New Extended Categories of Fixed Point Results for Weakly Compatible Contractions Mappings of Integral Type

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Abstract: The major goal of introduce article is to establish the uniqueness of different classes of common fixed point theorems under the influence of other enhanced sorts of extended contractions of integral type in $\mathcal{D}_d^*$-Symmetric spaces. The first main outcomes have been proved for pairs of faintly compatible maps utilizing the property of (E.A) under influence enhanced integral type of contraction conditions. While got our second major results by utilizing the property of common limit in the range of $\mathcal{G}$ ($\mathbb{CLR}_{\mathcal{G}}$), which is generalization of (E.A)property and doesn't need the closedness or completeness for space or sub-spaces and continuity of utilizing maps. As well, our main outcomes will be expanded involving two pairs of self weakly compatible maps under influence same previous contraction conditions of integral type. Additionally, various suitable models that uphold our main outputs have been prepared. Finally, proposed outcomes are improving and exploring of different renowned analogous many recent results relative to these classes of common fixed point results for extended compatibility mappings existing in the literature.

1 INTRODUCTION

The generalized of common fixed-point theorems are considered highly crucial in generality branches of mathematics and have productive implementations in other branches of sciences. In 2009, L. B. Ćirić [1] presented the property of mixed $\mathcal{G}$-monotone as an extended of the idea of mixed monotone property. Next, fixed point issues have been considered in probabilistic partially ordered spaces [2], partially ordered cone [3] further fuzzy partially ordered [4]. After that, the idea of contractive condition of integral type was presented via A. Branciari [5] and investigated the existence of fixed points for maps which is defined on complete metric spaces and satisfies integral kind of contractive condition. Lately various authors have been confirmed numerous outcomes about common fixed point theory for deferent categories of maps on a metric spaces for additional information of these results we refer the author to [6].

On the other side, the usefulness of principle of Banach contraction extremely important instrument in verifying the existence of solution for integral equations. Based on this truth, there have a lot of efforts to extend the metric spaces and discussed various fixed-point outcomes on deferent extended metric spaces. In this consider particularly, the meaning of $\mathcal{D}^*$-Metric spaces have been verified by S. Shaban, et al. [7]. Following, various outcomes related to coupled fixed point have been presented in the literature [8]. Likewise, sintunavarat et al. [9] extended the property of (E.A) through presenting a novel idea namely;($\mathbb{CLR}_{\mathcal{G}}$)Property for pair of self-maps, which doesn't need the closedness for the range sub-spaces such as (E. A) Property. In 2012, some recent coupled fixed point results have been generalized by E. Karapınar, et al. [10]. After that, the conception of ($\mathbb{CLR}_{\mathcal{G}}$) expanded to two pair of self-maps like Property of ($\mathbb{CLR}_{\mathcal{P},\mathfrak{F}}$) via Karampur, et al. [11]. Further, M. Abbas, *et al* [12] generalized the idea of F-contraction condition.

In addition, Zada, A, *et al* [13] utilizing the idea of integral type and presented the notion of integral contraction in G-metric space. Therefore, the existence of common fixed points for hybrid maps are established in [14]. After that Z. Mustafa, et al [15] defined novel ideas of contractions maps and verified various novel common fixed point results in G-Complete spaces. Newly, Majid, et al. [16] verified numerous original results for multi-valued monotone maps in $\mathcal{D}^*$-Metric spaces, additionally different results of coupled fixed point theorems have been investigated. new extended kind of spaces has been presented in [17].

The main aim of suggest article is to confirm the existence outcomes of common fixed points for pair of faintly compatible maps of integral type by involving Property of $(\mathbb{CLR}_{\mathcal{G}})$ without applying the ideas of closedness or completeness and continuity. These outcomes are extensions and improvement of the different results offered in the literature.

2 PREREQUISITES

Our motivation of present this part, is to offer different ideas which needed in the sequel and that play necessary role in this work for proving our main consequences.

Definition 2.1: [17] A $\mathcal{D}_d^*$-Symmetric on a set $\mathcal{X}$ is a map $\mathcal{D}_d^*: \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ (s. t) $\forall\, \varkappa^*, \mathcal{y}^*, z^* \in \mathcal{X}$ the next conditions are satisfied:

$$(\mathcal{D}_{d_1}^*)\mathcal{D}_d^*(\varkappa^*, \mathcal{y}^*, z^*) \geq 0, \forall\, \varkappa^*, \mathcal{y}^*, z^* \in \mathcal{X},$$

$$(\mathcal{D}_{d_2}^*)\mathcal{D}_d^*(\varkappa^*, \mathcal{y}^*, z^*) = 0 \Leftrightarrow \varkappa^* = \mathcal{y}^* = z^*,$$

$$(\mathcal{D}_{d_3}^*)\mathcal{D}_d^*(\varkappa^*, \mathcal{y}^*, z^*) = \mathcal{D}_d^*(\mathcal{P}\{\varkappa^*, \mathcal{y}^*, z^*\}).$$

The map $\mathcal{D}_d^*$ is called $\mathcal{D}_d^*$-Symmetric and $(\mathcal{X}, \mathcal{D}_d^*)$ a $\mathcal{D}_d^*$-Symmetric space.

