Compactness and Lindelöfness in Special Neutrosophic Crisp Topological Spaces

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Keywords: Neutrosophic Crisp Open Cover, Neutrosophic Crisp Finite Open Cover, Neutrosophic Crisp Countable Open Cover, Neutrosophic Crisp Compactness, Neutrosophic Crisp Lindelöf Property.

Abstract: Algebraic operations in Neutrosophic Crisp Sets are diverse, involving multiple types of intersections, unions, complements, and inclusions, which makes the study of their topological properties more complex. This paper investigates neutrosophic crisp compactness and Lindelöf properties within a neutrosophic crisp topological space defined by first-type intersections and second-type unions and complements for all Neutrosophic Crisp Sets. Definitions for compactness and Lindelöfness are introduced based on these operations, and the validity of classical topological theorems is analysed in this context. Comparisons with classical topology reveal that some theorems hold, while others fail, but are valid under specific conditions that have been clarified. This study also examines the impact of specifying algebraic operations on the behaviour of compactness and Lindelöf properties. The theorems that are valid under the proposed framework are rigorously established within the proposed framework. These findings provide new insights into the structure of Neutrosophic Crisp topological spaces and highlight the critical role of algebraic operation selection in determining fundamental.

1 INTRODUCTION

Florentin Smarandache is widely recognized as the founder of neutrosophic crisp Sets, having classified these sets into three distinct types under specific conditions [1], [2]:

Let $Q^{n} = < Q_1, Q_2, Q_3 >$ be a neutrosophic crisp set ($\mathrm{N_UÇ} - set$) in Y.
Type I: $Q_1 \cap Q_2 = \Phi, Q_1 \cap Q_3 = \Phi, Q_2 \cap Q_3 = \Phi$.
Type II: $Q_1 \cap Q_2 = \Phi, Q_1 \cap Q_3 =, \; Q_2 \cap Q_3 = \Phi$, and $Q_1 \cup Q_2 \cup Q_3 = Y$.
Type III: $Q_1 \cap Q_2 \cap Q_3 = \emptyset$, and $Q_1 \cup Q_2 \cup Q_3 = Y$.

Additionally, he and his colleagues introduced four forms for the universal set and four forms for the empty sets as follows:

$\Phi_1^{n} = < \Phi, \Phi, Y>$, $\quad \Phi_2^{n} = < \Phi, Y, \Phi >$,
$\Phi_3^{n} = < \Phi, Y, Y >$, $\quad \Phi_4^{n} = < \Phi, Y, \Phi>$.
$Y_1^{n} = < Y, \Phi, \Phi, >$, $\quad Y_2^{n} = < Y, Y, \Phi>$,
$Y_3^{n} = < Y, \Phi, Y>$, $\quad Y_4^{n} = < Y, Y, Y>$.

As well as defined three types of complements as follows:

Let $Q^{n} = < Q_1, Q_2, Q_3 >$ be a $\mathrm{N_UÇ} - set$ in Y. its complements are defined as follows:
Type I: $(Q^{n})^{c_1} = < {Q_1}^c, {Q_2}^c, {Q_3}^c >$,
Type II: $(Q^{n})^{c_2} = < Q_3, Q_2, Q_1 >$,
Type III: $(Q^{n})^{c_3} = < Q_3, {Q_2}^c, Q_1 >$.

Thereby establishing a solid foundation for the study and applications of these sets.

Subsequently, numerous researchers contributed to the development of $\mathrm{N_UÇ} - sets$ and their associated topological spaces, as well as the study of relationships among them. Notably, in [3], new concepts of $\mathrm{N_UÇ} - set$ closed sets were introduced, including g-neutrosophic crisp closed sets, αg −neutrosophic crisp closed sets, $g\alpha$ −neutrosophic crisp closed sets, and $g\alpha g$ −neutrosophic crisp closed sets, together with their fundamental properties in neutrosophic crisp topological spaces. Moreover, new notions such as $g\alpha g$ −neutrosophic crisp closure and $g\alpha g$ −neutrosophic crisp interior were defined, and some of their properties were investigated.

In [4], the general approach for generating any stable neutrosophic crisp topology using a basis or the concept of stable neutrosophic crisp interior was clarified. This interior is closed under finite intersections but not under finite unions. Conversely, the stable neutrosophic crisp exterior is closed under finite unions but not under finite intersections. Necessary and sufficient conditions for both finite unions and finite intersections to be closed were obtained based on the concept of confused crisp sets.

In [5], we provided a neutrosophic crisp topological space was defined based on Type I intersection and Type II union and complement, Meanwhile, the researchers in [6]-[9] contributed to the development of separation axioms as well as the concepts of $Ɲ_{U}Ç$ −Semi-Compact and Semi-Lindelöf spaces, in the context of $Ɲ_{U}Ç - set$.

Based on the definition presented in [5], this manuscript introduces the concepts of $Ɲ_{U}Ç$ – compact and $Ɲ_{U}Ç$ – Lindelöf spaces using the $Ɲ_{U}Ç$ – open cover as defined in the same study (Table 1). A comparison with classical topology reveals that some theorems hold universally, while others are valid only under specific conditions, as detailed in this work. Type I intersection together with Type II union and complement are adopted here, following [5], because this particular combination yields a consistent lattice structure on the collection of neutrosophic crisp open sets that supports well-behaved notions of compactness and the Lindelöf property; other combinations of operation types do not preserve this structure and are left for future investigation. This paper aims to clarify the challenges associated with determining the appropriate form of algebraic operations and propose solutions, thereby opening future avenues for identifying optimal forms when studying properties of neutrosophic crisp topological spaces.

Table 1: List of symbols.

Symbol	Descriptions
$Ɲ_{U}Ç$	Neutrosophic Crisp
$Ɲ_{U}Ç\mathcal{T}_{(1,2)}$ – space	Neutrosophic Crisp Topological Spaces
$Ɲ_{U}ÇÇ$ – set	Neutrosophic Crisp Closed Set
$Ɲ_{U}ÇO$ – set	Neutrosophic Crisp Open Set
$Ɲ_{s} - Ɲ_{U}Ç\mathcal{T}_{(1,2)}$ – space	subspace Neutrosophic Crisp Topology

2 PRELIMINARIES

2.1 Definition [10]

Let $Đ^{n} =< Đ_1, Đ_2, Đ_3 >$, and $Ł^{n} = < Ł_1, Ł_2,\ Ł_3 >$ be two $Ɲ_{U}Ç - sets$ in Y. Their union is defined as follows:

Type I:

$Đ^{n} \cup_1 Ł^{n} = < Đ_1 \cup Ł_1,\ Đ_2 \cup Ł_2,\ Đ_3 \cap Ł_3 >$

Type II:

$Đ^{n} \cup_2 Ł^{n} = < Đ_1 \cup Ł_1,\ Đ_2 \cap Ł_2,\ Đ_3 \cap Ł_3 >$

Similarly, their intersection is defined as follows:

Type I:
$Đ^{n} \cap_1 Ł^{n} = < Đ_1 \cap Ł_1,\ Đ_2 \cap Ł_2,\ Đ_3 \cup Ł_3 >$
Type II:
$Đ^{n} \cap_2 Ł^{n} = < Đ_1 \cap Ł_1,\ Đ_2 \cup Ł_2,\ Đ_3 \cup Ł_3 >$.

2.2 Definition 2.2 [10]

For any two $Ɲ_{U}Ç - sets$ $Đ^{n} = <Đ_1,\ Đ_2,\ Đ_3 >$, and $Ł^{n} = < Ł_1,\ Ł_2,\ Ł_3 >$ in Y, two types of subset relations are defined as follows:

Type I:
$Đ^{n} \subseteq_1 Ł^{n} \Leftrightarrow Đ_1 \subseteq Ł_1,\ Đ_2 \subseteq Ł_2,\ Ł_3 \subseteq Đ_3$,
Type II:
$Đ^{n} \subseteq_2 Ł^{n} \Leftrightarrow Đ_1 \subseteq Ł_1,\ Ł_2 \subseteq Đ_2,\ Ł_3 \subseteq Đ_3$.

2.3 Theorem 2.3 [1]

Assume that $\{M^{n}_{j} : j \in J\}$ is a family of $Ɲ_{U}Ç$ − subsets defined on a universe Y. under this assumption, we obtain that:

$$\cap_1 M^{n}_{j} = <\cap M_{j1},\ \cap M_{j2},\ \cup M_{j3} >,$$

$$\cup_2 M^{n}_{j} = <\cup M_{j1},\ \cap M_{j2},\ \cap M_{j3} >.$$

2.4 Definition [11]

If $Ɉ^{n} = < Ɉ_1,\ Ɉ_2,\ Ɉ_3 >$ is a $Ɲ_{U}Ç - set$ in Z, then the preimage of $Ɉ^{n}$ under h, defined by $h^{-1}(Ɉ^{n}) = < h^{-1}(Ɉ_1),\ h^{-1}(Ɉ_2),\ h^{-1}(Ɉ_3) >$, is a $Ɲ_{U}Ç - set$ in Y.

