Quasi-Subordination on Bi-Univalent Functions Involving a Generalized Operator Associated with Horadam Polynomials

Mustafa H. Abd Ali and Mohammed H. Saloomi
Department of Mathematics, College of Education for Pure Sciences, University of Kerbala, 56001 Kerbala, Iraq
mostafa.h@s.uokerbala.edu.iq, mohammed.hasan@uokerbala.edu.iq

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Abstract: Although numerous subclasses of bi-univalent functions have been investigated in recent years, coefficient estimates associated with generalized Q-operators and Horadam polynomials remain comparatively unexplored. Motivated by this gap, in this paper, we introduce and investigate a new subclass of the class Σ of analytic and bi-univalent functions defined in the open unit disk. The proposed subclass is constructed by means of the operator $\chi_{q,t}^{\lambda,\xi}$ in association with the Horadam polynomials $h_n(x)$, which play an important role in various areas of mathematical analysis. By employing this operator, we establish a novel framework that enables us to study the geometric properties of functions belonging to this subclass. In particular, we derive estimates for the initial Taylor–Maclaurin coefficients for functions in the considered class. These coefficient bounds contribute to the ongoing development of coefficient problems in the theory of bi-univalent functions. Furthermore, several known results are obtained as special cases of our main findings by suitably specializing the parameters involved. The results presented in this work not only generalize earlier studies but also provide new insights and potential directions for future research in this field.

1 INTRODUCTION

The theory of univalent and bi-univalent functions has long occupied a central position in geometric function theory, owing to its deep connections with coefficient problems, subordination theory, and special polynomial sequences. Since the class of bi-univalent functions was reintroduced, extensive research has been devoted to estimating the initial coefficients of functions belonging to its various subclasses.

Building on this foundation, numerous subclasses of bi-univalent functions have been introduced and studied by employing diverse tools, including subordination and quasi-subordination principles, q-calculus operators, and generalized special functions such as the Hurwitz-Lerch zeta function. These investigations have produced coefficient bounds and Fekete-Szegö inequalities for a wide range of subclasses, reflecting the richness of the underlying theory.

Horadam polynomials, in particular, have been actively employed in recent years to define and study new subclasses of bi-univalent functions. However, despite the growing interest in q-analogues of classical operators, coefficient estimates associated with a generalized q-Srivastava-Attiya-type operator in conjunction with Horadam polynomials remain comparatively unexplored.

Motivated by this gap, the main contribution of this paper is to introduce a new subclass of bi-univalent functions defined by means of a generalized q-Srivastava-Attiya-type operator associated with Horadam polynomials, and to derive estimates for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this subclass, together with several known results recovered as special cases. The necessary preliminary definitions and operators are presented below.

1.1 Preliminary Definitions and Operators

Let $\mathcal{A}$ denote the class of all holomorphic functions $Ŧ$ defined on the unit disk $\mathfrak{U}$.

$$\mathfrak{U} = \{\mathfrak{z}: |\mathfrak{z}|<1\},$$

normalized such that $Ŧ(0) = 0$ and $Ŧ'(0) = 1$. Any function $Ŧ \in \mathcal{A}$ in has the Taylor expansion:

$$Ŧ(ȝ) = ȝ + \sum_{n=2}^{\infty} a_n ȝ^n. \quad (1)$$

Let $Υ(ȝ)$ be holomorphic function in $\mathfrak{U}$ of the form:

$$Υ(ȝ) = k_{\circ} + k_1 ȝ + k_2 ȝ^2 + k_3 ȝ^3. \quad (2)$$

The class S consists of all holomorphic and univalent functions in $\mathfrak{U}$ by the Koebe-one quarter theorem, the image of the disk $\mathfrak{U}$ under any $Ŧ \in S$ contains a disk of radius at least $\frac{1}{4}$ [1]. Every univalent function Ŧ has an inverse $Ŧ^{-1}$ satisfying:

$$Ŧ^{-1}(Ŧ(ȝ)) = ȝ \in \mathfrak{U}, \quad (3)$$

and

$$Ŧ(Ŧ^{-1}(\omega)) = \omega, \left(|\omega| < r_{\circ}(Ŧ), r_{\circ}(Ŧ) > \frac{1}{4}\right), \quad (4)$$

A function Ŧ is called bi-univalent if both Ŧ and $Ŧ^{-1}$ are univalent in $\mathfrak{U}$, and the class of all such function is denoted by Σ of the form above, its invers $g = Ŧ^{-1}$ has the expansion:

$$g(\omega) = Ŧ^{-1}(\omega) = \omega - a_2\,\omega^2 + (2a_2^2 - a_3)\omega^3 - \cdots \quad (5)$$

Lewin [2] introduced the class of bi-univalent functions and proved that $|a_2| \leq 1.51$. Later, Brannan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$ and Netanyahu in [4] proved that $\max_{Ŧ\in\Sigma}|a_2| = \frac{4}{3}$.

Several different subclasses of Σ have since been studied, with estimates for $|a_2|$ and $|a_3|$ provided by multiple authors [5]-[17].