Example 2.2: [17] Assume that;$\mathcal{X} = [0,1]$ prepared with $\mathcal{D}_d^*$-Symmetric space and defined via $\mathcal{D}_d^*(\varkappa^*, \mathcal{y}^*, z^*) = (\varkappa^* - \mathcal{y}^*)^2 + (\mathcal{y}^* - z^*)^2 + (z^* - \varkappa^*)^2, \forall\, \varkappa^*, \mathcal{y}^*, z^* \in \mathcal{X}$. In that case, $(\mathcal{D}_d^*, \mathcal{X})$ is a $\mathcal{D}_d^*$-Symmetric space.

Definition 2.3: [17] Let $(\mathcal{X}, \mathcal{D}_d^*)$ be $\mathcal{D}_d^*$-Symmetric space,

1) $\{\varkappa_s^*\}$ is said to be $\mathcal{D}_d^*$-Converges to $\varkappa^* \in \mathcal{X} \Leftrightarrow \mathcal{D}_d^*(\varkappa_s^*, \varkappa_s^*, \varkappa^*) = \mathcal{D}_d^*(\varkappa^*, \varkappa^*, \varkappa_s^*) \longrightarrow 0$ as $s \longrightarrow \infty$. That is, for each $\sigma > 0$, $\exists\, \hbar \in \mathbb{N}$ such that, for each $s \geq \hbar \Longrightarrow \mathcal{D}_d^*(\varkappa^*, \varkappa^*, \varkappa_s^*) < \sigma$.
2) $\{x_s^*\}$ is called $\mathcal{D}_d^*$-Cauchy if $\forall\, \sigma > 0, \exists\, \hbar \in \mathbb{N}$ (s.t), $\forall$ s, $r \geq \hbar$, $\mathcal{D}_d^*(\varkappa_s^*, \varkappa_s^*, \varkappa_r^*) < \sigma$.
3) $(\mathcal{X}, \mathcal{D}_d^*)$ is $\mathcal{D}_d^*$-Complete if for each $\mathcal{D}_d^*$-Cauchy sequence $\{x_s^*\}$, there exist $\varkappa^* \in \mathcal{X}$ with $\lim_{s\to\infty} \mathcal{D}_d^*(\varkappa_s^*, \varkappa^*, \varkappa^*) = 0$.

Definition 2.4: [17] Assume that, $\mathcal{F}, \mathcal{G}: (\mathcal{X}, \mathcal{D}_d^*) \longrightarrow (\mathcal{X}, \mathcal{D}_d^*)$ are single-valued self-maps. *If* $\mathcal{F}(\varkappa^*) = \mathcal{G}(\varkappa^*) = \varkappa^*$ for some $\varkappa^* \in \mathcal{X}$, then $\varkappa^*$ is said to be common fixed (Concisely, C.F.P) of $\mathcal{F}$ and $\mathcal{G}$.

Definition2.5: [18] Assume that, $\mathcal{F}, \mathcal{G}: (\mathcal{X}, \mathcal{D}_d^*) \longrightarrow (\mathcal{X}, \mathcal{D}_d^*)$ are single-valued self-maps on $\mathcal{X}$. *If* $\mathcal{F}(\varkappa^*) = \mathcal{G}(\varkappa^*) = w^*$ for some $\varkappa^*$ in $\mathcal{X}$, so $\varkappa^*$ is called coincidence point of $\mathcal{F}$ & $\mathcal{G}$.

Definition2.6: [17] A self-maps $(\mathcal{F}, \mathcal{G})$ in $(\mathcal{X}, \mathcal{D}_d^*)$ is called weakly compatible(Shortly, W.C-Map) if $(\mathcal{F}, \mathcal{G})$ commute at their coincidence points.

Definition 2.7: [17] A self-maps $\mathcal{F}$ and $\mathcal{G}$ in $(\mathcal{X}, \mathcal{D}_d^*)$ are satisfying (E. A) Property if $\exists\, \{\varkappa_s^*\}$ in $\mathcal{X}$ where $\lim_{s\to\infty} \mathcal{F}\, \varkappa_s^* = \lim_{s\to\infty} \mathcal{G}\, \varkappa_s^* = t$ for some $t \in \mathcal{X}$.

Definition 2.8: [19] Assume, $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{P}, \mathfrak{F})$ are pairs of self-maps of $(\mathcal{X}, d)$. So, we say$(\mathcal{F}, \mathcal{G})$ and $(\mathcal{P}, \mathfrak{F})$ satisfies (E.A) common property, if $\exists\, \{\varkappa_s^*\}$ and $\{\mathcal{y}_s^*\}$ (s. t),

$$\lim_{s\to\infty} \mathcal{F}\varkappa_s^* = \lim_{s\to\infty} \mathcal{G}\varkappa_s^* = \lim_{s\to\infty} \mathcal{P}\mathcal{y}_s^* = \lim_{s\to\infty} \mathfrak{F}\mathcal{y}_s^* = \ell$$

for $\ell \in \mathcal{X}$.

Definition 2.9: [9] Suppose that $\mathcal{F}$ & $\mathcal{G}$ are self maps described on $(\mathcal{X}, d)$. So, $(\mathcal{F}, \mathcal{G})$ are satisfying in the Range of $\mathcal{G}$ the Property of Common Limit $(\mathbb{CLR}_{\mathcal{G}})$, if $\exists\, \{\varkappa_s^*\}$ where $\lim_{s\to\infty} \mathcal{F}\varkappa_s^* = \lim_{s\to\infty} \mathcal{G}\varkappa_s^* = \mathcal{G}\ell$ for some $\ell \in \mathcal{X}$.

