Certain Classes of Extended Continuous Mappings Involving New Extended Classes of Omega-Open Sets

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Abstract: In this paper, we introduce and analyse new classes of generalized continuous mappings in topological spaces using newly defined classes of ω-open sets. Specifically, ω-h-continuous and ω-h*-continuous functions are defined and investigated; a new generalized class, ω-h-irresolute functions is also proposed within the same framework. A few fundamental properties and characterizations of these mappings are described. Moreover, the associations among the proposed classes and some previously known types of generalized continuous and irresolute functions are analysed systematically. This study supports some inclusion results between the associated classes of generalized open and closed sets, and employs them to infer corresponding implications between the introduced types of mappings. Additionally, it discusses characterization theorems for ω-h-continuous and ω-h-irresolute functions with respect to inverse images, closures and interiors. Not only that, we research how these functions behave under restriction and composition, and we learn some of the conditions which contribute to preserving the newly introduced continuity properties. Some examples are provided to show that certain converse implications do not in general hold, and to elucidate the independence of elements and sharpness of the reported results. Results derived from that research expand and synthesize a number of the ideas already built in generalized topology, as well as providing a larger framework for examining generalized continuity in topological spaces generally.

1 INTRODUCTION

Generalized continuous (cont., for short) mappings and of course generalized forms of irresolute functions are the most important research concepts in theoretical and applied mathematical sciences, and they have been presented and discussed over the course of numerous years. Therefore, H. Z. Hdeib [1] defined and studied the concept of ω -open sets (ω -op.-sts) and ω-closed sets (ω-clo.-sts, for short) as well he presented in 1989 [2] the idea of ω -continuous (ω -cont., for short) functions as a generalized of continuity. Afterward, novel types of extended clo.-sts namely, regular extended ω-clos.-sts and regular extended ω-cont. functions have been presented via Al-Omari and Noorani in [3] and they are studied their fundamental characterizations. Also, other weak forms of cont. functions have been introduced and discussed by A. Al-Omari et al, [4] and studied some of their properties. Next, C. Carpintero et al, [5] described the notion of faint ω -continuity and presented novel characterizations related to this kind of continuity. Subsequently new kind expanded of ω_β -clo. -sts has been offered via H. Aljarrah et al, [6] and studied the necessary characterizations for this type of ω -clo.-sts. After that, B. J. Waqas and H. J. Ali [7] studied another form of cont. functions namely, contra ω pre-cont. functions, via the notion of ω pre-op.-sts. As well in the same year P. Sasmaz and M. Ozkoc [8], defined the notion of δω-op.-sts and presented some kinds of continuity based on δω-op.- sts.

Next, F. Abbas [9] defined new types of open sets namely, h -open sets (h -op.-sts, for short), also he studied some properties of h -interior, h -closure, and presented several types of h-functions in topological spaces (top-sp,s, for short) such as h-cont., involving original expanded ω-op.-sts and several basic properties related these kinds of expanded spaces have been discussed. In addition, Haza, Z. M. [10] explored a new type of mappings, namely; w-proper mappings.

In article, we will study continue and kinds of functions via new types of ω-op.-sts in top.-sp,s.

The aim of the present this article is to present and study types of generalized of cont. functions namely, $\omega_{\hbar}$-cont. and $\omega_{\hbar^*}$-cont. functions. Several motivating characterizations and fundamental properties related to these new kinds of functions are obtained. Additionally, the relationships among such of these kinds of functions are studied.

2 PRELIMINARIES

Throughout this paper, $\mathcal{X}$ and Υ denote top.-sp's. For any subset $\mathcal{A}$ of top.-sp's $\mathcal{X}$, the $\boldsymbol{\omega}_{\hbar}$-interior and $\boldsymbol{\omega}_{\hbar}$-closure are symbolizes, $\mathrm{Int}_{\boldsymbol{\omega}_{\hbar}}(\mathcal{A})$ and $\mathcal{C}\mathrm{l}_{\boldsymbol{\omega}_{\hbar}}(\mathcal{A})$ respectively. In this section, we study some definitions and the essential concepts that play important role in our study.

Definition 2.1: **[1]** Let $\mathcal{U}$ be a subset of a top.-sp's $\mathcal{X}$.The set $\mathcal{U}$ is called $\boldsymbol{\omega}$-op.-st, if for any point $x \in \mathcal{U}$ there is an op.-st V containing x , such that $\mathrm{V} - \mathcal{U}$ is countable, the complement of $\boldsymbol{\omega}$-op.-st is $\boldsymbol{\omega}$-clo.-st, the set off all of $\boldsymbol{\omega}$-op.-sts denoted by $\boldsymbol{\omega}\mathrm{O}(\mathcal{X})$.

Definition 2.2: **[9]** Suppose G of a top.-sp's $\mathcal{X}$ is $\hbar$-op.-st, if for any nonempty op.-st V in $\mathcal{X}$, $\mathrm{V}\neq \mathcal{X}$, $G \subseteq Int\ (G \cup \mathrm{V})$, the complement of $\hbar$-op. is called $\underline{\hbar}$-clo.-st and the set of all $\hbar$-op.-sts is symbolized by $\underline{\hbar o(\mathcal{X})}$.

Remark 2.3: The relation between $\boldsymbol{\omega}$-op.-st and $\underline{\hbar}$-op.-st are independent see **[5]**.

Propositions 2.4: The following statements hold for any sp. $\mathcal{X}$:

(i) Every op.-set is $\boldsymbol{\omega}$-op.-set. **[1]**

(ii) Every op.-set is $\hbar$-op.-set. **[9]**

Remark 2.5: The converses of propositions (2. 4) are not true see **[1]** and **[9]**, respectively.

