Expansion for Double ZZ-Shehu Transformation to Complex Kernel

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Abstract: The essential value of the double transform is how it can take the partial differential equation (PDE) that describes the physical process occurring in a complex differential domain and transform the PDE to a much simpler algebraic form. This procedure converts the partial derivatives of the PDE into algebraic operations and provides an efficient means for the simultaneous coupling of both the spatial and temporal parameters (variables) involved. On the other hand, the single transform has significant limitations; it only partially simplifies the PDE into ODE form and still requires a complex analytical solution to an ODE. Also, the single transform has great difficulty combining both the boundary and initial conditions into one unified analytical framework (solution), but the double transform allows both conditions to be contained within the solution of the algebraic equation, which significantly reduces the potential for analytical error. The present study introduces a new extension of the Double ZZ Shehu Transform that employs a special complex kernel. This extension was analytically proven to exist and have a unique solution as well as possessing significant analytical properties that are different from the double ZZ Shehu Transform. To showcase the real world application of this extension, the new transform was applied to simulate the flow of fluid through cylindrical conduits. Specifically, hydraulic scenarios that were investigated consist of fluid flow in a pipe with a stationary versus a moving pipe wall.

1 INTRODUCTION

Differential equations are considered one of the most important mathematical applications in other sciences such as engineering, physics, medicine, and financial accounting [18]-[26], [31], [34], [35]. Therefore, solution techniques have developed to suit them [1]-[3], [14], [27], [29]. One of the physical applications is fluid flow in tubes, which is represented by the one-dimensional Navier-Stokes equation [4], [15], [32], [33].

Single and double integral transforms are powerful methods that can be used to solve differential equations. A kernel function is used in conjunction with specific integration limits to define the properties of an integral transform. The single transform uses one variable and is referred to as transforms (for example, Laplace, Elzaki, Novel, etc.) to solve ordinary differential equations. Single transforms work on one variable [5]-[7], [30], while double integral transforms can be used to solve partial differential equations by addressing multiple independent variables. This is why double integral transforms are an effective way to solve these equations more efficiently than single integral transforms [8]-[11], [28].

Transformations by means of complex double integrals extend this concept further than that by using 2-dimensional integrals of a given function in terms of two complex kernels [12], [13], [16], [17]. The motivation for integrating the ZZ and Shehu transforms is due to their ability to work well together as they have synergistic properties. Because of this, the ZZ transform can bridge the gap between the Laplace and Sumudu transforms by being highly flexible, and the Shehu transform allows for easier differentiation of functions at different orders. As a result, this paper will seek to find solutions to linear partial differential equations via an innovative means using complex double integral transformations.

2 PRELIMINARIES

This section presents the fundamental definitions of the single complex ZZ transform, the single complex Shehu transform, and their double complex ZZ-

Shehu counterpart, together with its inverse, which underlie the derivations that follow.

- Definition (1). Let $£(t)$ be a function defined for all $t \geq 0$; then the complex ZZ transform of $£(t)$ is the function defined by:

$$Z\,(i\theta,\omega) = Z\{£(t)\} = \frac{\theta}{\omega}\int_0^{\infty} £(t)\, e^{-i\frac{\theta}{\omega}t}\, dt.$$

- Definition (2): The single complex Shehu transformation of a real-valued function with respect to the variable w, is defined by:

$$S=[\{£(¥)] = A(i\delta,\mu) = \int_0^{\infty} £(w)e^{-i\frac{\delta}{\mu}w}dw,\ \delta,\mu\ > 0.$$

- Definition (3): The double Complex ZZ-Shehu of function £(t,w) is denoted by the operator Ψ_2^c and given by:

$$\Psi_2^c\{£(t,w)\}=ZS[(i\theta,i\delta),(\omega,\mu)] = \frac{\theta}{\omega}\int_0^{\infty}\int_0^{\infty} e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}\, £(t,w)dtdw.$$

The inverse double ZZ− Shehu transform of Ψ_2 is defined as:

$$\Psi_2^{-1}\{\Psi_2\{Ⱥ(n,m)\}\} = ZS^{-1}\,(ZS[(Ҥ,ɟ),(в,\mu)]),$$

$$= \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{Ҥ}{в}Z[(Ҥ,ɟ),(в,\mu)]e^{-\frac{Ҥ}{в}n}dв\ .\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{1}{\mu}\, S[(Ҥ,ɟ),(в,\mu)]e^{-\frac{ɟ}{\mu}m}\,dɟ.$$

Where ZS^{-1} is the inverse of transform operator.

3 DOUBLE COMPLEX ZZ-SHEHU TRANSFORM FOR SOME FUNCTIONS

Using the definitions above, the double complex ZZ-Shehu transform is now derived for several elementary and commonly encountered functions.

