and analysis of these graphs. The proposed approach is illustrated through several illustrative examples involving finite groups, particularly Dihedral groups and Klein four-group, demonstrating how distinct topologies arise naturally and reflect the underlying algebraic structure of the groups. These results contribute to strengthening the interaction between algebra and topology through graphical techniques, providing a unified framework with potential applications to broader classes of finite groups and graphs structure analysis. Furthermore, we introduce a new binary relation on finite groups, called the factorial relation. This relation is precisely defined, and its associated graph is constructed and studied. We prove that the topology induced by the factorial relation is always a discrete topology on the set of elements of the underlying group. This result confirms a strong correspondence between the algebraic properties of the factorial relation and the resulting topological structure.

1 INTRODUCTION

Group theory is one of the cornerstones of mathematics, providing a framework for the study of symmetric structures in algebra, geometry, and physics [1]. Finite groups, in particular, are widely used in the study of molecular symmetries, cryptography, and number theory, and also play a pivotal role in the classification of simple finite groups [2]. The discovery of the relationship between group theory and graphs was made decades ago. Some specialists represent finite groups graphically, while others do the opposite. The concept of group graphs was introduced in [3]. The elements of a group represent vertices, and a relationship is defined to form the edges that connect them. Graphs are not just structural structures; they can be considered topological spaces when provided with appropriate structures. This has led to studies in topological graph theory that link algebraic topology and graph theory [4], [5].

Recently, some researchers have been able to construct a topological space on the graph and subgraph in several ways. In 2013,SNF Al-Khafajii [6] Generated topology from graph and subgraph, and in 2018, KA Abdu [7] constructed and applied a topology on binary graphs by associating two topologies with the set of edges of any directed graph, called conjugate and non-conjugate edge topologies.While the binary relationship between (groups ⇔ graphs) and (graphs ⇔ topology) is well-known, the direct triple connection (groups ⇔ graphs ⇔ topology) via the conjugate relationship has not received its due in previous studies [8].

In this work, we introduce a framework that goes beyond previous studies by connecting group theory, graph theory, and topology through group-theoretic relations. The methodology we adopt does more than represent groups graphically; it systematically constructs topologies derived from these graphs, offering a novel perspective that reveals deep interactions between algebraic and topological structures.

The main contribution of this study is the development of an algorithm for generating a graph from a finite group using conjugacy relations and then deriving topologies that reflect the underlying group structures and expose new properties. In Section,5 we define a new algebraic relation on finite groups, present its axiomatic formulation, and construct the corresponding graph. We demonstrate that the topology induced by this graph satisfies the axioms of a topological space and exhibits a well-defined structural form, thereby creating a specific class of graph-generated topologies that further illuminate the interplay between algebraic relations and topological properties.

2 PRELIMINARIES AND NOTATIONS

In this section we cover some important definitions and essential basics related to finite group theory, graph theory and topological space.

Definition 1. If the number of elements in a group is finite, then it is called a finite group [9].

Example 1. A cyclic group of order n, denoted by C_n, is a group that consists of n elements and can be generated by a single element g, called a generator, such that:

$$C_n = \langle g \rangle = \{e, g\ \ , g^2, \dots, g^{n-1}\}, \text{ where } g^n = e$$

Example 2. The dihedreal group D_n is the group of symmetric of a regular $n - gon$. It has $2n$ elements and is defined by the presentation [9]-[11]:

$$D_n = \langle r, s \mid r^n = e,\ \ s^2 = e,\ \ srs = r^{-1} \rangle.$$

In particular, if $n = 4$, then D_4 is the symmetry group of the square ($order$ 8). It has the following presentation:

$$D_4 = \langle r, s \mid r^4 = e,\ \ s^2 = e,\ \ srs = r^{-1} \rangle.$$

Every element of D_4 can be uniquely written as either Figure(1):

$$r^k\ \ (0 \le k \le 4)\ \ or\ \ sr^k\ (0 \le k \le 4).$$

Hence,

$$D_4 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}.$$

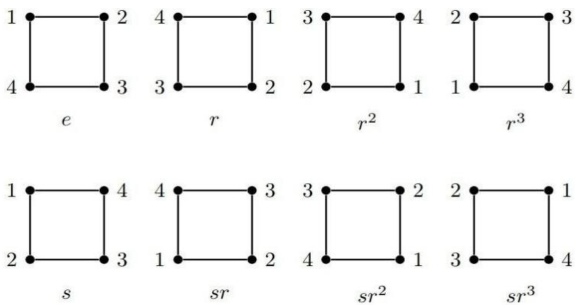

Figure 1: Complete classification of the elements of the finite D_4 symmetry group into rotation r^k and reflection sr^k elements, reflecting the algebraic structure defined by the presentation.