El-Ashwah and Kanas [18] defined two subclasses:

$$S*_q(\gamma, \phi) = \{ Ŧ \in \mathcal{A}: \frac{1}{\gamma}(\frac{ȝŦ'(ȝ)}{Ŧ(ȝ)} - 1) \prec_q \phi(ȝ) - 1;\ ȝ \in \mathfrak{U}, 0 \neq \gamma \in \mathbb{C}\}.$$

$$K_q(\gamma, \phi) = \left\{ Ŧ \in \mathcal{A}: \frac{1}{\gamma}\frac{ȝŦ''(ȝ)}{Ŧ'(ȝ)} \prec_q \phi(ȝ) - 1;\ ȝ \in \mathfrak{U}, 0 \neq \gamma \in \mathbb{C} \right\}.$$

Where $\prec_q$ denotes quasi-subordination. When $h(z) \equiv 1$, these reduce to the classical Ma-Minda starlike $S*_q(\gamma, \phi)$ and convex functions $K_q(\gamma, \phi)$ classes [4].

For two functions Ŧ and F in with expansions:

$$F(ȝ) = ȝ + \sum_{n=2}^{\infty} a_n ȝ^n.$$

Their convolution is:

$$(Ŧ * F)(ȝ) = ȝ + \sum_{2}^{\infty} a_n\, b_n ȝ^n, \qquad (ȝ \in \mathfrak{U}). \quad (6)$$

A function Ŧ is quasi-subordination to g, written $Ŧ(ȝ) \prec_q g(ȝ), (ȝ \in \mathfrak{U})$, if there exist holomorphic functions h and $\hbar$ with $|h(ȝ)| \leq 1$, $\hbar(0) = 0$ and $|\hbar(ȝ)| < 1$, such that:

$$Ŧ(ȝ) = h(ȝ)g(\hbar(ȝ)), (ȝ \in \mathfrak{U}). \quad (7)$$

Where $h(ȝ) = 1$, this reduces to classical subordination $Ŧ(ȝ) \prec g(ȝ)$.

In [13], [14] Jackson introduced the q −derivative operator $\mathfrak{D}_q$ defined by:

$$\mathfrak{D}_q Ŧ(ȝ) = \frac{Ŧ(ȝ) - Ŧ(qȝ)}{(1-q)ȝ}\ 0 < q < 1, \qquad ȝ \neq 0$$

Which converges to $Ŧ^{-1}$ as $q \to 1$. For $Ŧ \in \mathcal{A}$ these yields:

$$\mathfrak{D}_q Ŧ(ȝ) = 1 + \sum_{2}^{\infty} [n]_q\, a_n\, ȝ^{n-1}, \quad (8)$$

where $[n]_q = \frac{1-q^n}{1-q'}$ is q −analogue of $n \in N$ as $q \to 1$ the q −analogue $[n]_q$ approaches n with $[0]_q = 0$.

The general Hurwitz-Lerch Zeta function is defined by:

$$\Phi(ȝ, \lambda, a) := \sum_{n=0}^{\infty} \frac{ȝ^n}{(n+a)^{\lambda}}, \quad (9)$$

$(a \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}, \lambda \in \mathbb{C}, Re\, \lambda > 1$ and $|ȝ| = 1$, [19]).

Shah and Noor [20] introduced a q −analogue of this function:

$$\Phi_q(\lambda, t, ȝ) = \sum_{n=0}^{\infty} \frac{ȝ^n}{[n+t]_q^{\lambda}}. \quad (10)$$

Where $t \in C\backslash ȝ, \lambda \in C$ when $|ȝ| < 1$ and $\Re(\lambda) > 1$ when $|ȝ| = 1$.

From this, one can define:

$$\Theta_q(\lambda, t, ȝ) = [1 + t]_q^{\lambda}\{\Phi_q(\lambda, t, ȝ) - [t]_q^{-\lambda}\} = ȝ + \sum_{n=2}^{\infty}\left(\frac{[1 + t]_q}{[n+t]_q}\right)^{\lambda} ȝ^n. \quad (11)$$

The function $\mathcal{K}_S$ is given by [21]:

$$\mathcal{K}_{\xi}(\mathfrak{z}) = \mathfrak{z} + \sum_{n=2}^{\infty}(n^2\xi - \xi + 1)a_n\,\mathfrak{z}^n, 0 \le \xi \le 1. \quad (12)$$

Using Hadamard product and the above the operator:

$$\chi_{q,t}^{\lambda,\xi}: \mathcal{A} \longrightarrow \mathcal{A},$$

$$\chi_{q,t}^{\lambda,\xi}\mathrm{F}(\mathfrak{z}) = \Theta_q(\lambda, t, \mathfrak{z}) * \mathcal{K}_{\xi}(\mathfrak{z}) = \mathfrak{z} + \sum_{n=2}^{\infty}(n^2\xi - \xi + 1)\left(\frac{[1+t]_q}{[n+t]_q}\right)^{\lambda} a_n\mathfrak{z}^n.$$