Definition 2.6: **[2]** Let $F\colon \mathcal{X} \longrightarrow \Upsilon$ be a mapping. The function, F is $\boldsymbol{\omega}$-cont., if for every $\boldsymbol{\omega}$- op.-st V in Υ , the inverse image $F^{-1}(\mathrm{V})$ is $\boldsymbol{\omega}$-op.-set. in $\mathcal{X}$.

Definition 2.7: **[9]** A mapping $F\colon \mathcal{X} \longrightarrow \Upsilon$ is called $\hbar$ -cont. if the inverse image every op.-st. of Υ is an $\hbar$ - op.-set in $\mathcal{X}$.

Definition 2.8: **[9]** **A** mapping $F\colon \mathcal{X} \longrightarrow \Upsilon$ is said to be $\hbar$-irresolute, if for all $\hbar$-op.-set $\mathcal{U}$ in Υ , $F^{-1}\ (\mathcal{U})$ is $\hbar$ - op.-set in $\mathcal{X}$.

Proposition 2.9: **[9]** Every cont. is $\hbar$-cont. function.

The following example to show the convers of Proposition (2.9), is false**.**

Example 2.10: **[9]** Assume that $\mathcal{X} = \{1,2,3\}, \mathrm{T} = \{\emptyset, \mathcal{X}, \{1,2\}\}$ and $\Upsilon = \{4, 5, 6\}, T_{\Upsilon} = \{\emptyset, \Upsilon, \{4\}, \{4,5\}\}$ with $F\colon \mathcal{X} \to \Upsilon$ defined by $F\ (1)\ =\ 4, F\ (2)\ =\ 6, F\ (3)\ =\ 5$, so F is $\underline{\hbar}$ -cont. but is not cont.

Definition 2.11: **[11]** Let $F\colon \mathcal{X} \longrightarrow \Upsilon$ and $\mathrm{G} : \Upsilon \longrightarrow Z$ be functions defined as for all $x \in Z$, $F(x)\ =\ \mathrm{G}(x)$ is the restriction F on Z and $F|_Z\colon Z \longrightarrow \Upsilon$ where $G\ =\ F|_Z$.

3 SOME CHARACTERIZATIONS OF $\boldsymbol{\omega}_{\hbar}$-CONTINUOUS FUNCTIONS

This section, devoted to present various characterizations and important properties related $\omega_{\hbar}$-open set with some properties also new types of continuous functions via $\omega_{\hbar}$-open. set which are $\omega_{\hbar}$-cont. and $\omega_{\hbar^*}$-cont. functions, as well to verify some certain properties of $\omega_{\hbar}$-irresolute functions in top.-sp's (Fig. 1).

Definition 3.1: Let $\mathcal{X}$ be a top.-sp's. A subset $\mathcal{A} \subseteq \mathcal{X}$ is called $\omega_{\hbar}$-op. if for every $x \in \mathcal{A}$ there exists an $\hbar$-op.-st G_x containing x such that $G_x - \mathcal{A}$ is countable. Equivalently, the complement of $\omega_{\hbar}$-op.-st.is $\omega_{\hbar}$-clo. st.

Example 3. 2: Consider the set $\mathcal{X} = \{a, b, c\}$ with topology $\mathrm{T} = \{\ \emptyset, \mathcal{X}, \{a\}, \{a, b\}\}$. Then The collection of $\omega_{\hbar}$ - op.-st in $\mathcal{X}$ can be written as

$$\mathrm{T}_{\hbar} = \{\emptyset, \mathcal{X}, \{a\}, \{b\}, \{a, b\}, \{b, a\}\}.$$

Propositions 3. 3: Let $\mathcal{X}$ be top.-sp's, then:

1) Every op.-st (respectively, clo.-st) is $\omega_{\hbar}$-op.-st (respectively, $\omega_{\hbar}$-clo.-st).
2) Every ω-op.-st (respectively, ω- clo.-st) is $\omega_{\hbar}$-op.-st (respectively, $\omega_{\hbar}$-clo.-st)
3) Every $\hbar$-op.-st (respectively, $\hbar$-clo.-st) is $\omega_{\hbar}$-op.-st (respectively, $\omega_{\hbar}$-clo.-st).

Definition 3. 4: A point $x \in G$ is $\omega_{\hbar}$-interior point of G iff there is $\omega_{\hbar}$-op.-set H_x containing x such that $x \in H_x \subseteq \mathrm{G}$. All $\omega_{\hbar}$-interior point of G is called $\omega_{\hbar}$-interior set of G and symbolizes by $Int_{\omega_{\hbar}}(G)$. That is $Int_{\omega_{\hbar}}(G) = \{\ x \in G, \exists\ \ \omega_{\hbar}\text{-op.-st}\ H\colon\ \mathrm{x} \in H \subseteq G\}$.

Proposition 3. 5: Let G, H be two subsets of a top.-sp $\mathcal{X}$, then:

1- $Int_{\omega_{\hbar}}(G) \subseteq G$.

2- $Int_{\omega_{\hbar}}(G))$ is $\omega_{\hbar}$-op.-set.

1- $Int_{\omega_{\hbar}}(G) = G$ *iff* G is $\omega_{\hbar}$-op.-set.

Proof:-1and 2 satisfy from Definition 3. 4.