$$\langle r, s \mid r^4 = e,\ \ s^2 = e,\ \ srs = r^{-1} \rangle.$$

Definition 2. Conjugacy Relation on a Group. Let G be a group. The Conjugacy relation is a binary relation $\sim on\ G$ defined as follows:

$$For\ a, b \in G, \qquad a \sim b \iff \exists g \in G.$$
$$such\ that\ \ b = gag^{-1}$$

This relation is called conjugation, and if $a \sim b$, we say that a and b are conjugate in G [9].

Proposition 1. The conjugacy relation is an equivalence relation on G meaning it satisfies[9]:
- Reflexivity: $a \sim b, since\ a = eae^{-1}$.
- Symmetry: $If\ a \sim b, then\ b \sim a$.
- Transitivity: $If\ a \sim b\ and\ b \sim c,\ then\ a \sim c$.

Definition 3. The Conjugacy class of element $a \in G$ is the set [9]:

$$Cl(a) = \{gag^{-1} \mid g \in G\}.$$

Definition 4. Any group of order 4 is called klein four-group if every element of a group are self-inverse [10].

Definition 5. The graph of Γ is represented $\Gamma = (V, E)$,
where:
- V is a nonempty finite set of nodes.
- $E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$ is a set of edges, where each edge is an unordered pair of distinct vertices.

If the edges are instead ordered pairs, i.e. $E \subseteq V \times V$, then Γ is a say a directed graph [12].

Definition 6. Let $\Gamma = (V(G), E(G))$ be a simple graph with $V = \{v_1, v_2, \dots, v_n\}$, the adjacency matrix is defined as follows [13]:

$$M = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots \ddots & \vdots \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix}$$

and

$$a_{ij} = \begin{Bmatrix} 1, & for\ (v_i, v_j) \\ 0, & otherwise \end{Bmatrix}, \qquad with\ a_{ij} = a_{ji}$$

Notation 1.
(i) The i^{th} row of a matrix M denoted by A^i.
(ii) The set of rows A^i for which the entry a_{ij} is equal to one is denoted by $M^j = \{A^i : a_{ij} = 1\}$.

Definition 7. Let $(X, \mathcal{T})$ be a topological space, where $\mathcal{T}$ is a topology on X satisfying the usual axioms [14].

Theorem 1. Assume that X be a set and $\mathcal{B} \subseteq \mathcal{P}(X)$ be a collection of subsets of . Then $\mathcal{B}$ is a topological base on X if and only if:

1) $\bigcup_{B \in \mathcal{B}} B = X, and,$
2) $For\ any\ B_1, B_2 \in \mathcal{B}\ and, \forall\ x \in B_1 \cap B_2$
   $\exists\ B_3 \in \mathcal{B}\ suchthat\ x \in B_3 \subseteq B_1 \cap B_2.$

If these conditions are satisfied, then the topology T generated by $\mathcal{B}$ [14].

3 A TOPOLOGICAL SPACE INDUCED SIMPLE GRAPH

Our main result is: In these sections, we will construct a topological space generated from the graph of a finite group by conjugation relations.

Definition 8. Let $I = \{1, 2, \ldots, n\}$ be an index set for nodes set $V(G)$ of a graph . Each indicator i is meets to a subset of indicators, which can be obtained by the following function [15]:

$$\pi : I \to \mathcal{P}(I)$$
$$i \mapsto \bigoplus M^j.$$

$where\ \oplus\ means\ a\ direct\ sum, e.g.$

$$(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n) \oplus (\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_n) = (\mathfrak{a}_1 + \mathfrak{b}_1, \mathfrak{a}_2 + \mathfrak{b}_2, \ldots, \mathfrak{a}_n + \mathfrak{b}_n)$$

$here\ \mathfrak{a}_i, \mathfrak{b}_i \in \{0,1\}$

Notation 2. J_i^M is the set of all element locations $max\{\pi(i)\}\ in\ \pi(i)$.

Example 3. The following GAP code (provided in Appendix, Listing A.1) prepares the edge list and calculates the graph structure. The output of the computation is shown below:

```
gap> L:= [[1,2], [2,3], [3,4], [4,1],
[3,5], [5,6], [6,7], [7,8], [8,5]];
```

Figure 2 shows the corresponding graph of the calculated results L.