Similarly, If $K^{n} = < K_1, K_2, K_3 >$ is a $Ɲ_{U}Ç - set$ in Y, then the image of $Ɲ^{n}$ under h, defined by $h(K^{n}) = < h(K_1),\ h(K_2),\ h(K_3) >$, is a $Ɲ_{U}Ç - set$ in Z.

2.5 Corollary [2]

Let $h: Y \rightarrow Z$ be a function, and $K^{n} =< K_1, K_2, K_3 >, \mathcal{M}^{n} =< \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 >$ be $Ɲ_{U}Ç - sets$ in Y. Then:

$$\mathrm{K}^{n} \subseteq_1 \mathcal{M}^{n} \Leftrightarrow h(\mathrm{K}^{n}) \subseteq_1 h(\mathcal{M}^{n}),$$
$$\mathrm{K}^{n} \subseteq_2 \mathcal{M}^{n} \Leftrightarrow h(\mathrm{K}^{n}) \subseteq_2 h(\mathcal{M}^{n}).$$

2.6 Corollary [2]

Let $h: \mathrm{Y} \to Z$ be a function, and $\mathrm{B}^{n} =< \mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3 >, \mathrm{J}^{n} = < \mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_3 >$ be $\mathrm{N_U Ç} - sets$ in Z. Then:

$$\mathrm{B}^{n} \subseteq_1 \mathrm{J}^{n} \Leftrightarrow h^{-1}(\mathrm{B}^{n}) \subseteq_1 h^{-1}(\mathrm{J}^{n}),$$
$$\mathrm{B}^{n} \subseteq_2 \mathrm{J}^{n} \Leftrightarrow h^{-1}(\mathrm{B}^{n}) \subseteq_2 h^{-1}(\mathrm{J}^{n}).$$

2.7 Definition [5]

A pair (Y, T) is called a $\mathrm{N_U Ç}$ − topological space ($\mathrm{N_U ÇT_{(1,2)}}$ − space), if the following conditions are satisfied:

- $\Phi_1^{n} \in \mathrm{T}$, and $Y_1^{n} \in \mathrm{T}$.
- For any $\mathrm{Đ}^{n}, O^{n} \in \mathrm{T}$, the set $\mathrm{Đ}^{n} \cap_1 O^{n} \in \mathrm{T}$.
- If $\mathrm{Đ}^{n}{}_j \in \mathrm{T}\ \forall j \in J$, then $\cup_{j2}\ \mathrm{Đ}^{n}{}_j \in \mathrm{T}$.

Each $\mathrm{Đ}^{n} \in \mathrm{T}$, is called a $\mathrm{N_U Ç}$ − open set ($\mathrm{N_U ÇO}$ − set), and $(\mathrm{Đ}^{n})^{c_2}$ is called a $\mathrm{N_U Ç}$ − closed set ($\mathrm{N_U ÇÇ}$ − set).

2.8 Definition [5]

Let $h: (Y, \mathrm{T}_Y) \to (Z, \mathrm{T}_Z)$ be a mapping between two $\mathrm{N_U ÇT_{(1,2)}}$ − spaces. Then h is called $\mathrm{N_U Ç}$ − continuous$_{(1,2)}$ if for every $\mathrm{N_U ÇO}$ − set $O^{n} \in \mathrm{T}_Z$, the inverse image $h^{-1}(O^{n})$ is a $\mathrm{N_U ÇO}$ − set in Y.

2.9 Definition

A $\mathrm{N_U Ç} - set$ $\mathrm{Q}^{n} =< \mathrm{Q}_1, \mathrm{Q}_2, \mathrm{Q}_3 >$ in Y is called finite (countable) whenever all its component crisp sets $\mathrm{Q}_1, \mathrm{Q}_2$, and Q_3 are finite (countable).

2.10 Definition

Let (Y, T) be a $\mathrm{N_U ÇT_{(1,2)}}$ − space, $\mathrm{M}^{n} = < \mathrm{M}_1, \Phi, \mathrm{M}_3 > \subseteq_1 Y_1^{n}$. Then the neutrosophic crisp topology defined on M^{n} is termed a subspace neutrosophic crisp topology ($\mathrm{N_s}$ − $\mathrm{N_U ÇT_{(1,2)}}$ − space) denoted by ($\mathrm{T}_{\mathrm{M}^{n}}$) such that $\mathrm{T}_{\mathrm{M}^{n}} = \{\mathrm{M}^{n} \cap_1 \mathcal{N}^{n}: \mathcal{N}^{n} \in \mathrm{T}\}$.

2.11 Theorem

$(\mathrm{M}^{n}, \mathrm{T}_{\mathrm{M}^{n}})$ is a $\mathrm{N_U Ç}$ − topological space.

Proof: The proof follows directly from Definitions 7 and 10.

3 NEUTROSOPHIC CRISP COMPACT SPACE

The following section, we study $\mathrm{N_U Ç}$ − open covers, and finite open covers, within neutrosophic crisp topology. We also introduce the notion of $\mathrm{N_U Ç}$ − compactness based on the topology defined in Definition 2.7. Furthermore, a collection of fundamental theorems with their rigorous proofs is presented, along with several results that hold under additional conditions.

3.1 Definition

Let (Y, T) be a $\mathrm{N_U ÇT_{(1,2)}}$ − space, and let $\mathrm{Q}^{n} =< \mathrm{Q}_1, \mathrm{Q}_2, \mathrm{Q}_3 >$ be a $\mathrm{N_U Ç} - set$ in Y. A neutrosophic crisp covering of Q^{n} is a family of $\mathrm{N_U Ç} - sets$ $\mathrm{G} = \{\mathfrak{R}^{n}{}_\lambda =< \mathfrak{R}_{\lambda 1}, \mathfrak{R}_{\lambda 2}, \mathfrak{R}_{\lambda 3} >: \lambda \in \Lambda\}$ such that:

$$\mathrm{Q}_1 \subseteq \bigcup_{\lambda \in \Lambda} \mathfrak{R}_{\lambda 1}, \mathrm{Q}_2 \supseteq \bigcap_{\lambda \in \Lambda} \mathfrak{R}_{\lambda 2}, \mathrm{Q}_3 \supseteq \bigcap_{\lambda \in \Lambda} \mathfrak{R}_{\lambda 3}.$$

- A subcover of G is any subfamily $\mathrm{H} \subseteq \mathrm{G}$ that also a covering of Q^{n}.
- A covering G is called a $\mathrm{N_U Ç}$ − open cover if each $\mathfrak{R}^{n}{}_\lambda$ are $\mathrm{N_U ÇO}$ − set.
- A covering G is said to be $\mathrm{N_U Ç}$ − finite cover if it contains finitely many elements, i.e., $\mathrm{G} = \{\mathfrak{R}^{n}{}_i =< \mathfrak{R}_{i1}, \mathfrak{R}_{i2}, \mathfrak{R}_{i3} >, i = 1,2, \ldots, m\}$) satisfies, $\mathrm{Q}_1 \subseteq \bigcup_{i=1}^{m} \mathfrak{R}_{i1}$, $\mathrm{Q}_2 \supseteq \bigcap_{i=1}^{m} \mathfrak{R}_{i2}$, $\mathrm{Q}_3 \supseteq \bigcap_{i=1}^{m} \mathfrak{R}_{i3}$.
- The family G is called a $\mathrm{N_U Ç}$ − open cover of Y_1^{n} if and only if

$$\mathrm{Y} = \bigcup_{\lambda \in \Lambda} \mathfrak{R}_{\lambda 1}, \quad \Phi = \bigcap_{\lambda \in \Lambda} \mathfrak{R}_{\lambda 2}, \quad \Phi = \bigcap_{\lambda \in \Lambda} \mathfrak{R}_{\lambda 3}.$$

3.2 Remark

Let (Y, T) be a $\mathrm{N_U ÇT_{(1,2)}}$ − space, and $\mathrm{Q}^{n} =< \mathrm{Q}_1, \Phi, \mathrm{Q}_3 >$ be a $\mathrm{N_U Ç} - set$ in Y. A covering of Q^{n} is a collection of $\mathrm{N_U Ç} - sets$ $\mathrm{G} = \{\mathfrak{R}^{n}{}_\lambda =< \mathfrak{R}_{\lambda 1}, \emptyset, \mathfrak{R}_{\lambda 3} >: \lambda \in \Lambda\}$ such that

$$\mathrm{Q}_1 \subseteq \bigcup_{\lambda \in \Lambda} \mathfrak{R}_{\lambda 1}, \quad \mathrm{Q}_3 \supseteq \bigcap_{\lambda \in \Lambda} \mathfrak{R}_{\lambda 3}.$$

Because, $\mathrm{Q}_2 = \Phi$ according to Definition 3.1, it follows directly that the intersection of any family of open sets covering Q_2 is necessarily empty. Hence, this condition need not be stated separately.