- If $£\,(t,w) = 1$, then $\{\Psi_2^c(£(t,w)\} = ZS[(i\theta,i\delta),(\omega,\mu)]$

$$= \frac{\theta}{\omega}\int_0^{\infty}\int_0^{\infty} e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}\ dtdw$$

$$=\int_0^{\infty} e^{-i\frac{\delta}{\mu}w}dw = -\frac{\mu}{\delta}.$$

- If $£\,(t,w) = e^{(at+bw)}$, then $\{\Psi_2^c(£\,(t,w))\} = ZS[(i\theta,i\delta),(\omega,\mu)]$

$$= \frac{\theta}{\omega}\int_0^{\infty}\int_0^{\infty} e^{(at+bw)}\ e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}dtdw$$

$$= \frac{\theta}{\omega}\int_0^{\infty}\int_0^{\infty} e^{-(\frac{\theta}{\omega}i-a)t}\ \ e^{-\left(\frac{\delta}{\mu}i-b\right)w}\ dt\,dw$$

$$= \frac{\theta\ \mu}{(\theta i - a\omega)\ \ (\delta i - b\mu)}\ .$$

- If $£(t,w) = x^{\epsilon}\ y^{\rho}\quad \epsilon,\rho = 0,1,2,\ldots$, then $\{\Psi_2^c(£\,(t,w))\} = ZS[(\theta i,\delta i),(\omega,\mu)]$

$$= \frac{\theta}{\omega}\int_0^{\infty}\int_0^{\infty} x^{\epsilon}\ y^{\rho}\ \ e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}\ dtdw$$

$$= \frac{\theta}{\omega}\int_0^{\infty} e^{-i\frac{\theta}{\omega}t}\ x^{\epsilon}dt\ \int_0^{\infty} e^{-i\frac{\delta}{\mu}w}\ y^{\rho}\ dw$$

$$= \epsilon!\ \rho!\left(\frac{\omega}{\theta}i\right)^{\epsilon}\ \left(\frac{\delta}{\mu}i\right).^{\rho+1}$$

- If $£(t,w) = \cos(at+bw)$, then $\{\Psi_2^c\{(£\,(t,w))\} = ZS[(i\theta,i\delta),(\omega,\mu)]$

$$= \frac{\theta}{\omega}\int_0^{\infty}\int_0^{\infty}\cos(at + bw)\ \ e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}\ dtdw$$

$$= \frac{\theta}{\omega}\int_0^{\infty}\int_0^{\infty}\left(\frac{e^{i(at+bw)} + e^{-i(at+bw)}}{2}\right)\ e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}\ \ dtdw$$

$$= \frac{1}{2}\left(\frac{\theta}{\omega}e^{i(at+bw)}e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}dtdw + \frac{\theta}{\omega}\int_0^{\infty}\int_0^{\infty} e^{-i(at+bw)}e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}dtdw\right).$$

$$= \frac{-\theta\mu(\theta\delta + \alpha\mu\omega b)}{(\theta^2 - a^2\omega^2)\ (\delta^2 - b^2\mu^2)}$$

- If $£(t,w) = \sin(at+bw)$ $\{\Psi_2^c(£\,(t,w))\} = ZS[(\theta i,\delta i),(\omega,\mu)]$

$$= \frac{\theta}{\omega}\int_0^{\infty}\int_0^{\infty} sin(at + bw)\ \ \ e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}\ dtdw$$

$$= \frac{\theta}{\omega}\int_0^{\infty}\int_0^{\infty}\left(\frac{e^{i(at+bw)} - e^{-i(at+bw)}}{2i}\right)e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}dtdw$$

$$= \frac{1}{2i}(\frac{\theta}{\omega}\int\int_0^{\infty} e^{i(at+bw)}\ e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}dtdw - \frac{\theta}{\omega}\int_0^{\infty}\int_0^{\infty} e^{-i(at+bw)}e^{-\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}dtdw$$

$$= \frac{i\theta\mu(\theta\mu b + \delta\omega a)}{(\theta^2 - a^2\omega^2)\ (\delta^2 - \mu^2 b^2)}.$$