Definition 2.10: [11] Assume that, $(\mathcal{F}, \mathcal{G})$,$(\mathcal{P}, \mathfrak{F})$ are pairs of self-maps of $(\mathcal{X}, d)$. So, say $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{P}, \mathfrak{F})$ gratifies common limit range property (w. r. t, $\mathcal{P}$ & $\mathfrak{F}$), often indicated via Property of $(\mathbb{CLR}_{(\mathcal{P},\mathfrak{F})})$, if $\exists\, \{\varkappa_s^*\}$ & $\{\mathcal{y}_s^*\}$ (s. t)

$$\lim_{s\to\infty} \mathcal{F}\varkappa_s^* = \lim_{s\to\infty} \mathcal{P}\varkappa_s^* = \lim_{s\to\infty} \mathcal{G}\mathcal{y}_s^* = \lim_{s\to\infty} \mathfrak{F}\mathcal{y}_s^* = \ell \,\&\, \ell = \mathcal{P}\wp = \mathfrak{F}w^*; \ell, \wp, w^* \in \mathcal{X}.$$

Remark 2.11: If $\mathcal{F} = \mathcal{G}$ with $\mathcal{P} = \mathfrak{F}$, in this case Def-2.9, gives Property of $(\mathbb{CLR}_{\mathcal{G}})$.

Definition2.12: [20] Assume that, Φ is collection of auxiliary maps φ where $\varphi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is non-decreasing map satisfying $\lim_{s\to\infty} \varphi^s(\ell) = 0, \forall\, \ell \in (0, \infty)$. *If* $\varphi \in \Phi$, so φ is Φ-Mapping. Additional, $\varphi(\ell) < \ell, \forall\, \ell \in (0, \infty)$ with $\varphi(0) = 0$.

Definition 2.13: [20] Assume Ψ is collection of maps $\psi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, where ψ is lebesgue integrable mapping, nonnegative, summable $\forall\, \sigma, k_1, k_2 > 0$, $\int_0^{\sigma} \psi(\ell) d\ell > 0$ and:

$$\int_0^{k_1+k_2} \psi(\ell) d\ell \leq \int_0^{k_1} \psi(\ell) d\ell + \int_0^{k_2} \psi(\ell) d\ell.$$

3 NEW CATEGORIES OF COMMON FIXED POINT THEOREMS INVOLVING CONTRACTION CONDITIONS OF INTEGRAL TYPE

The motive for presenting segment is to confirm uniqueness of some classes of common fixed point outcomes by condition of integral type through applying the idea of Property (E.A) in $\mathcal{D}_d^*$-Symmetric spaces. Our main outcomes will be extended and improved involving two pairs of self weakly compatible maps. While got our other major outcomes by using in range of $\mathcal{G}$ the property of common limit($\mathbb{CLR}_{\mathcal{G}}$).

Theorem 3.1: Let $\mathcal{F}$ and $\mathcal{G}$ be two self maps on $(\mathcal{X}, \mathcal{D}_d^*)$ satisfying Property of (E.A). Assume $\varphi \in \Phi$ satisfactory:

$$\int_0^{\mathcal{D}_d^*(\mathcal{F}x^*,\mathcal{F}y^*,\mathcal{F}z^*)} \psi(\ell)d\ell \leq \varphi\left(\int_0^{\mathbb{L}(x^*,y^*,z^*)} \psi(\ell)d\ell\right)$$

$\forall\, x^*, y^*, z^* \in \mathcal{X}$,(s. t) $\psi \in \Psi$ with

$$\left\{\begin{array}{c} \mathbb{L}(x^*,y^*,z^*) = \\ max\left\{\begin{array}{c} \mathcal{D}_d^*(\mathcal{G}x^*,\mathcal{G}y^*,\mathcal{F}z^*), \\ \mathcal{D}_d^*(\mathcal{G}x^*,\mathcal{G}y^*,\mathcal{F}z^*), \\ \mathcal{D}_d^*(\mathcal{G}x^*,\mathcal{G}y^*,\mathcal{G}z^*), \\ \mathcal{D}_d^*(\mathcal{G}x^*,\mathcal{G}x^*,\mathcal{F}x^*), \\ \mathcal{D}_d^*(\mathcal{G}y^*,\mathcal{G}y^*,\mathcal{F}y^*), \\ \mathcal{D}_d^*(\mathcal{G}z^*,\mathcal{G}z^*,\mathcal{F}z^*) \end{array}\right\} \\ -min\left\{\begin{array}{c} \mathcal{D}_d^*(\mathcal{G}x^*,\mathcal{G}y^*,\mathcal{F}z^*), \\ \mathcal{D}_d^*(\mathcal{G}x^*,\mathcal{G}y^*,\mathcal{F}z^*), \\ \mathcal{D}_d^*(\mathcal{G}x^*,\mathcal{G}y^*,\mathcal{G}z^*), \\ \mathcal{D}_d^*(\mathcal{G}x^*,\mathcal{G}x^*,\mathcal{F}x^*), \\ \mathcal{D}_d^*(\mathcal{G}y^*,\mathcal{G}y^*,\mathcal{F}y^*), \\ \mathcal{D}_d^*(\mathcal{G}z^*,\mathcal{G}z^*,\mathcal{F}z^*) \end{array}\right\} \end{array}\right\}$$

If $\mathcal{G}(\mathcal{X})$ is closed, so $\mathcal{F}$ & $\mathcal{G}$ contain unique point of coincidence. Additional, *if* $\mathcal{F}$ & $\mathcal{G}$ are (W. C. M) in this case $\mathcal{F}$ & $\mathcal{G}$ contain unique (C.F.P).