G=

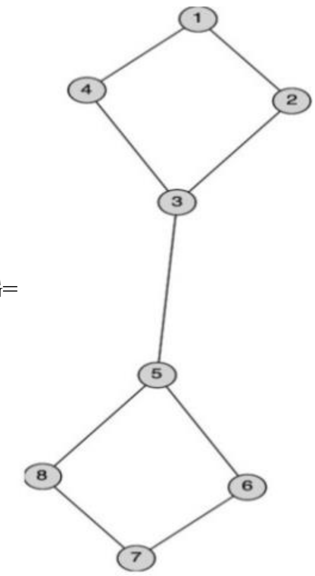

Figure 2: Connected simple graph.

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Then

$$M^1 = \{A^i : a_{i1} = 1\} = \{A^2, A^4\}$$
$$M^2 = \{A^i : a_{i2} = 1\} = \{A^1, A^3\}$$
$$M^3 = \{A^i : a_{i3} = 1\} = \{A^2, A^4, A^5\}$$
$$M^4 = \{A^i : a_{i4} = 1\} = \{A^1, A^3\}$$
$$M^5 = \{A^i : a_{i5} = 1\} = \{A^3, A^6, A^8\}$$
$$M^6 = \{A^i : a_{i6} = 1\} = \{A^5, A^7\}$$
$$M^7 = \{A^i : a_{i7} = 1\} = \{A^6, A^8\}$$
$$M^8 = \{A^i : a_{i8} = 1\} = \{A^5, A^7\}$$

And

$$J_1^M = \{1,3\}, \quad J_2^M = \{2,4\}, \quad J_3^M = \{3\},$$
$$J_4^M = \{2,4\}, \quad J_5^M = \{5\}, \quad J_6^M = \{6,8\},$$
$$J_7^M = \{5,7\}, \quad J_8^M = \{6,8\}.$$

Proposition 2. Let $\Gamma = (V(G), E(G))$ be a connected graph. The following collection of sets $\mathcal{B} = \{J_i^M \mid i \in I\}$ is a base for a topology on $I\ of\ V(G)$.

Example 4. The base for a topology via by the graph given in Example 3 is

$$\mathcal{B} = \{\emptyset, \{3\}, \{5\}, \{1,3\}, \{2,4\}, \{5,7\}, \{6,8\}\}.$$

$$\mathcal{T} = \left\{ \begin{array}{c} \Phi, I, \{3\}, \{5\}, \{1,3\}, \{2,4\}, \{3,5\}, \{5,7\}, \\ \{6,8\}, \{1,3,5\}, \{2,3,4\}, \{2,4,5\}, \{3,5,7\}, \\ \{3,6,8\}, \{5,6,8\}, \{1,2,3,4\}, \{1,3,5,7\}, \\ \{1,3,6,8\}, \{2,3,4,5\}, \{2,4,5,7\}, \\ \{2,4,6,8\}, \{3,5,6,8\}, \{5,6,7,8\}, \\ \{1,2,3,4,5\}, \{1,3,5,6,8\}, \{2,3,4,5,7\}, \\ \{2,3,4,6,8\}, \{2,4,5,6,8\}, \{3,5,6,7,8\} \\ , \{1,2,3,4,6,8\}, \{1,3,5,6,7,8\}, \\ \{2,3,4,5,6,8\}, \{1,2,3,4,5,7\}, \\ \{2,4,5,6,7,8\}, \{1,2,3,4,5,6,8\}, \\ \{2,3,4,5,6,7,8\} \end{array} \right\}.$$

4 GRAPHS OF FINITE GROUP

In this section, we describe the procedure for constructing a topology from a finite group. We begin by representing the group graphically using group-theoretic relations, such as the conjugacy relation, where each element is represented as a node and each relation between two elements as an edge. Once the graph is established, the topological basis is derived, and the corresponding topology is constructed. This approach transforms the theoretical structure of the group into a graphical model and then into a topology, which can be summarized with an illustrative diagram and applied to examples at the end of the section.

Definition 9. Let G be a finite group, the conjugacy graph of G, denoted by $\Gamma(G)$, is defined as:

1) $\Gamma(G) = \big(V(G), E(G)\big)$ *where*,
2) $V(G)$: *set nodes of* $\Gamma(G)$,
3) $E(G)$: *set of edges of* $\Gamma(G)$,
4) $E(G) = \{ g_i, g_j \mid g_i \text{ and } g_j \text{ are conjugate in } G \}$.