3.3 Definition

A $\mathrm{N_U Ç}$ – set Q^{n} in a $\mathrm{N_U ÇT_{(1,2)}}$ – space, is said to be a $\mathrm{N_U Ç}$ – compact provided that every $\mathrm{N_U Ç}$ – open cover of Q^{n} has a finite supcover.

- Y_1^{n} is a $\mathrm{N_U Ç}$ – compact if and only if $Y = \bigcup_{i=1}^{m} \mathfrak{R}_{i1}$, $\Phi = \bigcap_{i=1}^{m} \mathfrak{R}_{i2}$, $\Phi = \bigcap_{i=1}^{m} \mathfrak{R}_{i3}$;
- If Y_1^{n} is a $\mathrm{N_U Ç}$ – compact, then (Y, T) is a $\mathrm{N_U Ç}$ – compact space.

3.4 Example

Let $Y = N$ and $K_1^{n} =< \{2,3,\dots\}, \Phi, \Phi >$, $K_2^{n} =< \{3,4,\dots\}, \Phi, \{1\} >$, $K_3^{n} =< \{4,5 \dots\}, \Phi, \{1,2\} >$, $K_m^{n} =< \{m+1, m+2, \dots\}, \Phi, \{1,2,3,\dots, m-1\} >$.

Define $T = \{\Phi_1^{n}, Y_1^{n}\} \cup K_m^{n}$.

Then (Y, T) is a $\mathrm{N_U ÇT_{(1,2)}}$ – space (by Definition 2.7).

Moreover, every $\mathrm{N_U Ç}$ – open cover of N has $\mathrm{N_U Ç}$ – finite subcover $\bigcup_{i=1}^{m} K_{i1} = N$, $\bigcap_{i=1}^{m} K_{i1} = \Phi$, $\bigcap_{i=1}^{m} K_{i3} = \Phi$, so (Y, T) is a $\mathrm{N_U Ç}$ – compact space.

3.5 Example

Let $Y = (0,1)$, and $K_1^{n} =< \left(\frac{1}{3}, \frac{2}{3}\right), \Phi, \left(0, \frac{1}{3}\right) >$, $K_2^{n} =< \left(\frac{1}{4}, \frac{3}{4}\right), \Phi, \left(0, \frac{1}{4}\right) >$, $K_3^{n} =< \left(\frac{1}{5}, \frac{4}{5}\right), \Phi, \left(0, \frac{1}{5}\right) >$, $K_m^{n} =< \left(\frac{1}{m}, \frac{m-1}{m}\right), \Phi, \left(0, \frac{1}{m}\right) >$.

Define $T = \{\Phi_1^{n}, Y_1^{n}\} \cup \{K_m^{n}, m = 3,4,5,\dots\}$.

Then (Y, T) is a $\mathrm{N_U ÇT_{(1,2)}}$ – space (by Definition 2.7). Since,

$$Y = \bigcup_{m=3}^{\infty} \left(\frac{1}{m}, \frac{m-1}{m}\right), \bigcap_{m=3}^{\infty} \Phi = \Phi, \bigcap_{m=3}^{\infty} \left(0, \frac{1}{m}\right) = \Phi.$$

But, $\bigcup_{m=3}^{k} \left(\frac{1}{m}, \frac{m-1}{m}\right) \subset (0,1)$ *but* $\neq Y$, $m = 3,4,\dots,t$. Because all these intervals start at values greater than or equal to $\frac{1}{t}$. Hence, the interval $(0, \frac{1}{t})$ is not covered.

Therefore, no finite subcollection covers Y_1^{n}, and so (Y, T).is not $\mathrm{N_U Ç}$ – compact space.

3.6 Theorem

Let (Y, T) be a $\mathrm{N_U ÇT_{(1,2)}}$ – space and $Ħ^{n} =< Ħ_1, Ħ_2, Ħ_3 >$ be a finite $\mathrm{N_U Ç}$ – set in Y. Then $Ħ^{n}$ is a $\mathrm{N_U Ç}$ – compact.

Proof: Since, $Ħ^{n}$ is finite $\mathrm{N_U Ç}$ – set. Therefore $Ħ^{n} =< \{d_{1,1}, \dots, d_{1,t}\}, \{d_{2,1}, \dots, d_{2,l}\}, \{d_{3,1}, \dots, d_{3,k}\} >$ (by Definition 2.9)

Let $\{\mathfrak{R}^{n}_{\lambda} =< \mathfrak{R}_{\lambda 1}, \mathfrak{R}_{\lambda 2}, \mathfrak{R}_{\lambda 3} >: \lambda \in \mathfrak{X}\}$ be an $\mathrm{N_U Ç}$ – open cover of $Ħ^{n}$. Then

$$Ħ_1 \subseteq \bigcup_{\lambda \in \mathfrak{X}} \mathfrak{R}_{\lambda 1}, Ħ_2 \supseteq \bigcap_{\lambda \in \mathfrak{X}} \mathfrak{R}_{\lambda 2}, \quad Ħ_3 \supseteq \bigcap_{\lambda \in \mathfrak{X}} \mathfrak{R}_{\lambda 3}.$$

Therefore,

$$\forall d_1 \in Ħ_1 \exists \mathfrak{R}_{\lambda 1 d_1} \in \{\mathfrak{R}_{\lambda 1}\} \ni d_1 \in \mathfrak{R}_{\lambda 1 d_1}.$$

So, $d_{1,1} \in \mathfrak{R}_{\lambda 1 d_{1,1}}, d_{1,2} \in \mathfrak{R}_{\lambda 2 d_{1,2}}, \dots, d_{1,m} \in \mathfrak{R}_{\lambda 1 d_{1,l}}$.

Hence,

$$Ħ_1 \subseteq \bigcup_{i=1}^{m} \mathfrak{R}_{i1}. \qquad (1)$$

Since, $Ħ_2 \supseteq \bigcap_{\lambda} \mathfrak{R}_{\lambda 2}$ and $Ħ_2$ finite set.
So,

$$Ħ_2 \supseteq \bigcap_{i=1}^{m} \mathfrak{R}_{i2}. \qquad (2)$$

Since, $Ħ_3 \supseteq \bigcap_{\lambda} \mathfrak{R}_{\lambda 3}$ and $Ħ_3$ finite set.
So,

$$Ħ_3 \supseteq \bigcap_{i=1}^{m} \mathfrak{R}_{i3}. \qquad (3)$$

From (1), (2) and (3) we get $Ħ^{n}$ is a $\mathrm{N_U Ç}$ – compact.

3.7 Theorem

Let (Y, T) be a $\mathrm{N_U ÇT_{(1,2)}}$ – space and $Ƒ^{n} =< Ƒ_1, \Phi, Ƒ_1^{c} >$ be a $\mathrm{N_U ÇÇ}$ – *set* in Y. If (Y, T) is a $\mathrm{N_U Ç}$ – compact space then $Ƒ^{n}$ is a $\mathrm{N_U Ç}$ – compact.

Proof: Let $\{\mathfrak{R}^{n}_{\lambda} =< \mathfrak{R}_{\lambda 1}, \Phi, \mathfrak{R}_{\lambda 3} >: \lambda \in \mathfrak{X}\}$ be an $\mathrm{N_U Ç}$ – open cover of $Ƒ^{n}$. Then:

$Ƒ_1 \subseteq \bigcup_{\lambda \in \mathfrak{X}} \mathfrak{R}_{\lambda 1}$, $Ƒ_1^{c} \supseteq \bigcap_{\lambda \in \mathfrak{X}} \mathfrak{R}_{\lambda 3}$ (by Remark 3.2).

Since, $Ƒ^{n}$ is a $\mathrm{N_U ÇÇ}$ – set $\Rightarrow Ƒ^{n c_2}$ is a $\mathrm{N_U ÇO}$ – set.

Thus, $Y_1^{n} =< Y, \Phi, \Phi >= Ƒ^{n} \cup_2 Ƒ^{c_2} =< Ƒ_1 \cup Ƒ_1^{c}, \Phi \cap \Phi, Ƒ_1^{c} \cap Ƒ_1 >$. (by Definition 2.1)

So,

$$Y = Ƒ_1 \cup Ƒ_1^{c} \subseteq (\bigcup_{\lambda \in \mathfrak{X}} \mathfrak{R}_{\lambda 1}) \cup Ƒ_1^{c},$$

$$\Phi = Ƒ_1^{c} \cap Ƒ_1 \supseteq \left(\bigcap_{\lambda \in \mathfrak{X}} \mathfrak{R}_{\lambda 3}\right) \cap Ƒ_1.$$

Thus, $\{\mathfrak{R}^{n}_{\lambda}: \lambda \in \mathfrak{X}\}$, $\{Ƒ^{n c_2}\}$ are $\mathrm{N_U Ç}$ – open covering of Y_1^{n}. Since, Y_1^{n} is a $\mathrm{N_U Ç}$ – compact. Therefore,

$$Y = (\bigcup_{i=1}^{m} \mathfrak{R}_{i1}) \cup Ƒ_1^{c} \Rightarrow Ƒ_1 \subseteq \bigcup_{i=1}^{m} \mathfrak{R}_{i1}, \quad (1)$$
$$\Phi = (\bigcap_{i=1}^{m} \mathfrak{R}_{i3}) \cap Ƒ_1 \Rightarrow Ƒ_1^{c} \supseteq \bigcap_{i=1}^{m} \mathfrak{R}_{i1}. \quad (2)$$

From (1), and (2) we get $Ƒ^{n}$ is a $\mathrm{N_U Ç}$ – compact.