And

$$\chi_{q,t}^{\lambda,\xi}g(w) = w + \sum_{n=2}^{\infty}(n^2\xi - \xi + 1)\left(\frac{[1+t]_q}{[n+t]_q}\right)^{\lambda} a_n w^n.$$

Remark 1.1. The operator $\chi_{q,t}^{\lambda,\xi}$ is a generalization several known operators. for example:

- As $q \to 1$, $\Phi_q(\lambda, t, \mathfrak{z}) \to \Phi_q(\lambda, t, \mathfrak{z})$ recovering the classical Hurwitz-Lerch zeta function.
- For $\xi = 0$, it reduces the the operator $\Im_{q,t}^{s}$ [20].
- For $\xi = 0$ and $\lambda = 1$ it reduces to the q −Bernardi operator (see [22]).
- For $\xi = 0$ and $\lambda = t = 1$ it reduces to the q −Libera operator (see [23]).
- For $\xi = 0, \lambda = 1$ and $q \to 1$ it becomes to the classical Bernardi operator (see [23]).
- For $\xi = 0, \lambda = 1, t = 0$ and $q \to 1$ it becomes to the Alexander operator (see [24]).

1.2 Horadam Polynomials

The Horadam polynomial $h_n(x)$ satisfy the recurrence relation:

$$h_n(r) = pr\, h_{n-1}(r) + q\, h_{n-2}(r),\ r \in \mathbb{R}, n \in \aleph.$$

With initial conditions $h_1(r) = a, h_2(r) = br$, for some, $p.q, a, b \in \mathbb{R}$.

The characteristic equation is:

$$t^2 - prt - q = 0,$$

with roots:

$$\alpha = \frac{pr + \sqrt{(pr)^2 + 4q}}{2}, \beta = \frac{pr - \sqrt{(pr)^2 + 4q}}{2}.$$

For certain of parameters the Horadam polynomials generate well-known polynomial sequences. Their generating function is:

$$g(r, \mathfrak{z}) = \sum_{n=1}^{\infty} h_n(r)\mathfrak{z}^{n-1} = \frac{a + (b - ap)r\mathfrak{z}}{1 - pr\mathfrak{z} - q\mathfrak{z}^2}.$$

The polynomials have been applied in geometric function theory and complex analysis [25]-[33].

Definition 1.1. A function $\mathrm{F} \in \sum$ where $\mathrm{F}(\mathfrak{z}) = \mathfrak{z} + \sum_{n=2}^{\infty} a_n\mathfrak{z}^n$ is said to be in the class $\mathcal{F}_{\Sigma}^{q}(\varpi, \mu, \lambda, \xi, p, q, r, g)$ if it satisfies the following quasi-subordination conditions:

$$\frac{1}{\varpi}\left[(1-\mu)\frac{\mathfrak{z}(\chi_{q,t}^{\lambda,\xi}\mathrm{F}(\mathfrak{z}))'}{\chi_{q,t}^{\lambda,\xi}(\mathfrak{z})} + \mu\left(1 + \frac{\mathfrak{z}(\chi_{q,t}^{\lambda,\xi}\mathrm{F}(\mathfrak{z}))''}{(\chi_{q,t}^{\lambda,\xi}(\mathfrak{z}))'}\right) - 1\right] \prec_q g(r, \mathfrak{z}) - a. \quad (13)$$

$$\frac{1}{\varpi}\left[(1-\mu)\frac{w(\chi_{q,t}^{\lambda,\xi}g(w))'}{\chi_{q,t}^{\lambda,\xi}(w)} + \mu\left(1 + \frac{w(\chi_{q,t}^{\lambda,\xi}g(w))''}{(\chi_{q,t}^{\lambda,\xi}(w))'}\right) - 1\right] \prec_q g(r, \mathfrak{z}) - a \quad (14)$$

Where:

$$\varpi \in \mathbb{C} \setminus \{0\}, 0 \le \mu \le 1, r \in \mathbb{R}.$$

Remark 1.2.

- Setting $\xi = 0$ in the above class, we have:

$$\mathcal{F}_{\Sigma}^{q}(\varpi, \mu, \lambda, 0, p, q, r, g,) =: \mathcal{F}_{\Sigma}^{q}(\varpi, \mu, \lambda, p, q, r, g,).$$

- For $\varpi = a = 1, \lambda = 0, g(r, \mathfrak{z}) = \varphi(z) = 1 + B_1\mathfrak{z} + B_2\mathfrak{z}^2 + B_3\mathfrak{z}^3 + \cdots$, it reduces to $\mathcal{M}_{q,\Sigma}^{\mu}(\varphi) := \mathcal{M}_{\Sigma}^{\mu}(\varphi)$ [32], [33].
- Other special choices of λ, μ, ξ recover classes studied in previous works $\mathbb{Q}_{\Sigma}^{q}(\varpi, \xi, p, q, r, g)$ [32], [33].