Proof: Suppose $(\mathcal{F}, \mathcal{G})$ satisfies property of (E. A). In his case $\exists\, \{x_s^*\}$ in$\mathcal{X}$ such that:

$$\lim_{s\to\infty} \mathcal{F}x_s^* = \lim_{s\to\infty} \mathcal{G}x_s^* = \ell \text{ for some } \ell \in \mathcal{X}.$$

Because, $\mathcal{G}(\mathcal{X})$ closed, $\ell \in \mathcal{G}(\mathcal{X}) \Longrightarrow \ell = \mathcal{G}p$ for $p \in \mathcal{X}$.

Currently verify $\ell = \mathcal{F}p$. Assume that $\ell \neq \mathcal{F}p$, in this case consider.

$$\int_0^{\mathcal{D}_d^*(\mathcal{F}x_s^*,\mathcal{F}x_s^*,\mathcal{F}p)} \psi(\ell)d\ell \leq \varphi\left(\int_0^{\mathbb{L}(x_s^*,x_s^*,p)} \psi(\ell)d\ell\right) \quad (1)$$

Such that

$$\left\{\begin{array}{c} \mathbb{L}(x_s^*,x_s^*,p) = \\ max\left\{\begin{array}{c} \mathcal{D}_d^*(\mathcal{G}x_s^*,\mathcal{G}x_s^*,\mathcal{F}p), \\ \mathcal{D}_d^*(\mathcal{G}x_s^*,\mathcal{G}p,\mathcal{F}p), \\ \mathcal{D}_d^*(\mathcal{G}x_s^*,\mathcal{G}x_s^*,\mathcal{G}p), \\ \mathcal{D}_d^*(\mathcal{G}x_s^*,\mathcal{G}x_s^*,\mathcal{F}x_s^*) \\ ,\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}p), \\ \mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}p) \end{array}\right\} \\ -min\left\{\begin{array}{c} \mathcal{D}_d^*(\mathcal{G}x_s^*,\mathcal{G}x_s^*,\mathcal{F}p), \\ \mathcal{D}_d^*(\mathcal{G}x_s^*,\mathcal{G}p,\mathcal{F}p), \\ \mathcal{D}_d^*(\mathcal{G}x_s^*,\mathcal{G}x_s^*,\mathcal{G}p), \\ \mathcal{D}_d^*(\mathcal{G}x_s^*,\mathcal{G}x_s^*,\mathcal{F}x_s^*), \\ \mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}p), \\ \mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}p) \end{array}\right\} \end{array}\right\}$$

In this case,

$$\begin{aligned} &\lim_{s\to\infty} \mathbb{L}(x_s^*,x_s^*,p) \\ &= max\{\mathcal{D}_d^*(\mathcal{G}x^*,\mathcal{G}x^*,\mathcal{F}p),\mathcal{D}_d^*(\mathcal{G}p,\mathcal{F}p,\mathcal{F}p) \\ &= \mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}p)\} \end{aligned}$$

Letting s $\longrightarrow \infty$ in inequality (1) we obtain:

$$\int_0^{\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}p)} \psi(\ell)d\ell \leq \varphi\left(\int_0^{\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}p)} \psi(\ell)d\ell\right) < \int_0^{\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}p)} \psi(\ell)d\ell$$

Contradiction.

Therefore, $\ell = \mathcal{F}p = \mathcal{G}p$, which implies that p is coincidence of $\mathcal{F}$and $\mathcal{G}$. because, $\mathcal{F}$ & $\mathcal{G}$ are faintly compatible, obtain $\mathcal{F}\ell = \mathcal{F}\mathcal{G}p = \mathcal{G}\mathcal{F}p = \mathcal{G}\ell$.

Next, confirm $\ell = \mathcal{F}\ell$. Assume, $\ell \neq \mathcal{F}\ell$, in this case consider:

$$\int_0^{\mathcal{D}_d^*(\mathcal{F}\ell,\mathcal{F}\ell,\mathcal{F}p)} \psi(\ell)d\ell \leq \varphi\left(\int_0^{\mathbb{L}(\ell,\ell,p)} \psi(\ell)d\ell\right)$$

Such that:

$$\left\{\begin{array}{c} \mathbb{L}(\ell,\ell,p) = \\ max\left\{\begin{array}{c} \mathcal{D}_d^*(\mathcal{G}\ell,\mathcal{G}\ell,\mathcal{F}p),\mathcal{D}_d^*(\mathcal{G}\ell,\mathcal{G}p,\mathcal{F}p), \\ \mathcal{D}_d^*(\mathcal{G}\ell,\mathcal{G}\ell,\mathcal{G}p), \\ \mathcal{D}_d^*(\mathcal{G}\ell,\mathcal{G}\ell,\mathcal{F}\ell),\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}p), \\ \mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}p) \end{array}\right\} \\ -min\left\{\begin{array}{c} \mathcal{D}_d^*(\mathcal{G}\ell,\mathcal{G}\ell,\mathcal{F}p),\mathcal{D}_d^*(\mathcal{G}\ell,\mathcal{G}p,\mathcal{F}p), \\ \mathcal{D}_d^*(\mathcal{G}\ell,\mathcal{G}\ell,\mathcal{G}p), \\ \mathcal{D}_d^*(\mathcal{G}\ell,\mathcal{G}\ell,\mathcal{F}\ell),\mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}p), \\ \mathcal{D}_d^*(\mathcal{G}p,\mathcal{G}p,\mathcal{F}p) \end{array}\right\} \end{array}\right\}$$

Consequently, $\mathbb{L}(\ell,\ell,p) = \mathcal{D}_d^*(\mathcal{F}\ell,\mathcal{F}\ell,\mathcal{F}p)$.

$$\int_0^{\mathcal{D}_d^*(\mathcal{F}\ell,\mathcal{F}\ell,\mathcal{F}p)} \psi(\ell)d\ell \leq \varphi\left(\int_0^{\mathcal{D}_d^*(\mathcal{F}\ell,\mathcal{F}\ell,\mathcal{F}p)} \psi(\ell)d\ell\right) < \int_0^{\mathcal{D}_d^*(\mathcal{F}\ell,\mathcal{F}\ell,\mathcal{F}p)} \psi(\ell)d\ell$$

Contradiction.