- The above theorem may not hold in general if the set $Ƒ^{n}$ is taken as $Ƒ^{n} =< Ƒ_1, Ƒ_2, Ƒ_3 >$, unless an appropriate type of complement is used. In particular, if the complement is of type (II), the theorem cannot be satisfied, since in this case the universal set Y_1^{n} is not obtained from the union of $ꞙ^{ꞥ}$ with its complement, i.e., $ꞙ^{ꞥ} \cup_2 ꞙ^{c_2} =< ꞙ_1 \cup ꞙ_3, ꞙ_2 \cap ꞙ_2, ꞙ_3 \cap ꞙ_1 > \neq Y_1^{ꞥ}$. Therefore, the complement must be taken of type (I) in order for the theorem to remain valid, as in this case we obtain $ꞙ^{ꞥ} \cup_2 ꞙ^{c_1} =< ꞙ_1 \cup ꞙ_1{}^c, ꞙ_2 \cap ꞙ_2{}^c, ꞙ_3 \cap ꞙ_3{}^c > = Y_1^{ꞥ}$.

3.8 Theorem

If $Q^{ꞥ} =< Q_1, Q_2, Q_3 >$ and $Ł^{ꞥ} =< Ł_1, Ł_2, Ł_3 >$ are $Ɲ_U Ç -$ compact sets in $Ɲ_U ÇT_{(1,2)} -$ space, then $Q^{ꞥ} \cup_2 Ł^{ꞥ}$ is a $Ɲ_U Ç -$ compact set.

Proof: Let $\{\mathfrak{R}^{ꞥ}{}_\lambda = < \mathfrak{R}_{\lambda 1}, \mathfrak{R}_{\lambda 2}, \mathfrak{R}_{\lambda 3} >: \lambda \in \text{Ꭓ}\}$ be an $Ɲ_U Ç -$ open covering of $Q^{ꞥ} \cup_2 Ł^{ꞥ} =< Q_1 \cup Ł_1, Q_2 \cap Ł_2, Q_3 \cap Ł_3 >$. Then:

$$Q_1 \cup Ł_1 \subseteq \bigcup_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 1}, \quad Q_2 \cap Ł_2 \supseteq \bigcap_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 2}, \quad Q_3 \cap Ł_3 \supseteq \bigcap_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 3}.$$

Since, $Q_1 \subseteq Q_1 \cup Ł_1 \subseteq \cup_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 1}$, and:
$Ł_1 \subseteq Q_1 \cup Ł_1 \subseteq \cup_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 1} \Rightarrow Q_1 \subseteq \cup_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 1}$, and $Ł_1 \subseteq \cup_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 1}$. Since, $Q_2 \supseteq Q_2 \cap Ł_2 \supseteq \cap_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 2}$, and $Ł_2 \supseteq Q_2 \cap Ł_2 \supseteq \cap_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 2} \Rightarrow Q_2 \supseteq \cap_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 2}$, and $Ł_2 \supseteq \cap_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 2}$.

Since, $Q_3 \supseteq Q_3 \cap Ł_3 \supseteq \cap_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 3}$, and:
$Ł_3 \supseteq Q_3 \cap Ł_3 \supseteq \cap_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 3} \Rightarrow Q_3 \supseteq \cap_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 3}$, and $Ł_3 \supseteq \cap_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 3}$. Therefore,
$\{\mathfrak{R}^{ꞥ}{}_\lambda =< \mathfrak{R}_{\lambda 1}, \mathfrak{R}_{\lambda 2}, \mathfrak{R}_{\lambda 3} >: \lambda \in \text{Ꭓ}\}$ be an $Ɲ_U Ç -$ open covering of $Q^{ꞥ} and\ Ł^{ꞥ}$, but $Q^{ꞥ} and\ Ł^{ꞥ}$ are ƝᴜÇ – compact. There exist finite subfamilies: $\{\mathfrak{R}^{ꞥ}{}_{\lambda_i}\}_{i=1}^m$, $\{\mathcal{H}^{ꞥ}{}_{\alpha_j}\}_{j=1}^m$. Hence,

$$Q_1 \subseteq \bigcup_{i=1}^m \mathfrak{R}_{\lambda_i 1}, \quad Q_2 \supseteq \bigcap_{i=1}^m \mathfrak{R}_{\lambda_i 2}, \quad Q_3 \supseteq \bigcap_{i=1}^m \mathfrak{R}_{\lambda_i 3}$$

And:

$$Ł_1 \subseteq \bigcup_{j=1}^m \mathcal{H}_{\alpha_j 1}, \quad Ł_2 \supseteq \bigcap_{j=1}^m \mathcal{H}_{\alpha_j 2}, \quad Ł_3 \supseteq \bigcap_{j=1}^m \mathcal{H}_{i\alpha_j 3}.$$

Let: $\mathfrak{R} = \{\mathfrak{R}^{ꞥ}{}_{\lambda_i}\}_{i=1}^m \cup_2 \{\mathcal{H}^{ꞥ}{}_{\alpha_j}\}_{j=1}^m$. Then $\mathfrak{R}$ is finite. We obtain:

$$Q_1 \cup Ł_1 \subseteq \bigcup_{\mathfrak{R}^{ꞥ} \in \mathfrak{R}} \mathfrak{R}_1, \quad Q_2 \cap Ł_2 \supseteq \bigcap_{\mathfrak{R}^{ꞥ} \in \mathfrak{R}} \mathfrak{R}_2, \quad Q_3 \cap Ł_3 \supseteq \bigcap_{\mathfrak{R}^{ꞥ} \in \mathfrak{R}} \mathfrak{R}_3$$

Hence, $Q^{ꞥ} \cup_2 Ł^{ꞥ}$ is a $Ɲ_U Ç -$ compact set.

3.9 Corollary

Let $Q^{ꞥ} =< Q_1, Q_2, Q_3 >$ be a $Ɲ_U Ç -$ compact set and $ꞙ^{ꞥ} =< ꞙ_1, \Phi, ꞙ_1{}^c >$ be a ƝᴜÇÇ – set in a $Ɲ_U ÇT_{(1,2)} -$ space. Then $Q^{ꞥ} \cap_1 ꞙ^{ꞥ}$ is a $Ɲ_U Ç -$ compact set.

Proof: Let $\{\mathfrak{R}^{ꞥ}{}_\lambda =< \mathfrak{R}_{\lambda 1}, \Phi, \mathfrak{R}_{\lambda 3} >: \lambda \in \text{Ꭓ}\}$ be an $Ɲ_U Ç -$ open covering of:

$$Q^{ꞥ} \cap_1 ꞙ^{ꞥ} =< Q_1 \cap ꞙ_1, \Phi, Q_3 \cup ꞙ_1{}^c >.$$

So, $Q_1 \cap ꞙ_1 \subseteq \cup_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 1}$, $Q_3 \cup ꞙ_1{}^c \supseteq \cap_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 3}$ by Remark 3.2.

Since, $ꞙ^{ꞥ}$ is a ƝᴜÇÇ – $set \Rightarrow ꞙ^{ꞥ c_2}$ is a ƝᴜÇO – set.

Hence, $\{\mathfrak{R}^{ꞥ}{}_\lambda: \lambda \in \text{Ꭓ}\} \cup_2 \{ꞙ^{ꞥ c_2}\}$ forms an $Ɲ_U Ç -$ open covering of $Q^{ꞥ}$.

So,

$$Q_1 \subseteq \cup_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 1} \cup ꞙ_1{}^c, \quad Q_3 \supseteq \cap_{\lambda \in \text{Ꭓ}} \mathfrak{R}_{\lambda 3} \cap ꞙ_1.$$

Since, $Q^{ꞥ}$ is a ƝᴜÇ – compact, then:

$$Q_1 \subseteq (\bigcup_{i=1}^m \mathfrak{R}_{i1}) \cup ꞙ_1{}^c, \quad Q_3 \supseteq (\bigcap_{i=1}^m \mathfrak{R}_{i3}) \cap ꞙ_1.$$

Thus,

$$Q_1 \cap ꞙ_1 \subseteq \cup_{i=1}^m (\mathfrak{R}_{i1} \cap ꞙ_1) \cup ꞙ_1{}^c \subseteq \cup_{i=1}^m \mathfrak{R}_{i1} \quad (1)$$
$$Q_3 \cup ꞙ_1{}^c \supseteq \cap_{i=1}^m (\mathfrak{R}_{i3} \cup ꞙ_1{}^c) \cap ꞙ_1 \supseteq \cap_{i=1}^m \mathfrak{R}_{i3}. \quad (2)$$

- Hence, $Q^{ꞥ} \cap_1 ꞙ^{ꞥ}$ is a $Ɲ_U Ç -$ compact set. If the set $ꞙ^{ꞥ}$ is replaced by the general form $ꞙ^{ꞥ} = < ꞙ_1, ꞙ_2, ꞙ_3 >$, the theorem may fail when using the complement of type (II), due to the lack of the open covering property required to cover $Q^{ꞥ}$. Therefore, it is necessary to use the complement of type (I) to ensure the validity of the theorem when considering the general set $ꞙ^{ꞥ} =< ꞙ_1, ꞙ_2, ꞙ_3 >$, since the complement of type (I) guarantees the required covering property and allows the construction of an open covering of $Q^{ꞥ}$, which is essential for the compactness argument.