2 MAIN RESULT (COEFFICIENT BOUNDS)

2.1 Theorem

For $\mathrm{F}\epsilon\mathcal{F}_{\Sigma}^{q}(\omega,\mu,\lambda,\xi,p,q,r,g)$ the initial Taylor-Maclaurin coefficients satisfy:

$$|a_2| \le \frac{|br|[2+t]_q^{\lambda}\sqrt{\omega|br|k_{\circ}\{k_{\circ}+k_1\}[3+t]_q^{\lambda}}}{\sqrt{\begin{bmatrix}(br)^2\mathcal{T}(\omega,k_{\circ},\mu,\xi,\lambda,t,q)\\ -(br)^2\mathcal{Q}(\omega,k_{\circ},\mu,\xi,\lambda,t,q)-\\ pbr^2\Theta(\mu,\xi,\lambda,t,q)-\\ qa\Theta(\mu,\xi,\lambda,t,q)\end{bmatrix}}}.$$

$$|a_3| \le \frac{\omega^2 k_{\circ}^2(br)^2[2+t]_q^{2\lambda}}{(\mu+1)^2(1+3\xi)^2[1+t]_q^{2\lambda}} + \frac{\omega k_{\circ}|br|[3+t]_q^{\lambda}}{(4\mu+2)(1+8\xi)[1+t]_q^{\lambda}} + \frac{\omega k_1|br|[3+t]_q^{\lambda}}{(4\mu+2)(1+8\xi)[1+t]_q^{\lambda}}.$$

Where

$$\mathcal{T}(\omega,k_{\circ},\mu,\xi,\lambda,t,q) = \omega\, k_{\circ}\, h_2^2(r)(4\mu+2)(1+8\xi)[1+t]_q^{\lambda}[2+t]_q^{2\lambda}.$$

$$\mathcal{Q}(\omega,k_{\circ},\mu,\xi,\lambda,t,q) = \omega\, k_{\circ}\, h_2^2(r)(3\mu+1)(1+3\xi)^2[3+t]_q^{\lambda}[1+t]_q^{2\lambda}.$$

$$\Theta(\mu,\xi,\lambda,t,q) = h_3(r)(\mu+1)^2(1+3\xi)^2[3+t]_q^{\lambda}[1+t]_q^{2\lambda}.$$

Proof. Since $\mathrm{F}\in\mathcal{F}_{\Sigma}^{q}(\omega,\mu,\lambda,\xi,p,q,r,g)$, there are holomorphic functions η,θ belong to $\mathcal{A}$ and $\eta,\theta:\mathfrak{U}\to\mathfrak{U}$ given:

$$\eta(z) = \eta_1 z + \eta_2 z^2 + \eta_3 z^3 + \cdots \quad (z\in\mathfrak{U}),$$

$$\theta(w) = \theta_1 w + \theta_2 w^2 + \theta_3 w^3 + \cdots \quad (w\in\mathfrak{U}).$$

Such that $\eta(0)=\theta(0)=0$ and $\max\{|\eta(z)|,|\theta(w)|\}<1$, $(z,w\in\mathfrak{U})$ satisfying:

$$\frac{1}{\omega}\left[(1-\mu)\frac{z(\chi_{q,t}^{\lambda,\xi}\mathrm{F}(z))'}{\chi_{q,t}^{\lambda,\xi}\mathrm{F}(z)} + \mu\left(1+\frac{z(\chi_{q,t}^{\lambda,\xi}\mathrm{F}(z))''}{(\chi_{q,t}^{\lambda,\xi}\mathrm{F}(z))'}\right) - 1\right] = \Upsilon(z)[g(r,z)-a].$$

$$\frac{1}{\omega}\left[(1-\mu)\frac{w(\chi_{q,t}^{\lambda,\xi}g(w))'}{\chi_{q,t}^{\lambda,\xi}g(w)} + \mu\left(1+\frac{w(\chi_{q,t}^{\lambda,\xi}g(w))''}{(\chi_{q,t}^{\lambda,\xi}g(w))'}\right) - 1\right] = \Upsilon(w)[g(r,w)-a].$$

Equivalently:

$$\frac{1}{\omega}\left[(1-\mu)\frac{z(\chi_{q,t}^{\lambda,\xi}\mathrm{F}(z))'}{\chi_{q,t}^{\lambda,\xi}\mathrm{F}(z)} + \mu\left(1+\frac{z(\chi_{q,t}^{\lambda,\xi}\mathrm{F}(z))''}{(\chi_{q,t}^{\lambda,\xi}\mathrm{F}(z))'}\right) - 1\right] = k_{\circ} h_2(r)\eta_1 z + \left[k_{\circ}\{h_2(r)\eta_2 + h_3(r)\eta_1^2\} + k_1 h_2(r)\eta_1\right]z^2 + \cdots \tag{15}$$

$$\frac{1}{\omega}\left[(1-\mu)\frac{w(\chi_{q,t}^{\lambda,\xi}g(w))'}{\chi_{q,t}^{\lambda,\xi}g(w)} + \mu\left(1+\frac{w(\chi_{q,t}^{\lambda,\xi}g(w))''}{(\chi_{q,t}^{\lambda,\xi}g(w))'}\right) - 1\right] = k_{\circ} h_2(r)\theta_1 + \left[k_{\circ}\{h_2(r)\theta_2 + h_3(r)\theta_1^2\} + k_1 h_2(r)\theta_1\right]w^2 + \cdots \tag{16}$$