An edge exists between any two vertices g_i and g_j, if there exists an element $x \in G$ that satisfies the relation $g_j = x g_i x^{-1}$ [16].

4.1 Algorithms and Implementation

To obtain a graphical representation Γ of a finite group G such that each element of the group G represents a vertex, and every two vertices are connected by an edge Figure 3, the relation that determines the form of the connection between the vertices is a conjugation relation. This implementation follows the Algorithm 1.

```
Result: Graph Γ = (V,E)
Procedure:
V = Element(G);
E:= [ ];
Classes:= Relation classes(C);
for C in classes do
elts:= Element(C);
for i in [1..Length(elts)] do
    for j in [1..Length(elts)] do
        Add(E, [ elts[i], elts [j] ] );
    od;
od;
od;
return rec ( vertices:= V, edges:= E );
end;
```

Algorithm 1: A graph via a finite group by relation.

Example 5. Consider the following example of the graph of the finite group V_4 with vertex set $V = \{v_i,\ i = 1,2,3,4\}$.

The following GAP computation (presented in Appendix, Listing A.2) represents a list of edges as present:

```
gap> Elements:= [ <identity> of..., f_1, f_2, f_1*f_2 ],

gap> L:= [ [1, 2 ], [ 1, 3 ], [ 1, 4 ], [ 2, 3 ], [ 2, 4 ], [ 3, 4 ] ]
```

Therefore, the set of edges will be according to the final result of the code above is:

$$E = \{e_{1,2}, e_{1,3}, e_{1,4}, e_{2,3}, e_{2,4}, e_{3,4}\}$$
$$\text{where } e_{ij} \text{ join vertices } v_i \text{ and } v_j,\ i = 1,2,3,4.$$

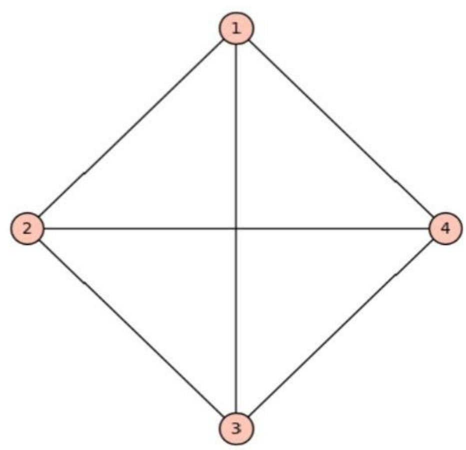

Figure 3: The graph $\Gamma(V_4)$ of this abelian group has a well-defined symmetric structure. The simplicity of this graph makes it a prime example for analyzing the overlap between group algebra, graph connectivity, and topology.

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$M^1 = \{A^i: a_{i1} = 1\} = \{A^2, A^3, A^4\}$$
$$M^2 = \{A^i: a_{i2} = 1\} = \{A^1, A^3, A^4\}$$
$$M^3 = \{A^i: a_{i3} = 1\} = \{A^1, A^2, A^4\}$$
$$M^4 = \{A^i: a_{i4} = 1\} = \{A^1, A^2, A^3\}$$

$$J_1^M = \{1\}, \quad J_1^M = \{2\}, \quad J_3^M = \{3\}, \quad J_4^M = \{4\},$$
$$\mathcal{B} = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{4\} \}$$

$$\mathcal{T} = \begin{Bmatrix} \emptyset, I, \{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\}, \\ \{2,3\},\{2,4\},\{3,4\} \{1,3,4\},\{1,2,4\},\{2,3,4\} \end{Bmatrix}$$

4.2 From Group Graphs to Topological Spaces

Proposition (Bridge between groups, graphs, and topology): Let G be a finite group and Γ_R is the graph of G with R a conjugacy relation, then $\mathcal{B}$ is a base for a topology.

Proof. Let G be a finite group, and R be conjugacy relation on . For g ∈ G So $C(a) = \{hgh^{-1}: h \in G\}$. Let $\mathcal{B} = \{C(g): g \in G\}$. Show that B a base for topology on G.