3- Suppose G is $\omega_{\hbar}$ -open set, from 1, we have $Int_{\omega_{\hbar}}(G) \subseteq G$.......(*)

Let $x \in G$, then $x \in G \subseteq G$, since G is $\omega_{\hbar}$-op.-st., so $x \in Int_{\omega_{\hbar}}(G)$. Hence, $G \subseteq Int_{\omega_{\hbar}}(G)$.......(**). From $(*)$ and $(**)$, we get $G = Int_{\omega_{\hbar}}(G)$

Conversely, suppose $Int_{\omega_{\hbar}}(G) = G$, and let $x \in G$, so $x \in Int_{\omega_{\hbar}}(G)$, from Definition 3. 4, there is $\omega_{\hbar}$-op.-st P_x containing x such that $x \in P_x \subseteq$ G.That is $x \in \bigcup_{x \in G} P_x \subseteq$ G and $G \subseteq \bigcup_{x \in G} P_x$, this implies $G = \bigcup_{x \in G} P_x$, so G is $\omega_{\hbar}$-op.-st.

Definition3.6: Let (X, T) be a top. sp. with $\mathcal{U} \subseteq X$, a point $x \in X$ is $\omega_{\hbar}$-limit point of $\mathcal{U}$, if for each $\omega_{\hbar}$-op.-set $V, x \in V$, then $\mathcal{U} \cap V - \{x\} \neq \emptyset$.

The set of all $\omega_{\hbar}$-limit points of $\mathcal{U}$ is the $\omega_{\hbar}$-derived set of $\mathcal{U}$ and denoted by $D_{\omega_{\hbar}}(\mathcal{U})$.

Proposition3.7 : Let $\mathcal{X}$ be top.-sp., and $G, H \subseteq \mathcal{X}$. Then the following properties are hold:

1- If $G \subseteq H$, then $D_{\omega_{\hbar}}(G) \subseteq D_{\omega_{\hbar}}(H)$.

2- G is $\omega_{\hbar}$-clo.-st. iff $D_{\omega_{\hbar}}(G) \subseteq G$.

Proof:-1- Let $x \in D_{\omega_{\hbar}}(G)$, then by definition 3,6 ,for each $\omega_{\hbar}$-op.-set $V, x \in V$, then $G \cap V - \{x\} \neq \emptyset$. then since $G \subseteq H$, we get for each $\omega_{\hbar}$-op.-set $V, x \in V$, then $H \cap V - \{x\} \neq \emptyset$. That is $x \in D_{\omega_{\hbar}}(H)$. Therefore, $D_{\omega_{\hbar}}(G) \subseteq D_{\omega_{\hbar}}(H)$.

3- Let $x \notin G$, then $x \in G^c$, so there is $\omega_{\hbar}$-op.-st. N, with $x \in N$ and $N \subseteq G^c$, that is $N \cap G = \emptyset$, then $N - \{x\} \cap G = \emptyset, x \notin D_{\omega_{\hbar}}(G)$, therefore $D_{\omega_{\hbar}}(G) \subseteq G$.

Conversely, let $x \in G^c$, then $x \notin G$, from hypothesis, $D_{\omega_{\hbar}}(G) \subseteq G$, hence $x \notin D_{\omega_{\hbar}}(G)$, so there is $\omega_{\hbar}$-op.- st. $M, x \in M$ with $M - \{x\} \cap G = \emptyset$, then $M \subseteq G^c$, that is G is $\omega_{\hbar}$-clo.-st.

Definition 3.8: Let $\mathcal{X}$ be top.-sp and $G \subseteq \mathcal{X}$, the $\omega_{\hbar}$-closure of G is the intersection of all $\omega_{\hbar}$-clo.-sts in x containing G, denoted by $\mathcal{C}l_{\omega_{\hbar}}(G)$. As well, $\mathcal{C}l_{\omega_{\hbar}}(G)$ is $\omega_{\hbar}$-clo.-set for every subset G of $\mathcal{X}$. That is $\mathcal{C}l_{\omega_{\hbar}}(G) = G \cup D_{\omega_{\hbar}}(G)$.

Proposition 3. 9: Let $\mathcal{X}$ be top.-sp., and $G, H \subseteq \mathcal{X}$. Then the following properties are hold:

1) $G \subseteq \mathcal{C}l_{\omega_{\hbar}}(G)$.
2) If $G \subseteq H$, then $\mathcal{C}l_{\omega_{\hbar}}(G) \subseteq \mathcal{C}l_{\omega_{\hbar}}(H)$.
3) G^c is $\omega_{\hbar}$-op. iff $\mathcal{C}l_{\omega_{\hbar}}(G) = G$.
4) 4 - $\mathcal{C}l_{\omega_{\hbar}}(G) \subseteq \mathcal{C}l(G)$.

Proof:- 1- From Definition 3.8, $\mathcal{C}l_{\omega_{\hbar}}(G) = G \cup D_{\omega_{\hbar}}(G)$., then $G \subseteq \mathcal{C}l_{\omega_{\hbar}}(G)$.

2-From proposition3.7, part(1), $D_{\omega_{\hbar}}(G) \subseteq D_{\omega_{\hbar}}(H)$. So, $G \cup D_{\omega_{\hbar}}(G) \subseteq$ H $\cup D_{\omega_{\hbar}}(H)$, that is $\mathcal{C}l_{\omega_{\hbar}}(G) \subseteq \mathcal{C}l_{\omega_{\hbar}}(H)$.

3-Let G^c is $\omega_{\hbar}$-op.-st., then G is $\omega_{\hbar}$- clo. st. and from proposition3.7, part(2), $D_{\omega_{\hbar}}(G) \subseteq G$. so $D_{\omega_{\hbar}}(G) \cup G \subseteq G$, hence $\mathcal{C}l_{\omega_{\hbar}}(G) \subseteq G$ and from (1) of this proposition, we get $\mathcal{C}l_{\omega_{\hbar}}(G) = G$.

Conversely, since $\mathcal{C}l_{\omega_{\hbar}}(G) = G$, that is $\mathcal{C}l_{\omega_{\hbar}}(G)$ is $\omega_{\hbar}$-clo., so G is $\omega_{\hbar}$-clo.