For this reason, $\mathcal{F}\ell = \ell = \mathcal{G}\ell$. Hence, ℓ is (C.F.P) of $\mathcal{F}$and $\mathcal{G}$.

Subsequently instance, which is expanding to that in [19] shows validity of Theorem-3.1:

Example 3.2: Assume $\mathcal{X} = [0,10]$ and $\mathcal{D}_d^*: \mathcal{X} \times \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}^+$ described via $\mathcal{D}_d^*(\varkappa^*, y^*, z^*) = 0$, $if\ \varkappa^* = y^* = z^*$; $\mathcal{D}_d^*(\varkappa^*, y^*, z^*) = max\{\varkappa^*, y^*, z^*\}$ in each other statuses. Subsequently $(\mathcal{X}, \mathcal{D}_d^*)$ is $\mathcal{D}_d^*$-Symmetric.

Assume $\mathcal{F}$ and $\mathcal{G}$ are two self-maps on $\mathcal{X}$ described via $\mathcal{F}\varkappa^* = \frac{1}{5}\varkappa^*$ & $\mathcal{G}x^* = \frac{1}{3}\varkappa^*, \forall\, \varkappa^* \in \mathcal{X}$.

Describe, $\varphi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ via $\varphi(\ell) = \frac{3}{5}\ell$. Herein, $\mathcal{F}$ and $\mathcal{G}$ are (W.C) maps gratifies the property (E.A). Regarde,$\{\varkappa_s^*\} = \left\{\frac{1}{s+1}\right\}, \forall$ s. Subsequently, $\mathcal{F}\varkappa_s^* \longrightarrow 0, \mathcal{G}\varkappa_s^* \longrightarrow 0$. Assume, $z^* \leq y^* \leq \varkappa^*$(Without loss of generality).

In that case,

$$\int_0^{\mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}y^*,\mathcal{F}z^*)} \psi(\ell)d\ell \leq \varphi\left(\int_0^{\mathbb{L}(\varkappa^*,y^*,z^*)} \psi(\ell)d\ell\right)$$

$\forall\, \varkappa^*, y^*, z^* \in \mathcal{X}$, (s. t) $\psi \in \Psi$, as well:

$$\left\{\begin{array}{c} \mathbb{L}(x^*, y^*, z^*) = \\ max\left\{\begin{array}{c} \mathcal{D}_d^*(\mathcal{G}\varkappa^*, \mathcal{G}y^*, \mathcal{F}z^*), \\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*, \mathcal{G}y^*, \mathcal{F}z^*), \\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*, \mathcal{G}y^*, \mathcal{G}z^*), \\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*, \mathcal{G}\varkappa^*, \mathcal{F}\varkappa^*), \\ \mathcal{D}_d^*(\mathcal{G}y^*, \mathcal{G}y^*, \mathcal{F}y^*), \\ \mathcal{D}_d^*(\mathcal{G}z^*, \mathcal{G}z^*, \mathcal{F}z^*) \end{array}\right\} \\ -min\left\{\begin{array}{c} \mathcal{D}_d^*(\mathcal{G}\varkappa^*, \mathcal{G}y^*, \mathcal{F}z^*), \\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*, \mathcal{G}y^*, \mathcal{F}z^*), \\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*, \mathcal{G}y^*, \mathcal{G}z^*), \\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*, \mathcal{G}\varkappa^*, \mathcal{F}\varkappa^*), \\ \mathcal{D}_d^*(\mathcal{G}y^*, \mathcal{G}y^*, \mathcal{F}y^*), \\ \mathcal{D}_d^*(\mathcal{G}z^*, \mathcal{G}z^*, \mathcal{F}z^*) \end{array}\right\} \end{array}\right\}$$

Additional, $\mathcal{F}$ and $\mathcal{G}$ gratifies Theo-3.1, we have unique (C.F.P) $\varkappa^* = 0$.

Subsequent, generalized Theo-3.1, by involving the $(\mathbb{CLR}_{\mathcal{G}})$ property.