1) $Covering: \forall x \in G \; lies \; in \; C(x) \in \mathcal{B}, so \; G = \cup \mathcal{B}.$
2) $Local \; intersection \; property:$

$If \; x \in C(\mathfrak{a}) \cap C(\mathfrak{b}), \forall \mathfrak{a}, \mathfrak{b} \in G. Then \; x \; is \; conjugate \; to \; be \; both \; a \; and \; b. \; Since \; conjugacy \; is \; an \; equivalence \; relation, so \; that \; C(\mathfrak{a}) \cap C(\mathfrak{b}) = \phi \; or \; C(\mathfrak{a}) = C(\mathfrak{b}).$
$If \; contains \; x, then \; C(\mathfrak{a}) = C(\mathfrak{b}). So \; choosing \; C(\mathfrak{a}) \in \mathcal{B}, gives \; x \in C(a) \subset C(a) \cap C(b).$
$Therefore \; \mathcal{B} \; is \; a \; base \; for \; a \; topology \; on \; G.$

Example7. $If \; G = Q_8 \; a \; finite \; group, then \; \Gamma(G)$ $by \; Algorithm \; 1. with \; (e_1 \times e_3 = e_3 \times e_2)$. Consider the following finite group (G). Applying Algorithm 1 with the conjugacy relation, we obtain the graph shown in Figure 4. The graph illustrates the interconnections between the elements of (G) and provides the basis for constructing the corresponding topology.

So the adjacent matrix:

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathcal{B} = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\} \}$$

$$\mathcal{T} = \mathcal{P}(I).$$

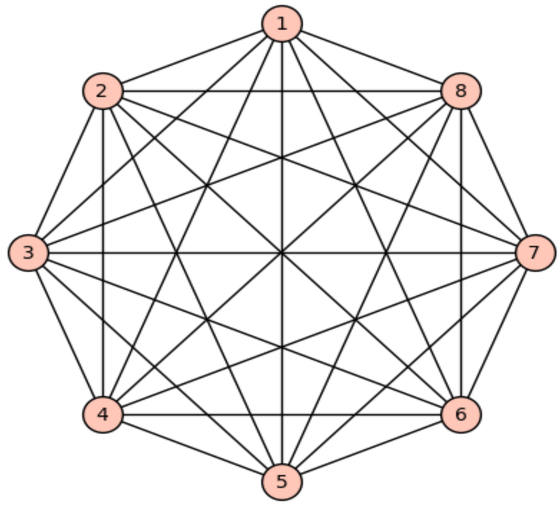

Figure 4: The graph of the finite group G.

Example 9. If G is a dihedral group of $order$ 10 with vertex $set \; V = \{v_i, i = 1, 2, \ldots, 10\}$ and the set of edges:

$$E = \begin{Bmatrix} e_{1,2}, e_{1,3}, \ldots, e_{1,10}, e_{2,1}, e_{3,1}, e_{3,5}, e_{3,7} \\ e_{3,9}, e_{4,1}, e_{5,1}, e_{5,3}, e_{5,7}, e_{5,9}, e_{6,1}, e_{7,1}, e_{7,3} \\ e_{7,5}, e_{7,9}, e_{8,1}, e_{9,1}, e_{9,3}, e_{9,5}, e_{9,7}, e_{10,1} \end{Bmatrix}$$

where e_{ij} join vertices $v_i \; and \; v_j, \; i = 1, 2, \ldots, 10$

By using the Algorithm 1. With the relation

$$(e_1 \times e_2 = e_2 \times e_1)$$

We get the following Figure 5:

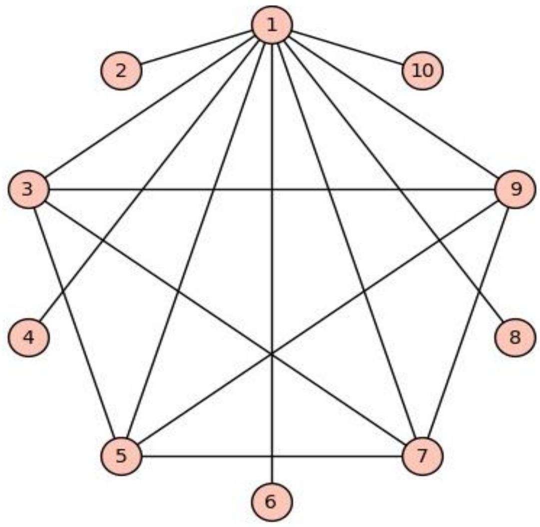

Figure 5: The graph $\Gamma(G)$ of a finite group D_{10}.