3.10 Theorem

Let (Y, T_Y) and (Z, T_Z) be $Ɲ_U ÇT_{(1,2)} -$ spaces and $h: Y \to Z$ be a $Ɲ_U Ç -$ continuous$_{(1,2)}$ bijective. If $O^{ꞥ} =< O_1, O_2, O_3 >$ is a $Ɲ_U Ç -$ compact in Y, then $h(O^{ꞥ})$ is a $Ɲ_U Ç -$ compact in Z.

Proof: Let $\{Ꜧ^{ꞥ}{}_\lambda =< Ꜧ_{\lambda 1}, Ꜧ_{\lambda 2}, Ꜧ_{\lambda 3} >: \lambda \in \text{Ꭓ}\}$ be an forms an $Ɲ_U Ç -$ open cover of $f(O^{ꞥ})$. Then:

$$h(\mathrm{O}_1) \subseteq \bigcup_{\lambda\in\mathfrak{X}} \text{Ꜧ}_{\lambda 1}, \qquad h(\mathrm{O}_2) \supseteq \bigcap_{\lambda\in\mathfrak{X}} \text{Ꜧ}_{\lambda 2}, \quad h(\mathrm{O}_3) \supseteq \bigcap_{\lambda\in\mathfrak{X}} \text{Ꜧ}_{\lambda 3}.$$

Since, $\{\text{Ꜧ}^{\text{ꞑ}}{}_{\lambda}\}$ are ƝUÇO – sets in $Z \Rightarrow h^{-1}(\{\text{Ꜧ}^{\text{ꞑ}}{}_{\lambda}\})$ are ƝUÇO – sets in Y (by Definition 2.8). since, h is a bijective. Then:

$$O_1 = h^{-1}(f(O_1)) \subseteq h^{-1}(\cup_\lambda \text{Ꜧ}_{\lambda 1}) = \cup_\lambda h^{-1}(\text{Ꜧ}_{\lambda 1}),$$
$$O_2 = h^{-1}(h(O_2)) \supseteq h^{-1}(\cap_\lambda \text{Ꜧ}_{\lambda 2}) = \cap_\lambda h^{-1}(\text{Ꜧ}_{\lambda 2}),$$
$$O_3 = h^{-1}(h(O_3)) \supseteq h^{-1}(\cap_\lambda \text{Ꜧ}_{\lambda 3}) = \cap_\lambda h^{-1}(\text{Ꜧ}_{\lambda 3}).$$

Since, $O^{\text{ꞑ}}$ is a ƝUÇ – compact in Y.
Therefore, $O_1 \subseteq \cup_{i=1}^{m} h^{-1}(\text{Ꜧ}_{i1})$,
$O_2 \supseteq \cap_{i=1}^{m} h^{-1}(\text{Ꜧ}_{i2})$, $\quad O_3 \supseteq \cap_{i=1}^{m} h^{-1}(\text{Ꜧ}_{i3})$. Hence,

$$h(O_1) \subseteq \cup_{i=1}^{m} h(h^{-1}(\text{Ꜧ}_{i1})),$$
$$h(O_2) \supseteq \cap_{i=1}^{m} h(h^{-1}(\text{Ꜧ}_{i2})),$$
$$h(O_3) \supseteq \cap_{i=1}^{m} h(h^{-1}(\text{Ꜧ}_{i3})).$$

Therefore, $h(O^{\text{ꞑ}})$ is a $\text{Ɲ}_{\mathrm{U}}\text{Ç}$ – compact in Z.

3.11 Corollary

Let (Y, T_Y) and (Z, T_Z) be $\text{Ɲ}_{\mathrm{U}}\text{ÇT}_{(1,2)}$ – spaces and $h: Y \to Z$ be a $\text{Ɲ}_{\mathbf{U}}\text{Ç}$ – continuous$_{(1,2)}$ bijective. If (Y, T_Y) is a $\text{Ɲ}_{\mathrm{U}}\text{Ç}$ – compact then (Z, T_Z) is a $\text{Ɲ}_{\mathrm{U}}\text{Ç}$ – compact.

Proof: The demonstration unfolds in a manner mirroring the approach adopted in Theorem 3.10.

3.12 Theorem

Let $(\text{Ɱ}^{\text{ꞑ}}, \mathrm{T}_{\text{Ɱ}^{\text{ꞑ}}})$ be a Ɲ_s – $\text{Ɲ}_{\mathrm{U}}\text{ÇT}_{(1,2)}$ – space and $\text{Ł}^{\text{ꞑ}} =< \text{Ł}_1, \Phi, \text{Ł}_3 > \subseteq_1 \text{Ɱ}^{\text{ꞑ}} =< \text{Ɱ}_1, \Phi, \text{Ɱ}_3 >$. Then $\text{Ł}^{\text{ꞑ}}$ is a $\text{Ɲ}_{\mathrm{U}}\text{Ç}$ – compact in (Y, T_Y) if and only if $\text{Ł}^{\text{ꞑ}}$ is a $\text{Ɲ}_{\mathrm{U}}\text{Ç}$ – compact in $(\text{Ɱ}^{\text{ꞑ}}, \mathrm{T}_{\text{Ɱ}^{\text{ꞑ}}})$.

Proof: Let $\{\text{Ꜧ}^{\text{ꞑ}}{}_{\lambda} =< \text{Ꜧ}_{\lambda 1}, \Phi, \text{Ꜧ}_{\lambda 3} >: \lambda \in \mathfrak{X}\}$ be an $\text{Ɲ}_{\mathrm{U}}\text{Ç}$ – open cover of $\text{Ł}^{\text{ꞑ}}$ in $(\text{Ɱ}^{\text{ꞑ}}, \mathrm{T}_{\text{Ɱ}^{\text{ꞑ}}})$,

$\text{Ꜧ}^{\text{ꞑ}}{}_{\lambda} = \mathfrak{R}^{\text{ꞑ}}{}_{\lambda} \cap_1 \text{Ɱ}^{\text{ꞑ}} \ni \mathfrak{R}^{\text{ꞑ}}{}_{\lambda} \in \mathrm{T}_Y$.

Then $\text{Ł}_1 \subseteq \cup_{\lambda\in\mathfrak{X}} \text{Ꜧ}_{\lambda 1}$, $\text{Ł}_3 \supseteq \cap_{\lambda\in\mathfrak{X}} \text{Ꜧ}_{\lambda 3}$.

Thus, $\text{Ł}_1 \subseteq \cup_{\lambda\in\mathfrak{X}}(\mathfrak{R}_{\lambda 1} \cap \text{Ɱ}_1) = \cup_{\lambda\in\mathfrak{X}} \mathfrak{R}_{\lambda 1} \cap \text{Ɱ}_1$, $\text{Ł}_3 \supseteq \cap_{\lambda\in\mathfrak{X}}(\mathfrak{R}_{\lambda 3} \cup \text{Ɱ}_3) = \cap_{\lambda\in\mathfrak{X}} \mathfrak{R}_{\lambda 3} \cup \text{Ɱ}_3$

So, $\text{Ł}_1 \subseteq \cup_{\lambda\in\mathfrak{X}} \mathfrak{R}_{\lambda 1}$, $\text{Ł}_3 \supseteq \cap_{\lambda\in\mathfrak{X}} \mathfrak{R}_{\lambda 3}$

Hence, $\{\mathfrak{R}^{\text{ꞑ}}{}_{\lambda} =< \mathfrak{R}_{\lambda 1}, \Phi, \mathfrak{R}_{\lambda 3} >: \lambda \in \mathfrak{X}\}$ be an $\text{Ɲ}_{\mathrm{U}}\text{Ç}$ – open cover of $\text{Ł}^{\text{ꞑ}}$ in (Y, T_Y).

Since, $\text{Ł}^{\text{ꞑ}}$ is a $\text{Ɲ}_{\mathrm{U}}\text{Ç}$ – compact in (Y, T_Y), then $\text{Ł}_1 \subseteq \cup_{i=1}^{m} \mathfrak{R}_{i1}$, $\quad \text{Ł}_3 \supseteq \cap_{i=1}^{m} \mathfrak{R}_{i3}$.