- If $£\,(t,w) = \cos h(at+bw)$, then

$$\{\Psi_2^c\,(£(t,w))\} = ZS[(\theta i,\delta i),(\omega,\mu)] = \frac{\theta}{\omega}\int_0^\infty\int_0^\infty cosh(at + bw)\; e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}dtdw$$

$$= \frac{\theta}{\omega}\int_0^\infty\int_0^\infty \left(\frac{e^{(at+bw)}+e^{-(at+bw)}}{2}\right) e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}\,dtdw$$

$$= \frac{1}{2}\Big(\frac{\theta}{\omega}\int\int_0^\infty e^{(at+bw)}\,e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}dtdw + \frac{\theta}{\omega}\int_0^\infty\int_0^\infty e^{-((ax+b¥))}e^{-i\left(\frac{\theta}{\omega}x+\frac{\delta}{\mu}¥\right)}dxd¥\Big)$$

$$= \frac{\theta\mu(\alpha\mu\omega b-\theta\delta)}{(\theta^2+\alpha^2\omega^2)\,(\delta^2+b^2\mu^2)}.$$

- If $£(t,w) = s\,inh(at+bw)$, then

$$\{\Psi_2^c(£(t,w))\} = ZS[(\theta i,\delta i),(\omega,\mu)] = \frac{\theta}{\omega}\int_0^\infty\int_0^\infty sinh(at + bw)\quad e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}dtdw$$

$$= \frac{\theta}{\omega}\int_0^\infty\int_0^\infty \left(\frac{e^{(at+bw)}-e^{-(at+bw)}}{2}\right) e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}\,dtdw$$

$$= \frac{1}{2}\Big(\frac{\theta}{\omega}\int\int_0^\infty e^{(at+bw)}\,e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}dtdw - \frac{\theta}{\omega}\int_0^\infty\int_0^\infty e^{-(at+bw)}e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}dtdw\Big)$$

$$= \frac{i\theta\mu(\theta\mu b+a\omega\delta)}{(\theta^2+\alpha^2\omega^2)\,(\delta^2+b^2\mu^2)}.$$

4 SOME PROPERTY AND THEOREMS OF THE DOUBLE COMPLEX ZZ-SHEHU TRANSFORM

4.1 The Linear Properties of the Double Complex ZZ - Shehu Transform

Like its single-variable counterparts, the double complex ZZ-Shehu transform is a linear operator, as stated below.

$$\Psi_2^c\{a_1£_1(t,w) \pm a_2£_2(t,w) \pm \cdots \pm a_n£_n(t,w)\} = a_1\,\Psi_2^c\,(£_1(t,w)) \pm \cdots \pm a_n\,\Psi_2^c(£_n(t,w))$$

From define

$$\Psi_2^c\,\{a_1£_1\,(t,w) \pm \cdots \pm a_n£_n(t,w)\}$$

$$= \frac{\theta}{\omega}\int_0^\infty\int_0^\infty \begin{Bmatrix} a_1£_1(t,w)\} \pm \cdots \\ \pm a_n£_n(t,w) \end{Bmatrix} e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}\quad dtdw$$

$$= \frac{\theta}{\omega}\int_0^\infty\int_0^\infty \{a_1£_1(t,w)\}\,e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}\quad dtdw\ \pm\cdots$$

$$\pm\frac{\theta}{\omega}\int_0^\infty\int_0^\infty a_n£_n(t,w)\ \ e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}\ \ dtdw$$

$$= a_1\frac{\theta}{\omega}\int_0^\infty\int_0^\infty \{£_1(t,w)\}\,e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}\,dtdw\ \pm\cdots$$

$$\pm a_n\frac{\theta}{\omega}\int_0^\infty\int_0^\infty £_n(x,¥)\ \ e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}\ \ dtdw$$

$$= a_1\,\Psi_2^c(\,£_1(t,w) \pm \cdots \pm a_n\Psi_2^c(\,£_n(t,w))$$

4.2 Theorem (Convolution Theorem)

Let $f(t,w)$ and $h(t,w)$ have double complex ZZ-Shehu transforms, then $\Psi_2^c\{(f ** h)(\theta i,\delta i)(\omega,\mu)\}= \frac{\theta}{\omega}\Psi_2^c\{\,f(\theta,\delta)(\omega,\mu)\}\Psi_2^c\{h\,(\theta,\delta)(\omega,\mu)\}$.

Proof; By definition of double complex ZZ-Shehu transformation:

$$\Psi_2^c\,\{(f ** h)(\theta i,\delta i)(\omega,\mu)\}= \frac{\theta}{\omega}\int_0^\infty\int_0^\infty e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}(f ** h)(\theta,\delta)(\omega,\mu)\,dtdw$$

$$= \frac{\theta}{\omega}\int_0^\infty\int_0^\infty e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}\Big(\int_0^{¥}\int_0^{x} f(a,b)h(t-a)(w-b)dadb\Big)\,dtdw$$

$$\Psi_2^c\,\{(f ** h)(\theta i,\delta i)(\omega,\mu)\}= \frac{\theta}{\omega}\Psi_2^c\{\,f(\theta,\delta)(\omega,\mu)\}\Psi_2^c\{h\,(\theta,\delta)(\omega,\mu)$$

4.3 Theorem

Double complex ZZ-Shehu transformation of first and second order partial derivatives are in the form:

(1) $\Psi_2^c\left\{\frac{\partial £(t,w)}{\partial t}\right\} = \left(\frac{\theta}{\omega}\right)i\Psi_2^c\big(£(t,w)\big) - \left(\frac{\theta}{\omega}\right)S^c(£(0,w))$.