4- From Definition of $\omega_{\hbar}$-closure of G which is the intersection of all $\omega_{\hbar}$-clo.-sets in x containing G, and proposition 3.3, part(1). We get $\mathcal{C}l_{\omega_{\hbar}}(G) \subseteq \mathcal{C}l(G)$.

Definition 3.10: Assume that $F: \mathcal{X} \longrightarrow$ Y be a map between two top.-sp.s. The map F is called $\omega_{\hbar}$-cont. if the inverse. image. of all op.-st $\mathcal{A}$ in Y is an $\omega_{\hbar}$-op.-st in $\mathcal{X}$.

Proposition 3.11: Suppose $F: \mathcal{X} \longrightarrow$ Y is a map between top. sps. then:

i. Every constant funct. is $\omega_{\hbar}$-cont.

ii. If $\mathcal{X}$ is discrete sp.., then, F is $\omega_{\hbar}$-cont.

iii. If $\mathcal{X}$ finite set and τ any top., then F is $\omega_{\hbar}$-cont.

iv. If (Y, τ^*) is an indiscrete top. sp. then F is $\omega_{\hbar}$-cont.

Proof: i. since $F(x) = a$, for all $x \in \mathcal{X}$. Hence the inverse image of any op.-st. in Y is either $\mathcal{X}$ or $\emptyset$. Thus F is $\omega_{\hbar}$-cont.

ii. If $\mathcal{X}$ has the discrete. top., every subset of $\mathcal{X}$ is op. Hence the inverse. image. of any op. in Y is op. in $\mathcal{X}$ .Therefore F is $\omega_{\hbar}$-cont.

iii. If $\mathcal{X}$ finite set and τ any top., each subset satisfies the cond. of $\omega_{\hbar}$-openness. Thus F is $\omega_{\hbar}$-cont.

iv. If Y has the indiscr. top., the only op. sets are Y and . Their inverse. image. under F are $\mathcal{X}$. and . Thus F is $\omega_{\hbar}$-cont.

Proposition 3.12:

i - Every $\underline{\hbar}$ -cont. is $\omega_{\hbar}$-cont.

ii - Every cont. is $\omega_{\hbar}$-cont..

Proof: i - Let $F: \mathcal{X} \longrightarrow$ Y be an $\underline{\hbar}$-cont. and $\mathcal{A}$ op.-set in Y then, $F^{-1}(\mathcal{A})$ is $\underline{\hbar}$-op.-st in $\mathcal{X}$ thus is from Proposition (3.3) part (3), $F^{-1}(\mathcal{A})$ is $\omega_{\hbar}$-op.-set in $\mathcal{X}$, therefore F is $\omega_{\hbar}$-cont.

ii - Let F be a cont. function, and $\mathcal{A}$ op.-st. in Y then $F^{-1}(\mathcal{A})$ op.-st. in $\mathcal{X}$, Thus by Proposition (3.3) part (1) $F^{-1}(\mathcal{A})$ is $\omega_{\hbar}$-op.-st. in $\mathcal{X}$, therefore F is $\omega_{\hbar}$-cont.

The next example to show the convers of (3.12) is not true:-

Examples 3.13:

i- Take $\mathcal{X} = \{x, y, z\}$, with top. $T_x = \{\emptyset, \mathcal{X}, \{x\}, \{x, y\}\}$, and let $Y = \{a, b, c\}$ with top. $T_y = \{\emptyset, y, \{b\}\}$,. Define a map $F: \mathcal{X} \longrightarrow Y$ $F(x) = F(z) = a, F(y) = b$, thus F is ω_h-cont. but $F^{-1}(\{b\}) = \{y\}$ is not $\underline{\hbar}$ -op.-st. in $\mathcal{X}$, so F is not $\hbar$ -cont.

ii- Let $\mathcal{X} = \{1,2\}$ and $Y = \{a, b\}$. Let $\mathcal{X}$ be a discrete. And define a top. on Y by $T' = \{\theta, Y, \{a\}\}$.

Define a map $F: \mathcal{X} \rightarrow Y$ by $F(1) = a$, $F(2) = b$. Then F is $\omega_{\hbar}$-cont. but is fails to be cont.

Proposition 3.14: Consider a map $\mathcal{G}: \mathcal{X} \rightarrow Y$, the following statements are equivalent:

1- $\mathcal{G}$ is ω_h-cont.

2- $\mathcal{G}^{-1}\mathrm{Int}(\mathcal{M}) \subseteq \mathrm{Int}_{\omega_h}(\mathcal{G}^{-1}(\mathcal{M}))$ for every $E \subseteq Y$.

3- $\mathcal{G}^{-1}$ (E) is ω_h-clo.-st in $\mathcal{X}$ for all clo.-set $\mathcal{M}$ in Y.

4- $\mathcal{G}\left(\mathcal{C}l_{\omega_h}(B)\right) \subseteq G(\mathcal{C}l(B))$ for every $B \subseteq \mathcal{X}$.

4- $\mathcal{C}l_{\omega_h}(\mathcal{G}^{-1}(\mathcal{M})) \subseteq \mathcal{G}^{-1}\mathcal{C}l(\mathcal{M})$ for any $\mathcal{M} \subseteq Y$.

Proof:(2→1) Suppose that $\mathcal{M} \subseteq Y$ since $Int(\mathcal{M})$ op.-set in Y, since $\mathcal{G}$ is ω_h-cont., in that case $F^{-1}(Int(\mathcal{M}))$is ω_h-op.-set in $\mathcal{X}$, consequently, from Proposition (3.3), we obtain $\mathcal{G}^{-1}(Int(\mathcal{M})) = Int_{\omega_{\hbar}}(\mathcal{G}^{-1}(\mathcal{M}))$. Therefore, $\mathcal{G}^{-1}Int(\mathcal{M}) \subseteq Int_{\omega_{\hbar}}\left(\mathcal{G}^{-1}(\mathcal{M})\right)$.