Theorem 3.3: *If* $\mathcal{F}$ and $\mathcal{G}$ are self-maps on $(\mathcal{X}, \mathcal{D}_d^*)$ satisfying Property of $(\mathbb{CLR}_{\mathcal{G}})$. Assume that, $\varphi \in \Phi$ satisfying:

$$\int_0^{\mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}y^*,\mathcal{F}z^*)} \psi(\ell)d\ell \leq \varphi\left(\int_0^{\mathbb{L}(\varkappa^*,y^*,z^*)} \psi(\ell)d\ell\right)$$

$\forall\, \varkappa^*, y^*, z^* \in \mathcal{X}$, (s.t) $\psi \in \Psi$ with

$$\left\{\begin{array}{c} \mathbb{L}(\varkappa^*, y^*, z^*) = \\ max\left\{\begin{array}{c} \mathcal{D}_d^*(\mathcal{G}\varkappa^*, \mathcal{G}y^*, \mathcal{F}z^*), \\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*, \mathcal{F}y^*, \mathcal{F}z^*), \\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*, \mathcal{G}y^*, \mathcal{G}z^*), \\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*, \mathcal{G}\varkappa^*, \mathcal{F}\varkappa^*), \\ \mathcal{D}_d^*(\mathcal{G}y^*, \mathcal{G}y^*, \mathcal{F}y^*), \\ \mathcal{D}_d^*(\mathcal{G}z^*, \mathcal{G}z^*, \mathcal{F}z^*) \end{array}\right\} \\ -min\left\{\begin{array}{c} \mathcal{D}_d^*(\mathcal{G}\varkappa^*, \mathcal{G}y^*, \mathcal{F}z^*), \\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*, \mathcal{F}y^*, \mathcal{F}z^*), \\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*, \mathcal{G}y^*, \mathcal{G}z^*) \\ ,\mathcal{D}_d^*(\mathcal{G}\varkappa^*, \mathcal{G}\varkappa^*, \mathcal{F}\varkappa^*) \\ ,\mathcal{D}_d^*(\mathcal{G}y^*, \mathcal{G}y^*, \mathcal{F}y^*), \\ \mathcal{D}_d^*(\mathcal{G}z^*, \mathcal{G}z^*, \mathcal{F}z^*) \end{array}\right\} \end{array}\right\}$$

If $\mathcal{F}$ & $\mathcal{G}$ are (W. C) maps, so $\mathcal{F}$ & $\mathcal{G}$ have unique(C. F. P).

Proof: Suppose $(\mathcal{F}, \mathcal{G})$ satisfies $(\mathbb{CLR}_{\mathcal{G}})$ property. In this case there exists $\{\varkappa^*{}_s^*\}$ in $\mathcal{X}$ such that

$$\lim_{s\to\infty} \mathcal{F}\varkappa_s^* = \lim_{s\to\infty} \mathcal{G}\varkappa_s^* = \ell \text{ for some } \ell \in \mathcal{X}.$$

Following, the same steps as the evidence of Theorem 3.1, the rest of the evidence follows.

Next, investigate instance which is expanding to that in [19], to explain validity of Theorem-3.3

Example 3.4: Assume $\mathcal{X} = [1,10]$ and $\mathcal{D}_d^*: \mathcal{X} \times \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}^+$ described via $\mathcal{D}_d^*(\varkappa^*, y^*, z^*) = max\{d(\varkappa^*, y^*), d(y^*, z^*), d(z^*, \varkappa^*)\}$, (s. t) d is usual metric. In this case $(\mathcal{X}, \mathcal{D}_d^*)$ is $\mathcal{D}_d^*$-Symmetric.

Assume $\mathcal{F}$ and $\mathcal{G}$ are two self-maps described via

$\mathcal{F}(\varkappa^*) = \begin{cases} 5 & if\ \varkappa^* \leq 5 \\ 3 & if\ \ \varkappa^* > 5 \end{cases}$ and $\mathcal{G}(\varkappa^*) = \begin{cases} \frac{\varkappa^*+5}{2} & if\ \ \varkappa^* \leq 5 \\ 8 & if\ \ \varkappa^* > 5 \end{cases}$

Describe, $\varphi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ via $\varphi(\ell) = \frac{2}{3}\ell$. Herein $\mathcal{F}$ and $\mathcal{G}$ are (W.C) maps and satisfies Property of $(\mathbb{CLR}_{\mathcal{G}})$, now verify this fact, assume, $\{\varkappa_s^*\} = \left\{5 - \frac{1}{s}\right\}, \forall$ s. In this case $\mathcal{F}\varkappa_s^* \longrightarrow 5, \mathcal{G}\varkappa_s^* \longrightarrow 5 = \mathcal{G}(5)$. Additional, $\mathcal{F}$ & $\mathcal{G}$ satisfies each conditions of Theorem-3.3 and have unique(C.F.P) at a point $\varkappa^* = 5$.

Corollary 3.5: Assume, $\mathcal{F}$ and $\mathcal{G}$ are self-maps on $(\mathcal{X},\mathcal{D}_d^*)$ satisfying Property of $(\mathbb{CLR}_{\mathcal{G}})$. Assume $\Omega \in [0,1)$ with:

$$\int_0^{\mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}y^*,\mathcal{F}z^*)} \psi(\ell)d\ell \le \alpha\left(\int_0^{\mathbb{L}(\varkappa^*,y^*,z^*)} \psi(\ell)d\ell\right)$$