(2) $\Psi_2^c\left\{\frac{\partial^2 £(t,w)}{\partial^2 t}\right\}$
$$= \left(\frac{\theta^2}{\omega^2}\right)(-\Psi_2^c(£(t,w)) - i\left(\frac{\theta^2}{\omega^2}\right)S^c(£(0,w)) - \left(\frac{\theta}{\omega}\right)\left(\frac{\partial}{\partial x} S^c(£(0,w))\right)$$

(3) $\Psi_2^c\left\{\frac{\partial £(t,w)}{\partial w}\right\}$
$$= (\frac{\delta}{\mu})i\,\Psi_2^c(£(t,w)) - Z^C(£(t,0))$$

(4) $\Psi_2^c\left\{\frac{\partial^2 £(t,w)}{\partial y^2}\right\} = \left(\frac{\delta^2}{\mu^2}\right)(-\Psi_2^c(£(t,w)) - i\left(\frac{\delta}{\mu}\right)Z^C(£(t,0)) - Z\left(\frac{\partial}{\partial y} Z^C(£(t,0)\right).$

Proof. The proof can be found from definition of double ZZ -Shehu transform as:

(1) $\Psi_2^c\left\{\frac{\partial £(t,w)}{\partial t}\right\} = \left(\frac{\theta}{\omega}\right)i\Psi_2^c(£(t,w)) - (\frac{\theta}{\omega})S^c£(0,w)$
$$= \frac{\theta}{\omega}\int_0^\infty\int_0^\infty \frac{d}{dt}\, £(t,w)e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}\,dtdw$$
$$= \frac{\theta}{\omega}\int_0^\infty e^{-i\frac{\delta}{\mu}w}\,[\int_0^\infty e^{-i\frac{\theta}{\omega}t}\,\frac{d}{dt}£(t,w)\,dt]\,dw$$
$$= -£(0,w) + i\frac{\theta}{\omega}Z^c£(t,w)$$
$$= \frac{\theta}{\omega}\int_0^\infty e^{-i\frac{\delta}{\mu}w}\,[i\frac{\theta}{\omega}Z^c£(t,w) - £(0,w)]dw$$
$$= (\frac{\theta}{\omega})i\Psi_2^c\,£(t,w) - \frac{\theta}{\omega}S^C(£(0,w)).$$

(2) $\Psi_2^c\left\{\frac{\partial^2 £(t,w)}{\partial^2 w}\right\}$
$$= \frac{\theta}{\omega}\int_0^\infty\int_0^\infty \frac{d^2}{dt^2}\, £(t,w)\, e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}\,dtdw$$
$$= -\frac{d}{dx}£(0,w) + i\frac{\theta}{\omega}[iZ^C£(t,w) - £(0,w)]$$
$$= \frac{\theta}{\omega}\int_0^\infty\left[\frac{\theta}{\omega}(-Z^C£(x,y)) - i\varphi\frac{\theta}{\omega}£(0,w) - \frac{d}{dt}£(0,w)\right]e^{-i\frac{\delta}{\mu}w}\,dw$$
$$= \left(\frac{\theta^2}{\omega^2}\right)(-\Psi_2^c(£(t,w)) - i\left(\frac{\theta^2}{\omega^2}\right)S^c(£(0,w)) - \left(\frac{\theta}{\omega}\right)\left(\frac{\partial}{\partial t} S^c(£(0,w)\right).$$

(3) $\Psi_2^c\left\{\frac{\partial £(t,w)}{\partial w}\right\}$
$$= (\frac{\delta}{\mu})i\,\Psi_2^c(£(t,w)) - Z^C(£(t,0))$$
$$= \frac{\theta}{\omega}\int_0^\infty\int_0^\infty \frac{d}{dy}\, £(t,w)\, e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}\,dtdw$$
$$= \frac{\theta}{\omega}\int_0^\infty e^{-i\frac{\theta}{\omega}t}\,[\int_0^\infty e^{-i\frac{\delta}{\mu}w}\,dw]\,dt$$
$$= -£(t,0) + i\frac{\delta}{\mu}S^c£(t,w)$$
$$= \frac{\theta}{\omega}\int_0^\infty e^{-i\frac{\theta}{\omega}t}[i\frac{\delta}{\mu}S^c£(t,w) - £(t,0)]dt$$
$$= (\frac{\delta}{\mu})i\Psi_2^c\,£\,(t,w) - Z^C(£(t,0)).$$