Since, $\text{Ł}^{\text{ꞑ}} = \text{Ł}^{\text{ꞑ}} \cap_1 \text{Ɱ}^{\text{ꞑ}} =< \text{Ł}_1 \cap \text{Ɱ}_1, \Phi, \text{Ł}_3 \cup \text{Ɱ}_3 >$. Because, $\text{Ł}^{\text{ꞑ}} \subseteq_1 \text{Ɱ}^{\text{ꞑ}}$. Thus,

$\text{Ł}_1 = \text{Ł}_1 \cap \text{Ɱ}_1 \subseteq (\cup_{i=1}^{m} \mathfrak{R}_{i1}) \cap \text{Ɱ}_1 = \cup_{i=1}^{m}(\mathfrak{R}_{i1} \cap \text{Ɱ}_1) = \cup_{i=1}^{m} \text{Ꜧ}_{i1}$.

$\text{Ł}_3 = \text{Ł}_3 \cup \text{Ç}_3 \supseteq (\cap_{i=1}^{m} \mathfrak{R}_{i3}) \cup \text{Ɱ}_3 = \cap_{i=1}^{m}(\mathfrak{R}_{i3} \cup \text{Ɱ}_3) = \cap_{i=1}^{m} \text{Ꜧ}_{i3}$.

Hence, $\text{Ł}^{\text{ꞑ}}$ is a $\text{Ɲ}_{\mathrm{U}}\text{Ç}$ – compact in $(\text{Ɱ}^{\text{ꞑ}}, \mathrm{T}_{\text{Ɱ}^{\text{ꞑ}}})$.

Conversely, similarly, assume $\text{Ł}^{\text{ꞑ}}$ is a $\text{Ɲ}_{\mathrm{U}}\text{Ç}$ – compact in $(\text{Ɱ}^{\text{ꞑ}}, \mathrm{T}_{\text{Ɱ}^{\text{ꞑ}}})$.

Let $\{\mathfrak{R}^{\text{ꞑ}}{}_{\lambda} =< \mathfrak{R}_{\lambda 1}, \Phi, \mathfrak{R}_{\lambda 3} >: \lambda \in \mathfrak{X}\}$ be an $\text{Ɲ}_{\mathrm{U}}\text{Ç}$ – open cover of $\text{Ł}^{\text{ꞑ}}$ in (Y, T_Y).

Define $\text{Ꜧ}^{\text{ꞑ}}{}_{\lambda} = \mathfrak{R}^{\text{ꞑ}}{}_{\lambda} \cap_1 \text{Ɱ}^{\text{ꞑ}} \ni \mathfrak{R}^{\text{ꞑ}}{}_{\lambda} \in \mathrm{T}_Y$.

Then $\{\text{Ꜧ}^{\text{ꞑ}}{}_{\lambda} =< \text{Ꜧ}_{\lambda 1}, \Phi, \text{Ꜧ}_{\lambda 3} >$ be an $\text{Ɲ}_{\mathrm{U}}\text{Ç}$ – open cover in the subspace.

By compactness, a finite subcover exists, which implies a finite subcover in (Y, T_Y).

3.13 Definition

A family of $\text{Ɲ}_{\mathbf{U}}\text{Ç}$ – sets $\{\mathfrak{R}^{\text{ꞑ}}{}_{\lambda} =< \mathfrak{R}_{\lambda 1}, \mathfrak{R}_{\lambda 2}, \mathfrak{R}_{\lambda 3} > : \lambda \in \mathfrak{X}\}$ is known to have the finite intersection property (FIP) if and only if every finite subfamily $\{\mathfrak{R}^{\text{ꞑ}}{}_1, \mathfrak{R}^{\text{ꞑ}}{}_2, \dots, \mathfrak{R}^{\text{ꞑ}}{}_m\}$ satisfies $\cap_{1\,i=1}^{m} \mathfrak{R}^{\text{ꞑ}}{}_i \neq \Phi_1^{\text{ꞑ}}$.

3.14 Theorem

A $\text{Ɲ}_{\mathrm{U}}\text{ÇT}_{(1,2)}$ – space is ƝUÇ – compact if and only if for every family of ƝUÇO – $sets$ $\text{Ƒ}^{\text{ꞑ}}{}_{\lambda} =< \text{Ƒ}_{\lambda 1}, \text{Ƒ}_{\lambda 2}, \text{Ƒ}_{\lambda 3} >: \lambda \in \mathfrak{X}\}$ we have $\cap_{1\lambda} \text{Ƒ}^{\text{ꞑ}}{}_{\lambda} \neq \Phi_1^{\text{ꞑ}}$.

Proof: Suppose (Y, T) is a $\text{Ɲ}_{\mathrm{U}}\text{Ç}$ – compact and let $\{\text{Ƒ}^{\text{ꞑ}}{}_{\lambda} =< \text{Ƒ}_{\lambda 1}, \text{Ƒ}_{\lambda 2}, \text{Ƒ}_{\lambda 3} >: \lambda \in \mathfrak{X}\}$ family of ƝUÇÇ – $sets$ having the finite intersection property.

Assume that $\cap_{1\lambda} \text{Ƒ}^{\text{ꞑ}}{}_{\lambda} = \Phi_1^{\text{ꞑ}}$. Then $\cap_{1\lambda} \text{Ƒ}^{\text{ꞑ}}{}_{\lambda} =< \cap_{\lambda\in\mathfrak{X}} \text{Ƒ}_{\lambda 1}, \cap_{\lambda\in\mathfrak{X}} \text{Ƒ}_{\lambda 2}, \cup_{\lambda\in\mathfrak{X}} \text{Ƒ}_{\lambda 3} > =< \Phi, \Phi, Y >$.

Taking complements, we get: $\text{Ƒ}^{\text{ꞑ}}{}_{\lambda}{}^{c_2} =< \text{Ƒ}_{\lambda 3}, \text{Ƒ}_{\lambda 2}, \text{Ƒ}_{\lambda 1} >$ and hence,

$\cup_{2\lambda} \text{Ƒ}^{\text{ꞑ}}{}_{\lambda}{}^{c_2} = \; < \cup_{\lambda\in\mathfrak{X}} \text{Ƒ}_{\lambda 3}, \cap_{\lambda\in\mathfrak{X}} \text{Ƒ}_{\lambda 2}, \cap_{\lambda\in\mathfrak{X}} \text{Ƒ}_{\lambda 1} > =< Y, \Phi, \Phi > = Y_1^{\text{ꞑ}}$.

Since, $\text{Ƒ}^{\text{ꞑ}}{}_{\lambda}$ is a ƝUÇÇ – $sets$ $\forall \lambda$. Hence, $\cup_{2\lambda} \text{Ƒ}^{\text{ꞑ}}{}_{\lambda}{}^{c_2} =< \cup_{\lambda\in\mathfrak{X}} \text{Ƒ}_{\lambda 3}, \cap_{\lambda\in\mathfrak{X}} \text{Ƒ}_{\lambda 2}, \cap_{\lambda\in\mathfrak{X}} \text{Ƒ}_{\lambda 1} >$ is a ƝUÇO – $sets$ $\forall \lambda$.

So, $\{\text{Ƒ}^{\text{ꞑ}}{}_{\lambda}{}^{c_2}\}$ is a ƝUÇ – open of $\mathrm{Y}_1^{\text{ꞑ}}$. Hence,

Y=λ∈ẊƑλ3, Φ=λ∈ẊƑλ2, Φ=λ∈ẊƑλ1Y =
$\cup_{\lambda\in\mathfrak{X}} \text{Ƒ}_{\lambda 3}$, $\Phi = \cap_{\lambda\in\mathfrak{X}} \text{Ƒ}_{\lambda 2}$, $\quad \Phi = \cap_{\lambda\in\mathfrak{X}} \text{Ƒ}_{\lambda 1}$.

Since, $\mathrm{Y}_1^{\text{ꞑ}}$ is a ƝUÇ – compact. So,

$$\mathrm{Y} = \bigcup_{i=1}^{m} \text{Ƒ}_{i3}, \qquad \Phi = \bigcap_{i=1}^{m} \text{Ƒ}_{i2}, \qquad \Phi = \bigcap_{i=1}^{m} \text{Ƒ}_{i1}.$$

Thus, $\cap_{1\,i=1}^{m} \text{Ƒ}^{\text{ꞑ}}{}_{\lambda} = \; < \cap_{i=1}^{m} \text{Ƒ}_{i1}, \cap_{i=1}^{m} \text{Ƒ}_{i2}, \cup_{i=1}^{m} \text{Ƒ}_{i3} > =< \Phi, \Phi, \mathrm{Y} > = \Phi_1^{\text{ꞑ}}$, which is a contradiction.

Therefore, $\cap_{1\lambda} \text{Ƒ}^{\text{ꞑ}}{}_{\lambda} \neq \Phi_1^{\text{ꞑ}}$.