(2→3) Suppose that $\mathcal{M}$ be a clo.-set of Y, then $\mathcal{M}^c$ is op.-set in Y, thus $\mathcal{M}^c = Int(\mathcal{M}^c)$ thus from (2) $\mathcal{G}^{-1}(\mathcal{M}^c) = \mathcal{G}^{-1}\left(Int(\mathcal{M}^c)\right) \subseteq Int_{\omega_{\hbar}}(\mathcal{G}^{-1}(\mathcal{M}^c))$ but from Proposition (3. 5), $\mathrm{Int}_{\omega_{\hbar}}(\mathcal{G}^{-1}(\mathcal{M}^c)) \subseteq \mathcal{G}^{-1}(\mathrm{Int}(\mathcal{M}^c))$ so $\mathcal{G}^{-1}(\mathcal{M}^c) = \mathrm{Int}_{\omega_{\hbar}}(\mathcal{G}^{-1}(\mathcal{M}^c))$ Thus $\mathcal{G}^{-1}(\mathcal{M}^c) = (\mathcal{G}^{-1}(\mathcal{M}))^c$is ω_h-op.-set Therefore $\mathcal{G}^{-1}(\mathcal{M})$ is ω_h-clo.-set .

(3 → 4) Suppose that $B \subseteq \mathcal{X}$, then $\mathcal{C}l(\mathcal{G}(B))$ is clo.-st. in Y, as a result via (3) we obtain $\mathcal{G}^{-1}(\mathcal{C}l\,\mathcal{G}(B)))$ is ω_h-clo.-st in $\mathcal{X}$, containing B thus $\mathcal{C}l_{\omega_h}(B) \subseteq \mathcal{G}^{-1}(\mathcal{C}l(\mathcal{G}(B)))$, so from Proposition (3.9), part (4) $\mathcal{C}l_{\omega_{\hbar}}(\mathcal{G}(B)) \subseteq \mathcal{C}l(\mathcal{G}(B))$.

(4→5) Assume that $\mathcal{M} \subseteq Y$, in that case via (4) we get $(\mathcal{C}l_{\omega_{\hbar}}(\mathcal{G}(\mathcal{G}^{-1}(\mathcal{M}))) \subseteq \mathcal{C}l\,\mathcal{G}^{-1}(G(\mathcal{M}))$, thus from Proposition (3.9), part (4), $(\mathcal{C}l_{\omega_{\hbar}}(\mathcal{G}^{-1}(\mathcal{M})) \subseteq \mathcal{G}^{-1}(\mathcal{C}l(\mathcal{M}))$.

(5→1) Assume that V be open set in Y , then $V^c = \mathcal{C}l(V^c)$by hypothesis $\mathcal{C}l_{\omega_{\hbar}}(\mathcal{G}^{-1}(V^c)) \subseteq \mathcal{G}^{-1}(\mathcal{C}l\,\mathcal{G}(V^c))$consequently $\mathcal{C}l_{\omega_{\hbar}}(\mathcal{G}^{-1}(V^c)) \subseteq \mathcal{G}^{-1}(\mathcal{C}l\ (V^c)))$, hence $\mathcal{G}^{-1}$ (V) is $\omega_{\hbar}$-op.-st in $\mathcal{X}$, so $\mathcal{G}$ is an $\omega_{\hbar}$-cont. function.

Now, we introduce another type of continuous function, formulated in terms of the class of open sets already defined in Definition 2.2 and Definition 3.1 above.

Definition 3.15: Suppose $F: \mathcal{X} \longrightarrow Y$ is a map. The map F is called $\omega_{\hbar}$-irresolute if for every $\omega_{\hbar}$-op.-st $\mathcal{A}$ in Y, then $F^{-1}(\mathcal{A})$ is an $\omega_{\hbar}$-op.-st in $\mathcal{X}$.

Remark 3.16:

1-Every constant mapping is ω_h-irresolute.

2- If both $\mathcal{X}$ and Y are finite sps. and $F: \mathcal{X} \longrightarrow Y$, then F is an $\omega_{\hbar}$-cont.

Proposition 3.17: Every $\omega_{\hbar}$- irresolute function is $\omega_{\hbar}$-cont. function.

Proof: Let F be a function from $\mathcal{X}$ into Y, and H op. st in Y, then H is $\omega_{\hbar}$- op.-st. and F is $\omega_{\hbar}$-irresolute, thus $F^{-1}(H)$ is $\omega_{\hbar}$op. in X, then F is $\omega_{\hbar}$cont.

The following example verifies that the converse of proposition 3.17, fails in general.

Example 3.18: Suppose $\mathcal{U}$ be a usual top. on R and T be indiscrete top. on $Y = \{a, b\}$, presume $F: \mathbb{R} \longrightarrow Y$ by $F(x) = \begin{cases} a & if \quad x \in \mathbb{Q} \\ b & if \quad x \notin \mathbb{Q} \end{cases}$ in that case $F$is an ω_h-cont., but not $\omega_{\hbar}$-irresolute. Since, $\hbar$-op.-sets in Y are $\emptyset, Y, \{a\}, \{b\}$ And $\omega_{\hbar}$-op.-sts. in Y are $\emptyset, Y, \{a\}, \{b\}$. So, $F^{-1}\{(a)\} = \mathbb{Q}$ is not $\omega_{\hbar}$-op.-st.

Proposition 3.19: Every cont. function is $\omega_{\hbar}$-irresolute function.

Proof:- From Definition 3.15, and definition of continuous function we get the result.