(4) $\Psi_2^c\left\{\frac{\partial^2 £(t,w)}{\partial w^2}\right\}$
$$= \frac{\theta}{\omega}\int_0^\infty\int_0^\infty \frac{d^2}{dw^2}\, £(t,w)\, e^{-i\left(\frac{\theta}{\omega}t+\frac{\delta}{\mu}w\right)}\,dtdw$$
$$= -\frac{d}{dw}£(t,0) + i\frac{\theta}{\omega}\left[i\frac{\delta}{\mu}S^C£(t,w) - £(t,0)\right]$$
$$= \frac{\theta}{\omega}\int_0^\infty\left[\frac{\delta^2}{\mu^2}(-S^C£(t,w)) - i\frac{\delta}{\mu}£(t,0) - \frac{d}{dt}£(t,0)\right]e^{-i\frac{\delta}{\mu}w}\,dw$$
$$= \left(\frac{\delta^2}{\mu^2}\right)(-\Psi_2^c(£(t,w)) - i\left(\frac{\delta}{\mu}\right)Z^C(£(t,0)). - \left(\frac{\partial}{\partial w} Z^C£(t,0)\right).$$

4.4 The General Formula of Exact Solution to Second Order Partial Differential Equation

Consider the general k-th partial differential equation:

$$\mathrm{F}(D^k£(t), D^{k-1}£(t), \ldots\ldots, D£(t), t) = 0, t \in R^n. \quad (1)$$

Where k and n represent the order and dimension, respectively.

By taking k = 2, then (1) becomes:

$$\mathrm{F}(D^2£(t), D\,£(t), t) = 0, t \in R^n.$$

Such that:

$$D£(t) = £_{t1}, £_{t2}, £_{t3}, \ldots\ldots £_{tn}, D^2£(t) = £_{t1t1}, £_{t1t2}, \ldots\ldots £_{t1tn}, £_{t2t1}, £_{t2t2}, \ldots £_{t2tn}, \ldots\ldots £_{tnt1}, £_{tnt2}, \ldots\ldots, £_{tntn}.$$

On the otherwise, if n = 2 has the equation:

$$£_{tt} + £_{ww} + £_{tw} + £_{wt} + £_t + £_w + £ = \emptyset(t,w). \quad (2)$$

With

$£(t,0) = \alpha_1$ $\qquad$ $£(0,w) = \alpha_3$
$£_y(t,0) = \alpha_2$ $\qquad$ $£_t(0,w) = \alpha_4$
$£(0,0) = \alpha_5$

To find general solution for (2) under initial conditions by Via complex Shehu-ZZ transform, obtained:

$$(\frac{\theta^2}{\omega^2})(-\Psi_2^c(£(t,w)) - i(\frac{\theta^2}{\omega^2})S^c(£(0,w)) - (\frac{\theta}{\omega})S^c$$
$$(\frac{d}{dt}£(0,w) - (\frac{\delta^2}{\mu^2})$$
$$(-\Psi_2^c(£(t,w)) - i\left(\frac{\delta}{\mu}\right)Z^C(£(t,0) - Z^C(\frac{d}{dw}£(t,0) +$$
$$(\frac{\theta\delta}{\omega\mu})(\Psi_2^c(£(t,w)) - i\left(\frac{\theta\delta}{\omega\mu}\right)S^c(£(0,w)) -$$
$$i\left(\frac{\theta}{\omega}\right)Z^C(£(t,0)) + \left(\frac{\theta}{\omega}\right)£(0,0) + (\frac{\theta\delta}{\omega\mu})\Psi_2^c(£(t,w)) -$$
$$i\left(\frac{\theta}{\omega}\right)Z^C(£(t,0)) - i(\frac{\theta\delta}{\mu\omega})S^c(£(0,w)) + \left(\frac{\theta}{\omega}\right)£(0,0) +$$
$$\left(\frac{\theta}{\omega}\right)i\Psi_2^c(£(t,w)) - \left(\frac{\theta}{\omega}\right)S^c£(0,w) + \left(\frac{\delta}{\mu}\right)i\Psi_2^c(£(t,w)) -$$
$$Z^C(£(t,0)) + \Psi_2^c(£(t,w)) = \emptyset(t,w).$$