It follows from (15) and (16):

$$\omega k_{\circ} h_2(r)\eta_1 = \frac{(1+\mu)(1+3\xi)[1+t]_q^{\lambda}}{[2+t]_q^{\lambda}}a_2. \tag{17}$$

$$k_{\circ}\{h_2(r)\eta_2 + h_3(r)\eta_1^2\} + k_1 h_2(r)\eta_1 = (4\mu+2)(1+8\xi)\frac{[1+t]_q^{\lambda}}{[3+t]_q^{\lambda}}a_3 - (3\mu+1)(1+3\xi)^2\frac{[1+t]_q^{2\lambda}}{[2+t]_q^{2\lambda}}a_2^2. \tag{18}$$

$$\omega k_{\circ} h_2(r)\theta_1 = -\frac{(1+\mu)(1+3\xi)[1+t]_q^{\lambda}}{[2+t]_q^{\lambda}}a_2. \tag{19}$$

$$k_{\circ}\left\{\hbar_2(ɍ)\theta_2+\hbar_3(ɍ){\theta_1}^2\right\}+k_1\hbar_2(ɍ)\theta_1 = \left\{2(4ɥ+1)(1+8\xi)^2\frac{{[1+t]_q}^{\lambda}}{{[3+t]_q}^{\lambda}} - (3ɥ+1)(1+3\xi)^2\frac{{[1+t]_q}^{2\lambda}}{{[2+t]_q}^{2\lambda}}\right\}{a_2}^2 - (4ɥ+2)(1+8\xi)\frac{{[1+t]_q}^{\lambda}}{{[3+t]_q}^{\lambda}}a_3. \tag{20}$$

Equations (17) and (19), yield:

$$\eta_1=-\theta_1, \tag{21}$$

and

$$2(ɥ+1)^2(1+3\xi)^2\frac{{[1+t]_q}^{2\lambda}}{{[2+t]_q}^{2\lambda}}{a_2}^2 = \omega^2 {k_{\circ}}^2\hbar_2^2(ɍ)\left({\eta_1}^2+{\theta_1}^2\right). \tag{22}$$

Add (18) to (20) and using (22), we obtain:

$$\omega k_{\circ}\hbar_2(ɍ)(\eta_2+\theta_2) + \omega k_{\circ}\hbar_3(ɍ).\ 2(ɥ+1)^2(1+3\xi)^2\frac{{[1+t]_q}^{2\lambda}}{\omega^2 {k_{\circ}}^2\hbar_2^2(ɍ){[2+t]_q}^{2\lambda}}{a_2}^2 + \omega k_1\hbar_2(ɍ)(\eta_1+\theta_1) = \left[2(4ɥ+2)(1+8\xi)\frac{{[1+t]_q}^{\lambda}}{{[3+t]_q}^{\lambda}} - 2(3ɥ+1)(1+3\xi)^2\left(\frac{[1+t]_q}{[2+t]_q}\right)^{2\lambda}\right]{a_2}^2. \tag{23}$$

Further a computation using (22) in (23):

$$\omega\hbar_2(ɍ)\{k_{\circ}\} = 2\left[(4ɥ+2)(1+8\xi)\frac{{[1+t]_q}^{\lambda}}{{[3+t]_q}^{\lambda}} - \left\{\frac{\omega\ k_{\circ}\hbar_2^2(ɍ)(3ɥ+1)(1+3\xi)^2{[1+t]_q}^{\lambda} + \hbar_3(ɍ)(ɥ+1)^2(1+3\xi)^2{[1+t]_q}^{2\lambda}}{\omega k_{\circ}\hbar_2^2(ɍ){[2+t]_q}^{2\lambda}}\right\}\right]{a_2}^2.$$

$$2\omega\hbar_2^3(ɍ)k_{\circ}\left\{k_{\circ}(\eta_2+\theta_2)+k_1(\eta_1+\theta_1)\right\}{[3+t]_q}^{\lambda}{[2+t]_q}^{2\lambda} = 2\left[(4ɥ+2)(1+8\xi){[1+t]_q}^{\lambda}{[2+t]_q}^{2\lambda} - \{(3ɥ+1)(1+3\xi)^2 + \hbar_3(ɍ)(ɥ+1)^2(1+3\xi)^2\}{[3+t]_q}^{\lambda}{[1+t]_q}^{2\lambda}\right]{a_2}^2.$$