Conversely, suppose (Y, T) is not $\mathrm{N_U Ç}-$ compact. $\exists$ $\{\Re^{n}{}_{\lambda} =< \Re_{\lambda 1}, \Re_{\lambda 2}, \Re_{\lambda 3} >: \lambda \in \mathrm{X}\}$ be an $\mathrm{N_U Ç}-$ open cover of Y_1^{n}

Such that $\mathrm{Y} = \bigcup_{\lambda \in \mathrm{X}} \Re_{\lambda 1}$, $\Phi = \bigcap_{\lambda \in \mathrm{X}} \Re_{\lambda 2}$, $\Phi = \bigcap_{\lambda \in \mathrm{X}} \Re_{\lambda 3}$, but for every finite subfamily:

$$\mathrm{Y} \neq \bigcup_{i=1}^{m} \Re_{\lambda 1}, \quad \Phi \neq \bigcap_{i=1}^{m} \Re_{i2}, \quad \Phi \neq \bigcap_{i=1}^{m} \Re_{i3}.$$

Consider the family $\{\Re^{n}{}_{\lambda}{}^{c_2} =< \Re_{\lambda 3}, \Re_{\lambda 2}, \Re_{\lambda 1} > \}$.

Then:

$\bigcap_{1\lambda} \Re^{n}{}_{\lambda}{}^{c_2} =< \bigcap_{\lambda \in \mathrm{X}} \Re_{\lambda 3}, \bigcap_{\lambda \in \mathrm{X}} \Re_{\lambda 2}, \bigcup_{\lambda \in \mathrm{X}} \Re_{\lambda 1} > = < \Phi, \Phi, \mathrm{Y} > = \Phi_1^{n}$, but every finite intersection is non – empty. Thus, this family has the finite intersection property but empty total intersection, which is a contradiction. Hence, (Y, T) is a $\mathrm{N_U Ç}-$ compact.

4 NEUTROSOPHIC CRISP LINDELÖF SPACE

The following section, we study the Lindelöf property, and countable open covers in $\mathrm{N_U ÇT_{(1,2)}}-$ space. We present the definition of a neutrosophic crisp Lindelöf space with respect to the topology given in Definition (2.7). Moreover, a collection of fundamental theorems concerning this property are presented along with their rigorous proofs. In addition, several results under additional conditions are also discussed.

4.1 Definition

Let (Y, T) be a $\mathrm{N_U ÇT_{(1,2)}}-$ space, and $\mathrm{Q}^{n} =< \mathrm{Q}_1, \mathrm{Q}_2, \mathrm{Q}_3 >$ be a $\mathrm{N_U Ç}-$ set in Y. A family:

$\mathrm{G} = \{\Re^{n}{}_{\lambda} =< \Re_{\lambda 1}, \Re_{\lambda 2}, \Re_{\lambda 3} >: \lambda \in \mathrm{X}\}, \Re^{n}{}_{\lambda} \in \mathrm{T} \ni$

$$\mathrm{Q}_1 \subseteq \bigcup_{\lambda \in \mathrm{X}} \Re_{\lambda 1}, \mathrm{Q}_2 \supseteq \bigcap_{\lambda \in \mathrm{X}} \Re_{\lambda 2}, \quad \mathrm{Q}_3 \supseteq \bigcap_{\lambda \in \mathrm{X}} \Re_{\lambda 3}.$$

If, in addition, the index set X is countable, then G is called a countable $\mathrm{N_U Ç}-$open cover of Q^{n}.

4.2 Definition

A $\mathrm{N_U Ç}-$ set Q^{n} of a $\mathrm{N_U ÇT_{(1,2)}}-$ space is called $\mathrm{N_U Ç}-$ Lindelöf if $\forall$ $\{\Re^{n}{}_{\lambda} =< \Re_{\lambda 1}, \Re_{\lambda 2}, \Re_{\lambda 3} >: \lambda \in \mathrm{X}\}, \Re^{n}{}_{\lambda} \in \mathrm{T}$, and $\mathrm{Q}_1 \subseteq \bigcup_{\lambda \in \mathrm{X}} \Re_{\lambda 1}$, $\mathrm{Q}_2 \supseteq \bigcap_{\lambda \in \mathrm{X}} \Re_{\lambda 2}$, $\mathrm{Q}_3 \supseteq \bigcap_{\lambda \in \mathrm{X}} \Re_{\lambda 3} \Rightarrow \exists\{\lambda_i\} \subseteq \mathrm{X}, \mathrm{X}$ is countable such that:

$$\mathrm{Q}_1 = \bigcup_{i=1}^{\infty} \Re_{\lambda_i 1}, \mathrm{Q}_1 = \bigcap_{i=1}^{\infty} \Re_{\lambda_i 2}, \mathrm{Q}_1 = \bigcap_{i=1}^{\infty} \Re_{\lambda_i 3}.$$

- Y_1^{n} is a $\mathrm{N_U Ç}-$ Lindelöf if and only if

$$\mathrm{Y} = \bigcup_{i=1}^{\infty} \Re_{\lambda_i 1}, \quad \Phi = \bigcap_{i=1}^{\infty} \Re_{\lambda_i 2}, \Phi = \bigcap_{i=1}^{\infty} \Re_{\lambda_i 3}.$$

- If Y_1^{n} is a $\mathrm{N_U Ç}-$ Lindelöf, then $\mathrm{N_U ÇT_{(1,2)}}-$ space is a $\mathrm{N_U Ç}-$ Lindelöf space.

4.3 Example

Let Y = N and $K_1{}^{n} = < \{2,3, \ldots\}, \ \Phi, \ \Phi >, K_2{}^{n} =< \{3,4, \ldots\}, \ \Phi, \ \{1\} >, K_3{}^{n} =< \{4,5 \ldots\}, \ \Phi, \ \{1,2\} >$,

$K_m{}^{n} =< \{m+1, m+2, \ldots\}, \ \Phi, \ \{1,2,3, \ldots, m-1\}$.

Then $\mathrm{T} = \{\Phi_1^{n}, \mathrm{Y}_1^{n}\} \cup K_m{}^{n}$ is a $\mathrm{N_U ÇT_{(1,2)}}-$ space.

Step 1:

Consider the family $K^{n}{}_{m, m \in N}$.

First component: $\bigcup_{m \in N} K_{m1} = N$.

Second component: $\bigcap_{m \in N} K_{m2} = \Phi$, because $K_{m2} = \Phi, \ m \in N$

Third component: $\bigcap_{m \in N} K_{m3} = \Phi$, because $K_{m3} = \{1,2,3, \ldots, m-1\}, m \in N$

Step 1:

The family $K^{n}{}_{m, m \in N}$ is already countable.

Hence, we can choose the subfamily: $\{K^{n}{}_{m}\}_{m \in N}$

Therefore: $\mathrm{Y} = \bigcup_{m \in N} K_{m1}$, $\Phi = \bigcap_{m \in N} K_{m2}$, $\Phi = \bigcap_{m \in N} K_{m3}$.

Hence, every $\mathrm{N_U Ç}-$ open cover of Y_1^{n} admits a countable subcover. Therefore (Y, T) is a $\mathrm{N_U Ç}-$ Lindelöf space.

4.4 Theorem

Every $\mathrm{N_U Ç}-$ compact space is a $\mathrm{N_U Ç}-$ Lindelöf space

Proof: Suppose that Y_1^{n} is a $\mathrm{N_U Ç}-$ compact. Then every $\mathrm{N_U Ç}-$ open cover of Y_1^{n} has a finite $\mathrm{N_U Ç}-$ open subcover.

Since every finite set is countable, it follows that every such cover admits a countable $\mathrm{N_U Ç}-$ open subcover. Therefore, Y_1^{n} is a $\mathrm{N_U Ç}-$ Lindelöf space.

4.5 Remark

The implication stated in Theorem 4.4 cannot be extended to its converse. This is supported by Example 3.5, where the space (Y, T) is a $\mathrm{N_U Ç}-$ Lindelöf but not $\mathrm{N_U Ç}-$ compact.

4.6 Corollary

If $\mathcal{N}^{n}$ is finite $\mathrm{N_U Ç}$ − set in a $\mathrm{N_U ÇT_{(1,2)}}$ − space, then it is a $\mathrm{N_U Ç}$ − Lindelöf set.

Proof: Since every finite $\mathrm{N_U Ç}$ − set is $\mathrm{N_U Ç}$ − compact by Theorems 3.6, and Every $\mathrm{N_U Ç}$ − compact space is a $\mathrm{N_U Ç}$ – $\mathrm{N_U Ç}$ − Lindelöf by Theorems 4.4.

4.7 Corollary

Let (Y, T) be a $\mathrm{N_U ÇT_{(1,2)}}$ − space and $\mathcal{N}^{n} =< \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3 >$ be a $\mathrm{N_U Ç}$ – countable set in Y. Then $\mathcal{N}^{n}$ is a $\mathrm{N_U Ç}$ − Lindelöf.

Proof: Since $\mathcal{N}^{n}$ is a $\mathrm{N_U Ç}$ – countable set, for all $\mathrm{N_U Ç}$ − open cover of $\mathcal{N}^{n}$ has a countable subcover. Therefore, $\mathcal{N}^{n}$ is a $\mathrm{N_U Ç}$ − Lindelöf.