$$(-\frac{\theta^2}{\omega^2} - \frac{\delta^2}{\mu^2} + \frac{\theta\delta}{\omega\mu} + \frac{\theta\delta}{\omega\mu} + \frac{\theta}{\omega}i + \frac{\delta}{\mu}i + 1)\,\Psi_2^c(£(t,w)) =$$
$$(\frac{\theta^2\mu\alpha_3 i}{\omega^2\delta} + .\frac{\theta\mu\alpha_4}{\omega\delta} + \frac{\delta\alpha_1 i}{\mu} + \alpha_2 + \frac{\theta\delta\mu\alpha_3 i}{\omega\mu\delta} + \frac{\theta\alpha_1 i}{\omega} - \frac{\theta\alpha_5}{\omega} + \frac{\theta\alpha_1 i}{\omega} +$$
$$\frac{\theta\delta\mu\alpha_3 i}{\omega\mu\delta} - \frac{\theta\alpha_5}{\omega} + \frac{\theta\mu\alpha_3}{\omega\delta} + \alpha_1 + \emptyset(t,w).$$

$$\beta\{(\theta i,\delta i)(\omega,\mu)\}.\ \Psi_2^c(£(t,w)) = \mathrm{H}\ \{(\theta i,\delta i)(\omega,\mu);\ \alpha_i\} + \Psi_2^c\{\emptyset(t,w)\}.$$

Where $\beta\{(\theta i,\delta i)(\omega,\mu)\}$ and $H\{(\theta i,\delta i)(\omega,\mu);\ \alpha_i\}$ represent the coefficients of $\Psi_2^c(\varphi(x,y))$ and the substitution of initial conditions, respectively.

$$\Psi_2^c(£(t,w)) = \frac{\mathrm{H}\{(\theta i,\delta i)(\omega,\mu);\ \alpha_i\} + \emptyset(x,y)}{\beta\{(\theta i,\delta i)(\omega,\mu)\}}. \quad (3)$$

The exact solution of (2) obtained by taking $\Psi_2^{c\ -1}$ of both side for the above equation:

$$£(t,w) = \Psi_2^{c\ -1}[\frac{\mathrm{H}\{(\theta i,\delta i)(\omega,\mu);\ \alpha_i\} + \emptyset(x,y)}{\beta\{(\theta i,\delta i)(\omega,\mu)\}}]. \quad (4)$$

To apply this formula on the following equation:

$$£_{tt} = £_{ww} + 2£_w + £.$$

With

$£(t,0) = e^t$ $\quad$ $£_w = (t\ .0) = -2e^t$

$£(0,w) = e^{-2w}$ $\quad$ $£_t = (0,w) = e^{-2w}$

The exact solution can be get by using (4):

$$£(t,w) = \Psi_2^{c\ -1}[\frac{\mathrm{H}\{(\theta i,\delta i)(\omega,\mu);\ \alpha_i\} + \emptyset(x,y)}{\beta\{(\theta i,\delta i)(\omega,\mu)\}}]$$
$$= \Psi_2^{c\ -1}\ [\frac{\theta(-\theta^2\mu^2 - \mu^2\omega^2 + \omega^2\delta^2 - 2\omega^2\delta\mu i)}{\mu\omega^2(\delta i + 2\mu)(\theta i - \mu)}$$
$$\frac{\omega^2\mu^2}{(-\theta^2\mu^2 + \delta^2\omega^2 - \mu^2\omega^2 + -2\omega^2\delta\mu i)}]$$
$$= \Psi_2^{c\ -1}[\frac{\theta\mu}{(\theta i - \omega)(\delta i + 2\mu)}]$$

Taking the inverse double ZZ-Shehu transform:

$$£(t,w) = e^{t-2w}. \quad (5)$$

5 DEPENDING ON THE WALLS OF TUBE CAN BE DETERMINED THE FLOW

5.1 Couette Steady Flow

In the case of Couette flow, the velocity distribution takes the form of a linear profile driven only by viscous shear stress between two infinite parallel walls separated by a distance of 2J. This flow is governed by the Navier-Stokes equation in the x-direction under the assumptions of incompressible, one-dimensional, laminar flow with no body force and constant pressure, where the lower wall is fixed and the upper wall moves with constant velocity L_1. Under these assumptions, the Navier-Stokes equation reduces to the continuity equation $\partial£/\partial x = 0$ (6) and the momentum equation $m\!\!\!/\ \partial^2£/\partial y^2 = 0$ (7).

With $£ = 0\ at\ y = -J\ and\ £ = L_1\ at\ y = J$, and $£_\omega(x,0) = \frac{L_1}{2J},\ £(x,0) = \frac{L_1}{2}$.