$$\omega\hbar_2^3(ɍ)k_{\circ}\left\{k_{\circ}(\eta_2+\theta_2)+k_1(\eta_1+\theta_1)\right\}{[3+t]_q}^{\lambda}{[2+t]_q}^{2\lambda} = 2\left[\omega\ k_{\circ}\hbar_2^2(ɍ)(4k_{\circ}+2)(1+8\xi){[1+t]_q}^{\lambda}{[2+t]_q}^{2\lambda} - \left\{\omega k_{\circ}\hbar_2^2(ɍ)(3ɥ+1)(1+3\xi)^2 + \hbar_3(ɍ)(ɥ+1)^2(1+3\xi)^2\right\}{[3+t]_q}^{\lambda}{[1+t]_q}^{2\lambda}\right]{a_2}^2.$$

$$\mathcal{T}(\omega,k_{\circ},ɥ,\xi,\lambda,t,q) = \omega\ k_{\circ}\hbar_2^2(ɍ)(4ɥ+2)(1+8\xi){[1+t]_q}^{\lambda}{[2+t]_q}^{2\lambda}.$$

$$\mathcal{Q}(\omega,k_{\circ},ɥ,\xi,\lambda,t,q) = \omega\ k_{\circ}\hbar_2^2(ɍ)(3ɥ+1)(1+3\xi)^2{[3+t]_q}^{\lambda}{[1+t]_q}^{2\lambda}.$$

$$\Theta(ɥ,\xi,\lambda,t,q) = \hbar_3(ɍ)(ɥ+1)^2(1+3\xi)^2\,{[3+t]_q}^{\lambda}{[1+t]_q}^{2\lambda}.$$

$${a_2}^2 = \frac{\omega\hbar_2^3(ɍ)k_{\circ}\left\{\begin{matrix}k_{\circ}(\eta_2+\theta_2)+\\ k_1(\eta_1+\theta_1)\end{matrix}\right\}\begin{matrix}{[3+t]_q}^{\lambda}\ {}^{2\lambda}\\ [2+t]\ {}_q\end{matrix}}{2\begin{bmatrix}\hbar_2^2(ɍ)\mathcal{T}(\omega,k_{\circ},ɥ,\xi,\lambda,t,q)-\\ \hbar_2^2(ɍ)\mathcal{Q}(\omega,k_{\circ},ɥ,\xi,\lambda,t,q)-\\ \hbar_3(ɍ)\Theta(ɥ,\xi,\lambda,t,q)\end{bmatrix}}. \tag{24}$$

$$|a_2|^2 \le \frac{\omega\hbar_2^3(ɍ)k_{\circ}\{k_{\circ}(|\eta_2|+|\theta_2|)+k_1(|\eta_1|+|\theta_1|)\}\ {[3+t]_q}^{\lambda}{[2+t]_q}^{2\lambda}}{2\begin{bmatrix}\hbar_2^2(ɍ)\mathcal{T}(\omega,k_{\circ},ɥ,\xi,\lambda,t,q)-\\ \hbar_2^2(ɍ)\mathcal{Q}(\omega,k_{\circ},ɥ,\xi,\lambda,t,q)-\\ \hbar_3(ɍ)\Theta(ɥ,\xi,\lambda,t,q)\end{bmatrix}}.$$

In view of $\max\{|\eta(z)|,|\theta(w)|\}<1$ together with (24) yield the desired estimate on a_2:

$|a_2|^2 \le$

$$\frac{ϖ b ɏ (bɏ)^2 ϖ k_{\circ}\{k_{\circ}+k_1\}[3+t]_q^{\lambda}[2+t]_q^{2\lambda}}{\begin{bmatrix} (bɏ)^2\mathcal{T}(ϖ,k_{\circ},\mu,\xi,\lambda,t,q)- \\ (bɏ)^2\mathcal{Q}(ϖ,k_{\circ},\mu,\xi,\lambda,t,q)-\not{p}bɏ^2\Theta(\mu,\xi,\lambda,t,q) \\ -qa\Theta(\mu,\xi,\lambda,t,q)\end{bmatrix}}.$$

$|a_2| \le$

$$\frac{|bɏ|[2+t]_q^{\lambda}\sqrt{ϖ|bɏ|k_{\circ}\{k_{\circ}+k_1\}[3+t]_q^{\lambda}}}{\sqrt{\begin{bmatrix} (bɏ)^2\mathcal{T}(ϖ,k_{\circ},\mu,\xi,\lambda,t,q)- \\ (bɏ)^2\mathcal{Q}(ϖ,k_{\circ},\mu,\xi,\lambda,t,q)-\not{p}bɏ^2\Theta(\mu,\xi,\lambda,t,q) \\ -qa\Theta(\mu,\xi,\lambda,t,q)\end{bmatrix}}}.$$

In order to find a_3, we subtract (20) from (18) to obtain:

$$\begin{aligned}(4\mu+2)(1+8\xi)\frac{[1+t]_q^{\lambda}}{[3+t]_q^{\lambda}}(a_3-a_2^{\,2}) &= ϖ k_{\circ}\hbar_2(ɏ)(\eta_2-\theta_2) \\ &+ ϖ k_{\circ}\hbar_3(ɏ).\left(\eta_1^{\,2}-\theta_1^{\,2}\right) \\ &+ 2\,ϖ k_1\hbar_2(ɏ)\eta_1 .\end{aligned} \tag{25}$$