So the adjacent matrix:

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{gathered} M^1 = \{A^2, A^3, \ldots, A^{10}\}, \\ M^2 = M^4 = M^6 = M^8 = M^{10} = \{A^1\}, \\ M^3 = \{A^1, A^5, A^7, A^9\}, \\ M^5 = \{A^1, A^3, A^7, A^9\}, \\ M^7 = \{A^1, A^3, A^5, A^9\}, \\ M^9 = \{A^1, A^3, A^5, A^7\}, \end{gathered}$$

$J_1^M = \{1\}$, $J_3^M = \{3\}$, $J_5^M = \{5\}$, $J_7^M = \{7\}$, $J_9^M = \{9\}$,
$J_2^M = J_4^M = J_6^M = J_8^M = J_{10}^M = \{2,3,4,5,6,7,8,9,10\}$.

$\mathcal{B} = \{ \emptyset, \{1\}, \{3\}, \{5\}, \{7\}, \{9\}, \{2,3,4,5,6,7,8,9,10\} \}$.

$$\mathcal{T} = \left\{ \begin{gathered} \emptyset, I, \{1\}, \{3\}, \{5\}, \{7\}, \{9\}, \{1,3\}, \{1,5\}, \{1,7\}, \\ \{1,9\}, \{3,5\}, \{3,7\}, \{3,9\}, \{5,7\}, \{5,9\}\ \{7,9\}, \\ \{1,3,5\}, \{1,3,7\}, \{1,3,9\}, \{1,5,7\}, \{1,5,9\}, \\ \{1,7,9\}, \{3,5,7\}, \{3,5,9\}, \{3,7,9\}, \{5,7,9\}, \\ \{1,3,5,7\}, \{1,3,5,9\}, \{1,3,7,9\}, \{1,5,7,9\}, \\ \{3,5,7,9\}, \{1,3,5,7,9\}, \{2,3,4,5,6,7,8,9,10\} \end{gathered} \right\}$$

5 FACTORIAL RELATION

Definition 10. Let G be a finite group of order , We define the relation $\mathcal{R}_F$ on G as follows:
For any $x, y \in G$, $(x, y) \epsilon \mathcal{R}_F$ if and only if there exist natural numbers $n, m \in N$ such that $x^{n!} = y^{m!}$
This relation is called the factorial relation on G.

Lemma 1. The relation $\mathcal{R}_F$ is well-define on any finite group.

Definition 11. The graph $\Gamma(G, \mathcal{R}_F)$ is called the factorial graph of G.

Notation 2. The factorial relation $\mathcal{R}_F$ is not necessarily an equivalence relation, however naturally induces a simple graph on the group.

Example 10. If G is a dihedral group of $order$ 8 with vertex $set\ V = \{v_i, i = 1, 2, \ldots, 8\}$ and the set of edges:

$$E = \left\{ \begin{gathered} e_{1,2}, e_{1,3}, \ldots, e_{1,8}, e_{2,1}, \ldots, e_{2,8}, e_{3,1}, \ldots, \\ e_{8,9}, e_{4,1}, e_{4,2}, \ldots, e_{4,8}, e_{5,6}, e_{5,7}, e_{5,8}, e_{6,7} \\ e_{6,8}, e_{7,8} \end{gathered} \right\}$$

where e_{ij} join vertices v_i and v_j, $i = 1, 2, \ldots, 8$

By using the **Algorithm 1**. With the relation:

$$(x^{n!} = y^{m!})\forall n, m \in N.$$

We get the following Figure 6:

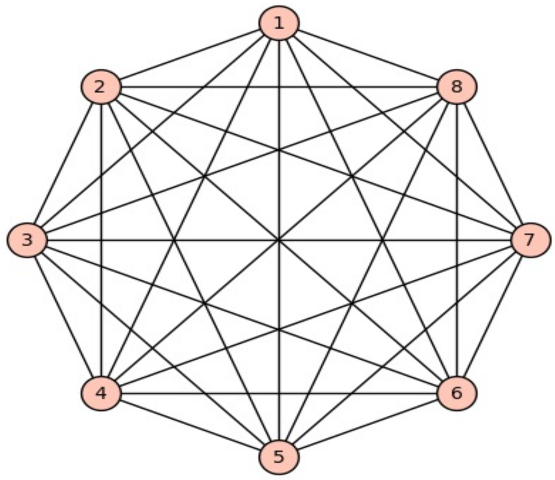

Figure 6: The graph $\Gamma(G, \mathcal{R}_F)$ of a finite group D_8.

So the adjacent matrix:

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathcal{B} = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\} \}.$$

$$\mathcal{T} = \mathcal{P}(I).$$

Example 11. If G is a dihedral group of order 22 with vertex $set\ V = \{v_i, i = 1, 2, \ldots, 22\}$.