Solution. Using formula (3) $\Psi_2^c(£)$ to solve (7) with initial and boundary conditions.

$$\left(\frac{\delta^2}{\mu^2}\right)(-\Psi_2^c(£) - \left(i\ \frac{\delta}{\mu}\right)Z^C(£(x,0)) -$$
$$\left(\frac{\partial}{\partial w}\ Z^C£(x,0)\right) = 0.$$

$$\left(\frac{\delta^2}{\mu^2}\right)\ (-\Psi_2^c(£) - \left(i\ \frac{\delta}{\mu}\right)Z^C\left(\frac{L_1}{2}\right) - Z^C\ \left(\frac{L_1}{2J}\right) = 0.$$

$$\Psi_2(£) = (-i\frac{\mu}{\delta})\frac{L_1}{2} + \frac{L_1}{2J}(-\frac{\mu^2}{\delta^2}).$$

By taking inverse $\Psi_2^{c-1}(£)$, the velocity distribution obtained:

$$£ = \Psi_2^{-1}((-i\frac{\mu}{\delta})\frac{L_1}{2} + \frac{L_1}{2J}\left(-\frac{\mu^2}{\delta^2}\right)) = \frac{L_1}{2}(\frac{y}{J} + 1).$$

In addition, the shear stress can be found (the transforms of the relevant special functions used throughout this derivation are summarized in Table 1)

$$\tau_{xx} = 2m\!\!\!/\frac{\partial£}{\partial x}0.$$

$$\tau_{xy} = \tau_{yx} = m\!\!\!/\left(\frac{\partial£}{\partial y} + \frac{\partial v}{\partial x}\right) = \mu\frac{L_1}{2J} = Const.$$

$$\tau_{xz} = \tau_{zx} = m\!\!\!/\left(\frac{\partial£}{\partial z} + \frac{\partial\omega}{\partial x}\right)$$
$= 0$, this case of flow state shear stress constant.

Table 1: Double complex ZZ-Shehu transformation for some special functions.

№	$£(\mathrm{t,w})$	$\Psi_2\{£(\mathrm{t,w})\}$	$\Psi_2^c\{£(\mathrm{t,w})\}$
1	1	$\frac{\mu}{\delta}$	$-\frac{\mu}{\delta}$
2	$e^{(at+bw)}$	$\frac{\mu}{(\theta-a\omega)\ (\delta-b\mu)}$	$\frac{\theta\ \mu}{(\theta i-a\omega)\ (\delta i-b\mu)}$
3	$t^{\epsilon}\ w^{\rho}$	$\epsilon!\ \rho!\left(\frac{\omega}{\theta}\right)^{\epsilon}\ \left(\frac{\delta}{\mu}\right)^{\rho+1}$	$\epsilon!\ \rho!\left(\frac{\omega}{\theta}i\right)^{\epsilon}\ \left(\frac{\delta}{\mu}i\right)^{\rho+1}$
4	$\cos(at+bw)$	$\frac{\theta\mu(\theta\delta+a\mu\omega b)}{(\theta^2+a^2\omega^2)\ (\delta^2+b^2\mu^2)}$	$\frac{-\theta\mu(\theta\delta+\alpha\mu\omega b)}{(\theta^2-a^2\omega^2)\ (\delta^2-b^2\mu^2)}$
5	$\sin(at+bw)$	$\frac{\theta\mu(\theta\mu b+\omega\delta a)}{(\theta^2+a^2\omega^2)\ (\delta^2+\mu^2b^2)}$	$\frac{i\theta\mu(\theta\mu b+\delta\omega a)}{(\theta^2-a^2\omega^2)\ (\delta^2-\mu^2b^2)}$
6	$\cos h(at+bw)$	$\frac{\theta\mu(\delta+\alpha\mu\theta b)}{(\theta^2-\alpha^2\omega^2)\ (\delta^2-b^2\mu^2)}$	$\frac{\theta\mu(\alpha\mu\omega b-\theta\delta)}{(\theta^2+a^2\omega^2)\ (\delta^2+b^2\mu^2)}$
8	$s\,inh(at+bw)$	$\frac{\theta\mu(\mu\theta b+a\omega\delta)}{(\theta^2-\alpha^2\omega^2)\ (\delta^2-b^2\mu^2)}$	$\frac{i\theta\mu(\theta\mu b+a\omega\delta)}{(\theta^2+a^2\omega^2)\ (\delta^2+b^2\mu^2)}$

5.2 Poiseuille Steady Flows

In circular tubes, the flow results from a pressure gradient along long cylindrical tubes. Under certain assumptions, namely zero gravity and a constant pressure gradient, the Navier-Stokes equation reduces to express the flow between two stationary walls in the tube, giving the continuity equation $\partial£/\partial x = 0$ (8) and the momentum equation $m\ \partial^2£/\partial y^2 - \partial p/\partial t = 0$ (9).