$\forall\, \varkappa^*, y^*, z^* \in \mathcal{X}$, (s. t) $\psi \in \Psi$ with

$$\left\{\begin{array}{c}\mathbb{L}(\varkappa^*,y^*,z^*) = \\ max\left\{\begin{array}{c}\mathcal{D}_d^*(\mathcal{G}\varkappa^*,\mathcal{G}y^*,\mathcal{F}z^*),\\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*,\mathcal{F}y^*,\mathcal{F}z^*),\\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*,\mathcal{G}y^*,\mathcal{G}z^*),\\ \mathcal{D}_d^*(\mathcal{G}\varkappa,\mathcal{G}\varkappa^*,\mathcal{F}x^*),\\ \mathcal{D}_d^*(\mathcal{G}y^*,\mathcal{G}y^*,\mathcal{F}y^*),\\ \mathcal{D}_d^*(\mathcal{G}z^*,\mathcal{G}z^*,\mathcal{F}z^*)\end{array}\right\} \\ -min\left\{\begin{array}{c}\mathcal{D}_d^*(\mathcal{G}\varkappa^*,\mathcal{G}y^*,\mathcal{F}z^*),\\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*,\mathcal{F}y^*,\mathcal{F}z^*),\\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*,\mathcal{G}y^*,\mathcal{G}z^*),\\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*,\mathcal{G}\varkappa^*,\mathcal{F}\varkappa^*),\\ \mathcal{D}_d^*(\mathcal{G}y^*,\mathcal{G}y^*,\mathcal{F}y^*),\\ \mathcal{D}_d^*(\mathcal{G}z^*,\mathcal{G}z^*,\mathcal{F}z^*)\end{array}\right\}\end{array}\right\}$$

If $\mathcal{F}$ and $\mathcal{G}$ are weakly compatible, in this case $\mathcal{F}$ and $\mathcal{G}$ have a unique (C.F.P).

Proof: This proof follows directly by putting $\varphi(\ell) = \Omega(\ell)$ (s. t) $\Omega \in [0,1)$ in Theorem 3.3.

Corollary 3.6: Assume $\mathcal{F}$ and $\mathcal{G}$ are two self-maps on $(\mathcal{X},\mathcal{D}_d^*)$ satisfying $(\mathbb{CLR}_{\mathcal{G}})$. Assume, $\Omega \in [0,1)$ and $\mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}y^*,\mathcal{F}z^*) \le \Omega(\mathbb{L}(\varkappa^*,y^*,z^*))$, $\forall\, \varkappa^*, y^*, z^* \in \mathcal{X}$, (s. t)

$$\left\{\begin{array}{c}\mathbb{L}(\varkappa^*,y^*,z^*) = \\ max\left\{\begin{array}{c}\mathcal{D}_d^*(\mathcal{G}\varkappa^*,\mathcal{G}y^*,\mathcal{F}z^*),\\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*,\mathcal{F}y^*,\mathcal{F}z^*)\\ ,\mathcal{D}_d^*(\mathcal{G}\varkappa^*,\mathcal{G}y^*,\mathcal{G}z^*),\\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*,\mathcal{G}\varkappa^*,\mathcal{F}\varkappa^*),\\ \mathcal{D}_d^*(\mathcal{G}y^*,\mathcal{G}y^*,\mathcal{F}y^*),\\ \mathcal{D}_d^*(\mathcal{G}z^*,\mathcal{G}z^*,\mathcal{F}z^*)\end{array}\right\} \\ -min\left\{\begin{array}{c}\mathcal{D}_d^*(\mathcal{G}\varkappa^*,\mathcal{G}y^*,\mathcal{F}z^*),\\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*,\mathcal{F}y^*,\mathcal{F}z^*),\\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*,\mathcal{G}y^*,\mathcal{G}z^*),\\ \mathcal{D}_d^*(\mathcal{G}\varkappa^*,\mathcal{G}\varkappa^*,\mathcal{F}\varkappa^*)\\ ,\mathcal{D}_d^*(\mathcal{G}y^*,\mathcal{G}y^*,\mathcal{F}y^*),\\ \mathcal{D}_d^*(\mathcal{G}z^*,\mathcal{G}z^*,\mathcal{F}z^*)\end{array}\right\}\end{array}\right\}$$

If $\mathcal{F}$ & $\mathcal{G}$ are (W.C), so $\mathcal{F}$ & $\mathcal{G}$ have unique (C. F. P).

Proof: It's follows instantly by putting $\varphi(\ell) = 1$and $\varphi(\ell) = \Omega(\ell)$(s.t) $\Omega \in [0,1)$ in Theorem 3.3.

Theorem 3.7: *If* $(\mathcal{F},\mathcal{P})$ and $(\mathcal{G},\mathfrak{F})$ are pairs of self-maps on $(\mathcal{X},\mathcal{D}_d^*)$ satisfying Property of common (E.A). Assume, $\varphi \in \Phi$ satisfying:

$$\int_0^{\mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}y^*,\mathcal{G}z^*)} \psi(\ell)d\ell \le \varphi\left(\int_0^{\mathbb{L}(\varkappa^*,y^*,z^*)} \psi(\ell)d\ell\right)$$

$\forall\, \varkappa^*, y^*, z^* \in \mathcal{X}$, (s. t) $\psi \in \Psi$ with

$$\left\{\begin{array}{c}\mathbb{L}(\varkappa^*,y^*,z^*) = \\ max\left\{\begin{array}{c}\mathcal{D}_d^*(\mathcal{P}\varkappa^*,\mathcal{P}y^*,\mathfrak{F}z^*)\\ ,\mathcal{D}_d^*(\mathcal{P}\varkappa^*,\mathcal{P}y^*,\mathcal{G}z^*),\\ \mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}\varkappa^*,\mathcal{P}x^*),\\ \mathcal{D}_d^*(\mathcal{G}y^*,\mathcal{G}y^*,\mathfrak{F}z^*),\\ \mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}y^*,\mathfrak{F}z^*)\end{array}\right\} \\ -\,min\left\{\begin{array}{c}\mathcal{D}_d^*(\mathcal{P}\varkappa^*,\mathcal{P}y^*,\mathfrak{F}z^*),\\ \mathcal{D}_d^*(\mathcal{P}\varkappa^*,\mathcal{P}y^*,\mathcal{G}z^*),\\ \mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}\varkappa^*,\mathcal{P}\varkappa^*),\\ \mathcal{D}_d^*(\mathcal{G}y^*,\mathcal{G}y^*,\mathfrak{F}z^*),\\ \mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}y^*,\mathfrak{F}z^*)\end{array}\right\}\end{array}\right\}$$

If $\mathcal{P}(\mathcal{X})$ and $\mathfrak{F}(\mathcal{X})$ closed, in this case $(\mathcal{F},\mathcal{P})$ & $(\mathcal{G},\mathfrak{F})$ have unique coincidence point.