Now the desired estimate on a_3 follows from (21), (22) and (25), at once:

$$\begin{aligned} a_3 = &\frac{ϖ^2 k_{\circ}^{\,2}\hbar_2^2(ɏ)(\eta_1^{\,2}+\theta_1^{\,2})[2+t]_q^{2\lambda}}{2(\mu+1)^2(1+3\xi)^2[1+t]_q^{2\lambda}} \\ &+\frac{ϖ k_{\circ}\hbar_2(ɏ)(\eta_2-\theta_2)[3+t]_q^{\lambda}}{2(4\mu+2)(1+8\xi)[1+t]_q^{\lambda}} \\ &+\frac{2\,ϖ k_1\hbar_2(ɏ)\eta_1[3+t]_q^{\lambda}}{2(4\mu+2)(1+8\xi)[1+t]_q^{\lambda}}.\end{aligned}$$

$$\begin{aligned} |a_3| \le &\frac{ϖ^2 k_{\circ}^{\,2}(bɏ)^2[2+t]_q^{2\lambda}}{(\mu+1)^2(1+3\xi)^2[1+t]_q^{2\lambda}} \\ &+\frac{ϖ k_{\circ}|bɏ|[3+t]_q^{\lambda}}{(4\mu+2)(1+8\xi)[1+t]_q^{\lambda}} \\ &+\frac{ϖ k_1|bɏ|[3+t]_q^{\lambda}}{(4\mu+2)(1+8\xi)[1+t]_q^{\lambda}}.\end{aligned}$$

By specializing the parameters λ, ξ, and μin the above theorem, we obtain several important corollaries corresponding to particular subclasses of the introduced family.

2.2 Corollary (Special Case for the Parameter λ)

If $Ŧ\epsilon\mathcal{F}_{\Sigma}^{q}(ϖ,\mu,\lambda,\not{p},q,ɏ,g,),ϖ\in\mathbb{C}\setminus\{0\}, 0\le\mu\le 1$, then:
$|a_2| =$

$$\frac{|bɏ|[2+t]_q^{\lambda}\sqrt{ϖ|bɏ|k_{\circ}\{k_{\circ}+k_1\}[3+t]_q^{\lambda}}}{\sqrt{\begin{bmatrix} (bɏ)^2\mathcal{T}(ϖ,k_{\circ},\mu,\lambda,t,q)- \\ (bɏ)^2\mathcal{Q}(ϖ,k_{\circ},\mu,,\lambda,t,q)-\not{p}bɏ^2\Theta(\mu,\lambda,t,q) \\ -qa\Theta(\mu,\lambda,t,q)\end{bmatrix}}},$$

$$\begin{aligned} |a_3| \le &\frac{ϖ^2 k_{\circ}^{\,2}(bɏ)^2[2+t]_q^{2\lambda}}{(\mu+1)^2[1+t]_q^{2\lambda}}+\frac{ϖ k_{\circ}|bɏ|[3+t]_q^{\lambda}}{(4\mu+2)\ [1+t]_q^{\lambda}} \\ &+\frac{ϖ k_1|bɏ|[3+t]_q^{\lambda}}{(4\mu+2)\ [1+t]_q^{\lambda}}.\end{aligned}$$

2.3 Corollary (Special Case for the Parameter ξ)

If $Ŧ\epsilon\mathcal{F}_{\Sigma}^{q}(ϖ,\mu,\xi,\not{p},q,ɏ,g,),ϖ\in\mathbb{C}\setminus\{0\}, 0\le\mu\le 1$, then:
$|a_2| =$

$$\frac{|bɏ|\sqrt{ϖ|bɏ|k_{\circ}\{k_{\circ}+k_1\}}}{\sqrt{\begin{bmatrix} (bɏ)^2\mathcal{T}(ϖ,k_{\circ},\mu,\xi,t,q)- \\ (bɏ)^2\mathcal{Q}(ϖ,k_{\circ},\mu,,\xi,t,q)-\not{p}bɏ^2\Theta(\mu,\xi,t,q) \\ -qa\Theta(\mu,\xi,t,q)\end{bmatrix}}},$$

$$\begin{aligned} |a_3| \le &\frac{ϖ^2 k_{\circ}^{\,2}(bɏ)^2}{(\mu+1)^2(1+3\xi)^2}+\frac{ϖ k_{\circ}|bɏ|}{(4\mu+2)(1+8\xi)} \\ &+\frac{ϖ k_1|bɏ|}{(4\mu+2)(1+8\xi)}.\end{aligned}$$