With $£ = 0\ at\ y = -J\ and\ y = J,\ £_w(x,0) = 0,\quad £(\mathrm{x},0) = \frac{-1}{2m}\frac{\partial p}{\partial x}.H^2$

Solution. Applying formula (3) with initial and boundary conditions:

$$\left(\frac{\delta^2}{\mu^2}\right)\left(-\Psi_2^c(£)-\left(i\,\frac{\delta}{\mu}\right)Z^c\big(£(x,0)\big)-\left(\frac{\partial}{\partial w}\,Z^c£(\,x,0)\right)\right)=\frac{1}{\mu}\frac{\partial p}{\partial x}\frac{\mu}{\delta}$$

$$\left(\frac{\delta^2}{\mu^2}\right)\left(-\Psi_2(£)-\left(i\,\frac{\delta}{\mu}\right)Z^c\left(\frac{-1}{2m}\frac{\partial p}{\partial x}.H^2\right)-\frac{1}{\mu}\frac{\partial p}{\partial x}\frac{\mu}{\delta}\right.$$

$$\Psi_2\big(£(\mathrm{x,y})\big)=\left(-i\,\frac{\mu}{\delta}\right)\left(\frac{-1}{2m}\frac{\partial p}{\partial x}.H^2\right)-\frac{1}{m}\frac{\partial p}{\partial x}\frac{\mu^3}{\delta^3}$$

$$£(\mathrm{x,y})=\Psi_2^{-1}(\left(-i\,\frac{\mu}{\delta}\right)Z\left(\frac{-1}{2\square}\frac{\partial p}{\partial x}.H^2\right)-\frac{1}{m}\frac{\partial p}{\partial x}\frac{\mu^3}{\delta^3})$$

$$£(\mathrm{x,y})=\frac{-1}{2m}\frac{\partial p}{\partial x}(H^2-y^2)$$

If y=0 then the maximum flow velocity occurs at the center

$$£(\mathrm{x,y})_{max}=\frac{-1}{2m}\frac{\partial p}{\partial x}(H^2)$$

shear stress is the opposite of velocity and can be calculated:

$$\tau_{xy=}\tau_{yx}=m\left(\frac{\partial£}{\partial y}+\frac{\partial v}{\partial x}\right)=\frac{\partial p}{\partial x}y=0.$$

6 CONCLUSIONS

The study rigorously verifies the fundamental properties of this new transform, including its existence, uniqueness, and specific linear characteristics.

The research utilizes the transform to solve the Navier-Stokes equation for Couette flow, accurately determining velocity distribution and shear stress when tube walls are moving or fixed; the resulting velocity profile is depicted in Figure 1.

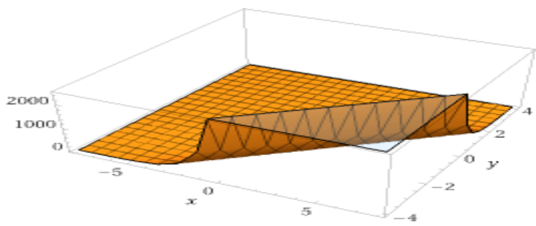

Figure 1: The solution of (5).

The technique effectively addresses Poiseuille flow resulting from pressure gradients, allowing for the calculation of maximum flow velocity at the center of circular tubes.

By connecting complex integral transforms to physical applications, the paper highlights the importance of these mathematical methods in fields such as engineering, physics, and medicine; the full sets of ZZ and Shehu transforms of common special functions used in the derivations are collected in Table 2 and Table 3, respectively.

Table 2: ZZ- Transformation for some special functions.

No	£(t)	Z(£(t)
1	1	1
2	e^{at}	$\frac{\theta}{(\theta - a\omega)}$
3	$\cos(at)$	$\frac{\theta^2}{(\theta^2 + a^2\omega^2)}$
4	$\sin(at)$	$\frac{a\omega\theta}{(\theta^2 + a^2\omega^2)}$
5	t	$\frac{\theta}{(\omega)}$

Table 3: Shehu transformation for some special functions.

No.	£(t)	S(£(t))
1	1	$\frac{\mu}{\delta}$
2	e^{at}	$\frac{\mu}{(\delta - a\mu)}$
3	$\cos(at)$	$\frac{\mu\delta}{(\delta^2 + a^2\mu^2)}$
4	$\sin(at)$	$\frac{\mu^2}{(\theta^2 + a^2\omega^2)}$
5	T	$\frac{\mu^2}{\delta^2}$

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