The following GAP computation (presented in Appendix, Listing A.3) represents a list of edges as present:

$$E = \left\{ \begin{gathered} e_{1,2}, e_{1,3}, \ldots, e_{1,22}, e_{2,1}, e_{2,3} \ldots, e_{2,22}, e_{3,1}, e_{3,2} \\ e_{3,4} \ldots e_{3,22}, e_{4,1}, e_{4,2}, e_{4,3}, e_{4,5}, \ldots, e_{4,22}, \ldots, \\ e_{22,1}, e_{22,2}, \ldots, e_{22,21} \end{gathered} \right\}$$

where e_{ij} join vertices v_i and v_j, $i = 1, 2, \ldots, 22$.
By using the Algorithm 1. With the relation:

$$(x^{n!} = y^{m!})\ \forall n, m \in N.$$

We get the following Figure 7.

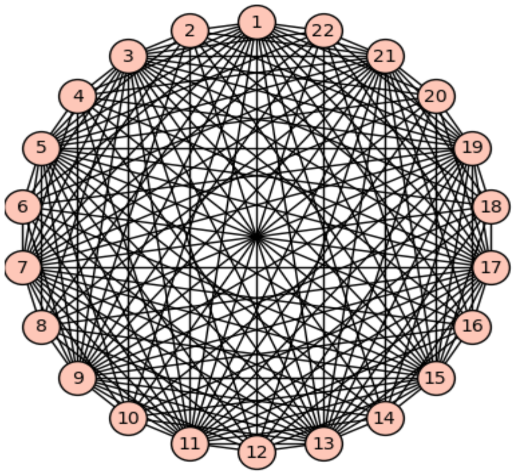

Figure 7: The graph $\Gamma(G, \mathcal{R}_F)$ of a finite group D_{22}.

So the adjacent matrix:

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 0 \end{bmatrix}_{22\times 22}$$

Then:

$$M^1 = \{A^i : a_{i1} = 1\} = \{A^2, A^3, \dots, A^{22}\},$$
$$M^2 = \{A^i : a_{i2} = 1\} = \{A^1, A^3, \dots, A^{22}\},$$
$$M^3 = \{A^i : a_{i3} = 1\} = \{A^1, A^2, A^4, \dots, A^{22}\},$$
$$M^4 = \{A^i : a_{i4} = 1\} = \{A^1, A^2, A^3, A^5, \dots, A^{22}\},$$
$$M^5 = \{A^i : a_{i5} = 1\} = \{A^1, \dots, A^4, A^6, \dots, A^{22}\},$$
$$\vdots$$
$$M^{22} = \{A^i : a_{i22} = 1\} = \{A^1, A^2, \dots, A^{21}\}.$$

And

$$J_1^M = \{1\},\quad J_2^M = \{2\},\quad J_3^M = \{3\},$$
$$J_4^M = \{4\},\quad J_5^M = \{5\},\quad J_6^M = \{6\},$$
$$J_7^M = \{7\},\quad J_8^M = \{8\}, \dots, J_{22}^M = \{22\}$$
$$\mathcal{B} = \{\emptyset, \{1\}, \{2\}, \{3\}, \cdots, \{22\}\}$$
$$\mathcal{T} = \mathcal{P}(I).$$

Proposition 3. Let $k_n = (V, E)$ be a complete graph, where $V = \{v_i \mid i = 1, 2, \dots, n\}$.Then the topology generated by $k_n$is a discrete topology on V [17].

Proof: The set of all positions of vertices $V = \{1, 2, \dots, n\}$ are as follows, respectively: $J_1^M = \{1\}$, $J_1^M = \{2\}$, $J_3^M = \{3\}$, $\cdots$, $J_n^M = \{n\}$.

Therefore, $K_n = \{\{v_i\} \mid v_i \in V\}$.

Hence, the topology generated by the complete graph is given by $\mathcal{T}_{K_n} = \mathcal{P}(V)$, where $\mathcal{P}(V)$denotes the power set of V.

It follows that $\mathcal{T}_{K_n}$ is a discrete topology on V. The following theorem is one of the main results of this paper.

Theorem 3. (Factorial Relation).

Every dihedral group of order 2n with a Factorial relation $(x^{n!} = y^{m!})\forall n, m \in N$, induces a Discrete topology.

Proof: Let $G = D_{2n}$ be a finite group of order 2n. Since G is finit, the order of every element $x \in G$ is finite and satisfies $ord(x)|2n$. Because the factorial number $n!$ *and* $m!$ are divisible by every positive integer less than or equal to $2n$, it follows that $x^{n!} = e$ and $y^{m!} = e$, $\forall x, y \in G$. Hence $x^{n!} = y^{m!}, \forall x, y \in G$.