Example 3.20: Let F: X→Y and X = {a, b, c}, $T_x = \{\emptyset, X, \{c\}\}$ and Y = {x, y}, $T_Y = \{\emptyset, Y, \{y\}\}$ with f(a) = F(c) = {y}, F(b) = {x}, the h-op.-sets of X are {∅, X, {c}, {a, b}} and $\omega_{\hbar}$-op.-sts in X are {∅, X, {c}, {a}, {b}, {a, b}, {b, c}, {a}}. well, $\hbar$-op.-sts in Y are ∅, Y, {x}, {y}, and $\omega_{\hbar}$-op.-sts in Y are ∅, Y, {x}, {y}. In that case, F is ω -cont., but not cont., not $\hbar$ -cont. not $\omega_{\hbar}$irresolute, and not $\omega_{\hbar}$cont.

Proposition 3.21: Let $F: \mathcal{X} \longrightarrow Y$, then F is an $\omega_{\hbar}$- irresolute if the inverse image. of every $\omega_{\hbar}$-clo.-st. in Y is an $\omega_{\hbar}$-clo.-st. in $\mathcal{X}$.

Proof: Assume that $\mathcal{A}$ is $\omega_{\hbar}$-clo.-st in Y, so $\mathcal{A}^c$ is ω_h-op.-st, in Y, Since $F\ is\ \omega_{\hbar}$- irresolute then $F^{-1}(\mathcal{A}^c)$ is $\omega_{\hbar}$-op.-st in $\mathcal{X}$, because $F^{-1}(\mathcal{A}^c) = (F^{-1}(\mathcal{A}))^C$, therefore, $F^{-1}(\mathcal{A})$ is $\omega_{\hbar}$-clo.-st in $\mathcal{X}$.

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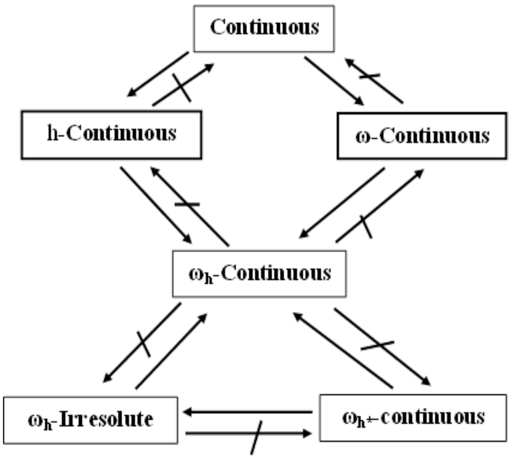

Figure 1: The relationships between some types of continuous functions.

Proposition 3.22: Assume $F: \mathcal{X} \longrightarrow \mathrm{Y}$ is an one to one $\hbar$ –irresolute, then $F$is an $\omega_{\hbar}$-irresolute

Proof: Suppose that $\mathcal{B}$ be an $\omega_{\hbar}$-op.-set in Y and suppose that $x \in F^{-1}(\mathcal{B})$. Then $F(x) \in \mathcal{B}$ and there exists an $\hbar$-op.-set C of Y with $F(x) \in \mathrm{C} \subset \mathcal{B}$ .Since F is $\hbar$-irresolute, so F^{-1} (C) is $\hbar$-op. in $\mathcal{X}$ and contains x and F^{-1} $(\mathrm{C} - \mathcal{B})$ is a countable set, then F^{-1} (C) $- F^{-1}(\mathcal{B})$ is a countable. that is F^{-1} $(\mathcal{B})$ is an $\omega_{\hbar}$-op. subset of $\mathcal{X}$.

Proposition 3.23: Consider a mapping $F: \mathcal{X} \longrightarrow \mathrm{Y}$. The following conditions are equivalent.

1- F is an $\omega_{\hbar}$-irresolute funct.

2- F $(\mathcal{C}\mathrm{l}_{\omega_{\hbar}}(\mathrm{N})) \subseteq \mathcal{C}\mathrm{l}_{\omega_{\hbar}}(F$ (N)), for every set N $\in \mathcal{X}$.

3- for all $\mathcal{M} \in \mathrm{Y} \subseteq F^{-1}(\mathcal{C}\mathrm{l}_{\omega_{\hbar}}(\mathcal{M})) \subseteq \mathcal{C}\mathrm{l}_{\omega_{\hbar}}(F^{-1}(\mathcal{M}))$.

Proof: 1→2, $\mathrm{N} \subseteq \mathcal{X}$, then $F(\mathrm{N}) \subseteq \mathrm{Y}, \mathcal{C}\mathrm{l}_{\omega_{\hbar}}(F(\mathrm{N}))$ is $\omega_{\hbar}$-clo.-set in Y, because F is an $\omega_{\hbar}$- irresolute thus from proposition(3.21), F^{-1} $(\mathcal{C}\mathrm{l}_{\omega_{\hbar}}F(\mathrm{N}))$ is $\omega_{\hbar}$-clo.-set in $\mathcal{X}$ from Proposition (3.9) $F(\mathrm{N}) \subseteq (\mathcal{C}\mathrm{l}_{\omega_{\hbar}}F(\mathrm{N}))$, hence F^{-1} $(F(\mathrm{N})) \subseteq F^{-1}(\mathcal{C}\mathrm{l}_{\omega_{\hbar}}F(\mathrm{N}))$, $\mathrm{N} \subseteq F^{-1}(F(\mathrm{N}))$, in that case $\mathrm{N} \subseteq F^{-1}(\mathcal{C}\mathrm{l}_{\omega_{\hbar}}F(\mathrm{N}))$. Since F^{-1} $(\mathcal{C}\mathrm{l}_{\omega_{\hbar}}$ $F(\mathrm{N}))$, is $\omega_{\hbar}$ -clo.-set thus $(\mathcal{C}\mathrm{l}_{\omega_{\hbar}}(\mathrm{N})) \subseteq F^{-1}$ $(\mathcal{C}\mathrm{l}_{\omega_{\hbar}}$ $F(\mathrm{N}))$, hence F $(\mathcal{C}\mathrm{l}_{\omega_{\hbar}}(\mathrm{N})) \subseteq F(F^{-1}(\mathcal{C}\mathrm{l}_{\omega_{\hbar}}$ $F(\mathrm{N})) \subseteq (\mathcal{C}\mathrm{l}_{\omega_{\hbar}}F(\mathrm{N}))$,so $F((\mathcal{C}\mathrm{l}_{\omega_{\hbar}}(\mathrm{N})) \subseteq (\mathcal{C}\mathrm{l}_{\omega_{\hbar}}F(\mathrm{N}))$.