2.4 Corollary (Special Case for $\mu = 1$)

If $Ŧ\epsilon\mathcal{F}_{\Sigma}^{q}(ϖ,\lambda,\xi,\not{p},q,ɏ,g,),ϖ\in\mathbb{C}\setminus\{0\}$, then:

$$|a_2| \le \frac{|bɏ|[2+t]_q^{\lambda}\sqrt{ϖ|bɏ|k_{\circ}\{k_{\circ}+k_1\}[3+t]_q^{\lambda}}}{\sqrt{\begin{bmatrix} (bɏ)^2\mathcal{T}(ϖ,k_{\circ},1,\xi,\lambda,t,q)- \\ (bɏ)^2\mathcal{Q}(ϖ,k_{\circ},1,\xi,\lambda,t,q)- \\ \not{p}bɏ^2\Theta(1,\xi,\lambda,t,q)-qa\Theta(1,\xi,\lambda,t,q)\end{bmatrix}}}.$$

$$\begin{aligned} |a_3| \le &\frac{ϖ^2 k_{\circ}^{\,2}(bɏ)^2[2+t]_q^{2\lambda}}{4(1+3\xi)^2[1+t]_q^{2\lambda}}+\frac{ϖ k_{\circ}|bɏ|[3+t]_q^{\lambda}}{6\,(1+8\xi)\,[1+t]_q^{\lambda}} \\ &+\frac{ϖ k_1|bɏ|[3+t]_q^{\lambda}}{6\,(1+8\xi)\,[1+t]_q^{\lambda}}.\end{aligned}$$

2.5 Corollary (Special Case for the Associated Polynomial Family)

If $Ŧ\epsilon\mathbb{Q}_{\Sigma}^{q}(ϖ,\xi,\not{p},q,ɏ,g),ϖ\in\mathbb{C}\setminus\{0\}$, then:

$$|a_2| = \frac{|b\text{ɍ}|\sqrt{\varpi|b\text{ɍ}|\mathrm{k}_{\circ}\{\mathrm{k}_{\circ}+\mathrm{k}_1\}}}{\sqrt{\begin{bmatrix} 2\varpi(b\text{ɍ})^2\mathrm{k}_{\circ}\hbar_2^2(\text{ɍ})(1+8\xi) - \\ \varpi\,\mathrm{k}_{\circ}\hbar_2^2(\text{ɍ})(1+3\xi)^2(b\text{ɍ})^2 - \\ \wp b\text{ɍ}^2\hbar_3(\text{ɍ})(1+3\xi)^2 - qa\hbar_3(\text{ɍ})(1+3\xi)^2 \end{bmatrix}}},$$

$$|a_3| \le \frac{\varpi^2\,\mathrm{k}_{\circ}^{\,2}(b\text{ɍ})^2}{(1+3\xi)^2} + \frac{\varpi\mathrm{k}_{\circ}|b\text{ɍ}|}{2\,(1+8\xi)} + \frac{\varpi\mathrm{k}_1|b\text{ɍ}|}{2\,(1+8\xi)}.$$

2.6 Corollary (Special Case for the $\mathcal{K}^q$-Class)

If $\text{Ŧ}\epsilon\mathrm{K}_{\Sigma}^{\mathrm{q}}(\varpi,\xi,\wp,q,\text{ɍ},g)$, $\varpi \in \mathbb{C}\setminus\{0\}$, then:

$$|a_2| = \frac{|b\text{ɍ}|\sqrt{\rho|b\text{ɍ}|\mathrm{k}_{\circ}\{\mathrm{k}_{\circ}+\mathrm{k}_1\}}}{\sqrt{\begin{bmatrix} \left(6\varpi\,\mathrm{k}_{\circ}\hbar_2^2(\text{ɍ})(1+8\xi)b\text{ɍ}\right)^2 - \\ 4\varpi\mathrm{k}_{\circ}\hbar_2^2(\text{ɍ})(1+3\xi)^2(b\text{ɍ})^2 - \\ 2\hbar_3(\text{ɍ})(1+3\xi)^2\wp b\text{ɍ}^2 - 2\hbar_3(\text{ɍ})(1+3\xi)^2 qa \end{bmatrix}}},$$

$$|a_3| \le \frac{\varpi^2\,\mathrm{k}_{\circ}^{\,2}(b\text{ɍ})^2}{2(1+3\xi)^2} + \frac{\varpi\mathrm{k}_{\circ}|b\text{ɍ}|}{6\,(1+8\xi)} + \frac{\varpi\mathrm{k}_1|b\text{ɍ}|}{6\,(1+8\xi)}.$$

3 CONCLUSIONS

In this paper, a new subclass of analytic and bi-univalent functions defined in the open unit disk has been introduced by means of a newly constructed operator in combination with the concept of quasi-subordination and Horadam polynomials. This formulation provides a flexible setting that includes several previously studied subclasses as special cases under appropriate choices of the involved parameters.

By applying this operator together with the properties of quasi-subordination and the generating function of Horadam polynomials, explicit bounds for certain Taylor–Maclaurin coefficients were obtained. In particular, the initial coefficients were estimated in a clear and unified manner.

The results indicate that the proposed operator serves as an effective tool in the study of coefficient problems for bi-univalent functions. In addition, the use of Horadam polynomials offers a useful connection between special functions and geometric function theory.

The approach presented here may be extended in future work to investigate higher-order coefficients or to study related subclasses defined through similar operators and other families of special polynomials.

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