Consequently, every pair of distinct vertices is connected by an edge: $\Gamma(G, \mathcal{R}_F) = K_n$

By Proposition 3. and since: $\Gamma(G, \mathcal{R}_F) = K_n$ the topology on the vertex set G is $\mathcal{T}_{\mathcal{R}_F} = \mathcal{P}(G)$ (is discrete topology). Remark: the argument above uses only the finiteness of G and does not rely on any property specific to dihedral groups; hence the same conclusion holds, by the identical proof, for the complete graph induced by the Factorial relation on any finite group. The dihedral case is highlighted here because it is the primary class of non-commutative groups examined throughout this paper.

6 CONCLUSIONS

This paper introduces the concept of generating a topological space from graphs generated from finite groups using a group relation that connects nodes to edges, resulting in a new topology. It focuses on how the algebraic structure of groups affects the topological properties of graphs. The results demonstrate a deep connection between algebraic equivalence and topological disjunction properties through examples D_4 and V_4. The results confirm the possibility of integrating finite group theory, graph theory, and topology, which we summarize in a new property that serves as a bridge between these three topics. Furthermore, we introduce a new algebraic relation whose associated graph induces a well-defined topology, thereby strengthening the structural link between group relations and graph-generated topological spaces.

Finally, it is essential to highlight that the findings established in this research are primarily applicable to complete graphs and specific classes of non-commutative groups, such as the finite dihedral group. Extending these results to incomplete graphs or a broader spectrum of non-commutative groups may require additional constraints on the group relationships. Subsequent investigations could define these limitations and potentially uncover novel structural properties.

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APPENDIX

This appendix provides the GAP implementations used to construct the graphs and compute the corresponding topological bases discussed in Sections 4 and 5. The first listing implements the procedure for obtaining a topological base from a graph, while the subsequent listings implement the relations considered in the examples, including the factorial relation. The code is provided to ensure the reproducibility of the computational results presented in the paper:

```
gap> EdgesToBase:= function(L)
>local   M,   B,   EdgesToIncidenceMatrix,
>IncidenceMatrixToBase;

########### % ###########
>EdgesToIncidenceMatrix:= function(L)
>local K, G, M;
>K:= SimplicialComplex(L);
>K:= SimplicialComplexToRegularCWComplex(K);
>G:= Graph(K);
>M:= G!.incidenceMatrix;
>return(M);
>end;

########### % ###########
>IncidenceMatrixToBase:= function(M)
>local k, N, i, j, x, n, R;
>k:= Length(M);
>N:= List([1..k], j -> []);
>for i in [1..k] do
  >for x in M do
     >if x[i] = 1 then
        >Add(N[i], x);
     >fi;
  >od;
>od;
>N:= List(N, x -> Sum(x));
>n:= List(N, x -> Maximum(x));
>R:=  List([1..k],   x   ->  [Positions(N[x],
n[x])]);
>return Union(R);
>end;

########### % ###########
>M:= EdgesToIncidenceMatrix(L);
>B:= IncidenceMatrixToBase(M);
>return B;
>end;
gap> B:=EdgesToBase(L);
```

Listing A.1: GAP implementation for computing a topological base from a graph.

```
gap> V4Pairs:= function()
gap> local G, E, M, e_1, e_2;
> G:= ElementaryAbelianGroup(4);
> E:= Elements(G);
> M:= [];
> for e_1 in E do
> for e_2 in E do
> if e_1 <> e_2 and e_1^Factorial(4)
= e_2^4 then
> Add(M, [ Position(E, e_1), Position(E,
e_2) ]);
> fi;
> od;
> od;
> return rec(
> Elements:= E,
> Pairs:= M
> );
> end;
> R:=V4Pairs();
```

Listing A.2: GAP implementation of the factorial relation for V_4.

```
gap>D22FactorialRelationNoDiag:=
function(m)
    >local G, E, n, f1, f2, i, j, pairs;

>G:= DihedralGroup(22);
>E:= Elements(G);
>n:= Size(G);
>f1:= Factorial(n);
>f2:= Factorial(m);

>pairs:= [];

>for i in [1..n] do
>for j in [1..n] do
>if i <> j and E[i]^f1 = >E[j]^f2 then
>Add(pairs, [i, j]);
>fi;
>od;
>od;

>return pairs;
>end;
```

Listing A.3: GAP implementation of the factorial relation for D_{22}.