2→3 Suppose that $F((\mathcal{C}\mathrm{l}_{\omega_{\hbar}}(\mathrm{N})) \subseteq (\mathcal{C}\mathrm{l}_{\omega_{\hbar}}F(\mathrm{N}))$, $\forall$ N $\subseteq \mathcal{X}$ and M $\subseteq$ Y,then $F^{-1}(\mathcal{M}) \subseteq \mathcal{X}, \mathcal{C}\mathrm{l}_{\omega_{\hbar}}$ $F^{-1}(\mathcal{M})) \subseteq \mathcal{C}\mathrm{l}_{\omega_{\hbar}}(F(F^{-1}(\mathcal{M}))$, Since $F(F^{-1}(\mathcal{M}) \subseteq \mathcal{M}$, hence, $\mathcal{C}\mathrm{l}_{\omega_{\hbar}}(F(F^{-1}(\mathcal{M})) \subseteq \mathcal{C}\mathrm{l}_{\omega_{\hbar}}(\mathcal{M})$, $F^{-1}(\mathcal{C}\mathrm{l}_{\omega_{\hbar}}$ $F\left(F^{-1}(\mathrm{N})\right) \subseteq F^{-1}\left(\mathcal{C}\mathrm{l}_{\omega_{\hbar}}(\mathcal{M})\right)$, so $\mathcal{C}\mathrm{l}_{\omega_{\hbar}}(F^{-1}(\mathcal{M})) \subseteq F^{-1}(\mathcal{C}\mathrm{l}_{\omega_{\hbar}}(\mathcal{M}))$.

1→3, Assume that $\mathcal{M}$ be $\omega_{\hbar}$ –clo.-st. of Y, then $\mathcal{M} = \mathcal{C}\mathrm{l}_{\omega_{\hbar}}(\mathcal{M}$) and $\mathcal{C}\mathrm{l}_{\omega_{\hbar}}F(\mathcal{M}) \subseteq F^{-1}$ $(\mathcal{C}\mathrm{l}_{\omega_{\hbar}}(\mathcal{M}))$, in that case $\mathcal{C}\mathrm{l}_{\omega_{\hbar}}F(\mathcal{M}) \subseteq F^{-1}$ $(\mathcal{M})$, because $F^{-1}(\mathcal{M}) \subseteq \mathcal{C}\mathrm{l}_{\omega_{\hbar}}F(\mathcal{M})$, hence F^{-1} $(\mathcal{M}) \subseteq \mathcal{C}\mathrm{l}_{\omega_{\hbar}}$ F^{-1} $(\mathcal{M})$ thus F^{-1} $(\mathcal{M})$ is $\omega_{\hbar}$ -clo.-st , then F is $\omega_{\hbar}$ -irresolute funct.

Remark 3.24: The composition of two $\omega_{\hbar}$ -cont. mappings is not necessarily ω_{h} -cont. function. The following example illustrates this remark.

Example 3.25: Assume that F: $(\mathrm{R}, \mathrm{T}_u) \longrightarrow (\{a, b\}, \mathrm{T}_{ind})$ defined as:-

$$F(x) = \begin{cases} n & if\ \mathcal{X}\ \in \mathrm{Q} \\ m & if\ \mathcal{X}\ \in Q^C \end{cases} \text{ and}$$

G: $(\{n, m\}, \mathrm{T}_{ind}) \longrightarrow (\{0,1\}, \mathrm{T}_D)$ such that $G(n) = 0, G(m) = 1$

Now F and G are $\omega_{\hbar}$ -cont. funct., since the inverse image. of every op.-st is $\omega_{\hbar}$ -op.-st of F and G respectively but the composition GoF is not $\omega_{\hbar}$ -cont., since $G^{-1}(\{1\})$ is op.-st in $(\{0,1\}, \mathrm{T}_D)$, but is not $\omega_{\hbar}$ -op.-st in $(\mathrm{R}, \mathrm{T}_u)$, that is $(\mathrm{G}oF)^{-1}(\{1\}) = Q^C$ is not $\omega_{\hbar}$ -op.-st in $(\mathrm{R}, \mathrm{T}_u)$.

Proposition 3.26: If $F: \mathcal{X} \longrightarrow \mathrm{Y}$ is an ω_{h}-cont., and G: Y $\longrightarrow \mathcal{Z}$ is cont., then G$oF: \mathcal{X} \longrightarrow \mathrm{Z}$ is $\omega_{\hbar}$ -cont.

Proof: Suppose that $\mathcal{B}$ is op.-st in $\mathcal{Z}$, then $G^{-1}(\mathcal{B})$ is op.-st in Y and F is $\omega_{\hbar}$ -cont., then $F^{-1}(\mathrm{G}^{-1}(\mathcal{B}))$ is $\omega_{\hbar}$ -op.-st in $\mathcal{X}$, so (Go $F)^{-1}(\mathcal{B})$ is $\omega_{\hbar}$ -op.-set in $\mathcal{X}$ therefore, G$oF: \mathcal{X} \rightarrow \mathcal{Z}$ is an $\omega_{\hbar}$ -cont.