4.8 Theorem

Let (Y, T) be a $\mathrm{N_U ÇT_{(1,2)}}$ − space and $F^{n} =< F_1, \Phi, F_1{}^{c} >$ be a $\mathrm{N_U ÇÇ}$ – *set* in Y. If (Y, T) is a $\mathrm{N_U Ç}$ − Lindelöf, then F^{n} is a $\mathrm{N_U Ç}$ − Lindelöf.

Proof: The demonstration unfolds in a manner mirroring the approach adopted in Theorem 3.7.

4.9 Theorem

Let (Y, T_Y) and (Z, T_Z) be $\mathrm{N_U ÇT_{(1,2)}}$ − spaces and let $h : Y \to Z$ be a $\mathrm{N_U Ç}$ − continuous$_{(1,2)}$ bijection. If K^{n} is a $\mathrm{N_U Ç}$ − Lindelöf in Y, then $h(K^{n})$ is a $\mathrm{N_U Ç}$ − Lindelöf in Z.

Proof: The reasoning mirrors the approach adopted in the proof of Theorem 3.10.

4.10 Corollary

Let (Y, T_Y) and (Z, T_Z) be $\mathrm{N_U ÇT_{(1,2)}}$ − spaces and let $h : Y \to Z$ be a $\mathrm{N_U Ç}$ − continuous$_{(1,2)}$ bijection. If (Y, T_Y) is a $\mathrm{N_U Ç}$ − Lindelöf, then (Z, T_Z) is a $\mathrm{N_U Ç}$ − Lindelöf.

Proof: The demonstration unfolds in a manner mirroring the approach adopted in Theorem 3.10.

4.11 Theorem

Let $(M^{n}, T_{M^{n}})$ be a $\mathrm{N_s} - \mathrm{N_U ÇT_{(1,2)}}$ − space, and let $Ł^{n} =< Ł_1, \Phi, Ł_3 > \subseteq_1 M^{n}$. Then $Ł^{n}$ is a $\mathrm{N_U Ç}$ − Lindelöf in $(M^{n}, T_{M^{n}})$ if and only if $Ł^{n}$ is a $\mathrm{N_U Ç}$ − Lindelöf in (Y, T_Y)

The proof follows the same approach as in Theorem 3.11.

4.12 Theorem

Let $Q^{n} =< Q_1, Q_2, Q_3 >$ and $H^{n} =< H_1, H_2, H_3 >$ be $\mathrm{N_U Ç}$ − Lindelöf sets in $\mathrm{N_U ÇT_{(1,2)}}$ − space. Then $Q^{n} \cup_2 H^{n}$ is a $\mathrm{N_U Ç}$ − Lindelöf set.

Proof: The proof follows the same approach as in Theorem 3.8.

4.13 Corollary

Let $Q^{n} =< Q_1, Q_2, Q_3 >$ be a $\mathrm{N_U Ç}$ − Lindelöf set, and let $F^{n} =< F_1, \Phi, F_1{}^{c} >$ be a $\mathrm{N_U ÇÇ}$ – *set* in $\mathrm{N_U ÇT_{(1,2)}}$ − space. Then $Q^{n} \cap_1 F^{n}$ is a $\mathrm{N_U Ç}$ − Lindelöf set.

Proof: Let $\{\Re^{n}{}_{\lambda} =< \Re_{\lambda 1}, \emptyset, \Re_{\lambda 3} >: \lambda \in \Lambda\}$ be an $\mathrm{N_U Ç}$ − open covering of $Q^{n} \cap_1 F^{n} =< Q_1 \cap F_1, \Phi, Q_3 \cup F_1{}^{c} >$.

Then:

$$Q_1 \cap F_1 \subseteq \bigcup_{\lambda \in \Lambda} \Re_{\lambda 1}, \qquad L_3 \cup F_1{}^{c} \supseteq \bigcap_{\lambda \in \Lambda} \Re_{\lambda 3}.$$

Since F^{n} is a $\mathrm{N_U ÇÇ}$ – *set* $\Rightarrow F^{n, c_2}$ is a $\mathrm{N_U ÇO}$ – *set*.

Hence, $\{\Re^{n}{}_{\lambda} : \lambda \in \Lambda\} \cup_2 \{F^{n, c_2}\}$ is a $\mathrm{N_U Ç}$ − open cover of Q^{n}.

Thus,

$$Q_1 \subseteq \bigcup_{\lambda \in \Lambda} \Re_{\lambda 1} \cup F_1{}^{c}, \quad Q_3 \supseteq \bigcap_{\lambda \in \Lambda} \Re_{\lambda 3} \cap F_1$$

Since, Q^{n} has the $\mathrm{N_U Ç}$ − Lindelöf property, each $\mathrm{N_U Ç}$ − open cover of Q^{n} contains a countable subcover such that:

$$Q_1 \subseteq (\bigcup_{i=1}^{\infty} \Re_{\lambda_i 1}) \cup F_1{}^{c}, \qquad Q_3 \supseteq (\bigcap_{i=1}^{\infty} \Re_{\lambda_i 3}) \cap F_1.$$

Thus,

$$Q_1 \cap F_1 \subseteq \bigcup_{i=1}^{\infty} (\Re_{\lambda_i 1} \cap F_1) \cup F_1{}^{c} \subseteq \bigcup_{i=1}^{\infty} \Re_{\lambda_i 1},$$

$$Q_3 \cup F_1{}^{c} \supseteq \bigcap_{i=1}^{\infty} (\Re_{\lambda_i 3} \cup F_1{}^{c}) \cap F_1 \supseteq \bigcap_{i=1}^{\infty} \Re_{\lambda_i 3}.$$

Hence, $Q^{n} \cap_1 F^{n}$ is a $\mathrm{N_U Ç}$ − Lindelöf set.

5 CONCLUSIONS

This paper has investigated the properties of compactness and Lindelöfness in neutrosophic crisp topological spaces defined by a topology based on Type-I intersection and Type-II union and complement. The obtained results demonstrate that the choice of algebraic operations in neutrosophic crisp sets plays a crucial role in determining the behavior of topological properties. In particular, it has been shown that while some classical results from general topology remain valid within this framework, others fail to hold or require additional conditions to be satisfied. This confirms that the considered structure is not merely a direct extension of classical topology; rather, it fundamentally depends on the nature of the underlying algebraic operations.

Furthermore, these findings contribute to strengthening the theoretical understanding of neutrosophic crisp topological spaces and indicate promising directions for future research. In particular, the investigation of related notions such as semi-compactness and semi-Lindelöfness, as well as the adoption of alternative algebraic structures, may provide deeper insights and lead to broader applications in this field.

REFERENCES

[1] A. Salama, F. Smarandache, and V. Kroumov, Neutrosophic Crisp Sets & Neutrosophic Crisp Topological Spaces. Infinite Study, 2014.

[2] A. Salama, F. Smarandache, and V. Kroumov, "Neutrosophic closed set and neutrosophic continuous functions," Neutrosophic Sets and Systems, vol. 4, pp. 2-8, 2014.

[3] M. M. Abdulkadhim, Q. H. Imran, A. H. Al-Obaidi, and S. Broumi, "On neutrosophic crisp generalized alpha generalized closed sets," International Journal of Neutrosophic Science, vol. 19, no. 1, pp. 107-115, 2022.

[4] D. N. Tomma and L. A. A. Al-Swidi, "Necessary and sufficient conditions for a stability of the concepts of stable interior and stable exterior via neutrosophic crisp sets," International Journal of Neutrosophic Science, vol. 24, no. 1, 2024.

[5] H. AL. Salman and R. D. Ali, "Special Properties of Operators in Neutrosophic Crisp Topological Spaces," F1000Research, vol. 15, p. 216, 2026.

[6] N. M. Easi, L. A. A. Jabar, and A. H. Al-Obaidi, "Kernel and Weakly Ultra-Separated Relationship with Separation Axioms in Stable Neutrosophic Crisp Topological Spaces," Neutrosophic Sets and Systems, vol. 97, pp. 1-8, 2026.

[7] A. B. AL-Nafee, R. K. Al-Hamido, and F. Smarandache, Separation Axioms in Neutrosophic Crisp Topological Spaces. Infinite Study, 2019.

[8] A. Salama and F. Smarandache, Neutrosophic Crisp Set Theory. Infinite Study, 2015.

[9] A. Salama, I. M. Hanafy, and M. S. Dabash, Semi-Compact and Semi-Lindelöf Spaces via Neutrosophic Crisp Set Theory. Infinite Study, 2018.

[10] X. Zhang, M. Li, and T. Lei, "On neutrosophic crisp sets and neutrosophic crisp mathematical morphology," Neutrosophic Sets and Systems, vol. 43, no. 1, p. 1, 2021.

[11] K. Hur, P. K. Lim, J. G. Lee, and J. Kim, "The category of neutrosophic crisp sets," Annals of Fuzzy Mathematics and Informatics, vol. 14, no. 1, pp. 43-54, 2017.