Proposition 3.27: Consider mappings $F: \mathcal{X} \longrightarrow \mathrm{Y}$ and G: Y $\longrightarrow \mathcal{Z}$ .If both if F is $\omega_{\hbar}$ -irresolute and G is $\omega_{\hbar}$ -cont. , then G$oF: \mathcal{X} \longrightarrow \mathcal{Z}$ is $\omega_{\hbar}$ -cont.

Proof: Take $\mathcal{B}$ is op.-st in $\mathcal{Z}$. Then $\mathrm{G}^{-1}(\mathcal{B})$is an $\omega_{\hbar}$ -op.-st in Y. Because F is $\omega_{\hbar}$ - irresolute then., $F^{-1}(\mathrm{G}^{-1}(\mathcal{B})) = (Go\ F)^{-1}(\mathcal{B})$ is $\omega_{\hbar}$ -op.-st in $\mathcal{X}$. Thus GoF is $\omega_{\hbar}$ -cont.

Proposition 3.28: Consider mappings $F: \mathcal{X} \longrightarrow \mathrm{Y}$ and G: Y $\longrightarrow$ Z. If both $F$and G are $\omega_{\hbar}$ - irresolute, then Go F is an $\omega_{\hbar}$ - irresolute.

Proof: Suppose that $\mathcal{M}$ is $\omega_{\hbar}$- op.-st in Z, $(\mathrm{G}o\,F)^{-1}(\mathcal{M}) = F^{-1}(\mathrm{G}^{-1}(\mathcal{M}))$ since G is $\omega_{\hbar}$-irresolute and $\mathcal{M}$ is $\omega_{\hbar}$-op.- st in Z, hence $\mathrm{G}^{-1}(\mathcal{M})$ is $\omega_{\hbar}$-op.- st in Y, and F is $\omega_{\hbar}$-irresolute, thus $F^{-1}(\mathrm{G}^{-1}(\mathrm{M}))$ is $\omega_{\hbar}$-op.- set in $\mathcal{X}$, but $(\mathrm{G}o\,F)^{-1}(\mathrm{M}) = F^{-1}(\mathrm{G}^{-1}(\mathcal{M}))$ so,
$(\mathrm{G}o\,F)^{-1}(\mathcal{M})$ is $\omega_{\hbar}$-op.- st in $\mathcal{X}$, then GoF is $\omega_{\hbar}$-irresolute.

Now, we study restriction of $\omega_{\hbar}$-conti. ($\omega_{\hbar^*}$-cont.) and ω_{h}-irresolute functions.

Proposition 3.29: Let $F: \mathcal{X} \longrightarrow \mathrm{Y}$, and $\mathcal{A} \subseteq \mathcal{X}$. Then:

(1) If F *is* $\omega_{\hbar}$-cont., then $F\,|_{\mathcal{A}}$ is $\omega_{\hbar}$-cont..

(2) If F *is* $\omega_{\hbar}$- irresolute, then $F|_{\mathcal{A}}$ is $\omega_{\hbar}$-irresolute.

(3) If F *is* $\omega_{\hbar^*}$-cont., then $F|_{\mathcal{A}}$ is $\omega_{\hbar^*}$-cont..

Proof: (1) Let $\mathcal{M}$ be any op.-set in Y. Since F is $\omega_{\hbar}$-cont. then, $F^{-1}(\mathcal{M})$ is $\omega_{\hbar}$-op.-st in $\mathcal{X}$, so $F^{-1}(\mathcal{M}) \cap \mathcal{A}$ is $\omega_{\hbar}$-op.-st in $\mathcal{A}$ and $(F\,|_{\mathcal{A}})^{-1}(\mathcal{M}) = F^{-1}(\mathcal{M}) \cap \mathcal{A}$ is $\omega_{\hbar}$-op.-st, therefore $F\,|_{\mathcal{A}}$ is $\omega_{\hbar}$-cont.

(2) Let $\mathcal{B}$ be an $\omega_{\hbar}$-op.- set in Y, but $F$is $\omega_{\hbar}$-cont. then $F^{-1}(\mathcal{B})$ is $\omega_{\hbar}$-op.-st in $\mathcal{X}$, $F^{-1}(\mathcal{B}) \cap \mathcal{A}$ is $\omega_{\hbar}$-op.-st in $\mathcal{A}$ and $(F|\mathcal{A}\,(\mathcal{B}))^{-1} = F^{-1}(\mathcal{B}) \cap \mathcal{A}$, then $(F|\mathcal{A}\,(\mathcal{B}))^{-1}$is $\omega_{\hbar}$-op.-st in $\mathcal{A}$, hence $F\,|_{\mathcal{A}}$is $\omega_{\hbar}$-cont.

(3)-The proof as the same as (2).

4 CONCLUSIONS

In this paper, we introduced and developed new generalized classes of functions in topological spaces, namely ω_{h}-continuity and ω_{h}-irresolute functions. We established several fundamental properties and provided precise characterizations describing their behavior with respect to ω_{h}-open sets. In particular, we investigated the stability of these functions under restriction and analyzed their interactions with other generalized forms of continuity. Moreover, we explored the relationships between the proposed concepts and various existing notions in the literature, showing that many known results can be extended and unified within this broader framework. To the best of our knowledge, the approach presented in this work offers a new perspective in the study of generalized continuity and enriches the current theory of topological structures. The results obtained in this paper contribute significantly to the development of generalized topology and open several promising directions for future research, including the study of stronger variants of ω_{h}-continuous, decomposition of topological properties, and potential applications in soft topological and related generalized spaces such as in reference [12].

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