If $(\mathcal{F},\mathcal{P})$ and $(\mathcal{G},\mathfrak{F})$weakly compatible, so $(\mathcal{F},\mathcal{P})$ & $(\mathcal{G},\mathfrak{F})$ contain unique (C. F. P).

Proof: Presume, $(\mathcal{F},\mathcal{P})$ with $(\mathcal{G},\mathfrak{F})$ satisfactory property of common (E.A). In this case there exist $\{\varkappa_s^*\}$ and $\{y_s^*\}$ in $\mathcal{X}$ where

$$\lim_{s\to\infty}\mathcal{F}\varkappa_s^* = \lim_{s\to\infty}\mathcal{P}\varkappa_s^* = \lim_{s\to\infty}\mathcal{G}y_s^* = \lim_{s\to\infty}\mathfrak{F}y_s^* = \ell$$

for some $\ell \in \mathcal{X}$.

Because, $\mathcal{P}(\mathcal{X})$ and $\mathfrak{F}(\mathcal{X})$ are closed, get $\ell = \mathcal{P}\varkappa^* = \mathfrak{F}y^*$ for some $\varkappa^*$ & y^* in $\mathcal{X}$.

Currently, verify $\ell = \mathcal{F}\varkappa^*$: consider:

$$\int_0^{\mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}\varkappa^*,\mathcal{G}y_s^*)} \psi(\ell)d\ell \le \varphi\left(\int_0^{\mathbb{L}(\varkappa^*,\varkappa^*,y_s^*)} \psi(\ell)d\ell\right) \quad (2)$$

Such that

$$\left\{\begin{array}{c}\mathbb{L}(\varkappa^*,y_s^*,y_s^*) = \\ max\left\{\begin{array}{c}\mathcal{D}_d^*(\mathcal{P}\varkappa^*,\mathcal{P}\varkappa^*,\mathfrak{F}y_s^*),\\ \mathcal{D}_d^*(\mathcal{P}\varkappa^*,\mathcal{P}\varkappa^*,\mathcal{G}y_s^*),\\ \mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}\varkappa^*,\mathcal{P}\varkappa^*),\\ \mathcal{D}_d^*(\mathcal{G}y_s^*,\mathcal{G}y_s^*,\mathfrak{F}y_s^*),\\ \mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}\varkappa^*,\mathfrak{F}y_s^*)\end{array}\right\} \\ -min\left\{\begin{array}{c}\mathcal{D}_d^*(\mathcal{P}\varkappa^*,\mathcal{P}\varkappa^*,\mathfrak{F}y_s^*),\\ \mathcal{D}_d^*(\mathcal{P}\varkappa^*,\mathcal{P}\varkappa^*,\mathcal{G}y_s^*),\\ \mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}\varkappa^*,\mathcal{P}\varkappa^*),\\ \mathcal{D}_d^*(\mathcal{G}y_s^*,\mathcal{G}y_s^*,\mathfrak{F}y_s^*),\\ \mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}\varkappa^*,\mathfrak{F}y_s^*)\end{array}\right\}\end{array}\right\}$$

Subsequently,

$$\lim_{s\to\infty}\mathbb{L}(\varkappa^*,\varkappa^*,y_s^*) = 0$$

Letting s → ∞ in inequality(2), obtain:

$$\int_0^{\mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}\varkappa^*,\ell)} \psi(\ell)d\ell \leq \varphi\left(\int_0^0 \psi(\ell)d\ell\right) = \varphi(0) = 0.$$

Consequently, $\mathcal{F}\varkappa^* = \ell = \mathcal{P}\varkappa^*, \Longrightarrow \varkappa^*$ coincidence point of $\mathcal{F}$& $\mathcal{P}$.

For the reason that, $\mathcal{F}$ and $\mathcal{P}$are (W. C) maps, obtain $\mathcal{F}\ell = \mathcal{F}\mathcal{P}\varkappa^* = \mathcal{P}\mathcal{F}\varkappa^* = \mathcal{P}\ell$.

In the same way we can establish y^* is coincidence point of $\mathcal{G}$ & $\mathfrak{F}$. As, $\mathcal{G}$ & $\mathfrak{F}$ are (W. C) maps, obtain $\mathcal{G}\ell = \mathcal{G}\mathfrak{F}y^* = \mathfrak{F}\mathcal{G}y^* = \mathfrak{F}\ell$. Therefore, $\ell = \mathcal{F}\varkappa^* = \mathcal{P}\varkappa^* = \mathcal{G}y^* = \mathfrak{F}y^*$. At this instant confirm $\ell = \mathcal{F}\ell$: Assume, $\ell \neq \mathcal{F}\ell$, in that case consider:

$$\int_0^{\mathcal{D}_d^*(\mathcal{F}\ell,\mathcal{F}\ell,\ell)} \psi(\ell)d\ell = \int_0^{\mathcal{D}_d^*(\mathcal{F}\ell,\mathcal{F}\ell,\mathcal{G}y^*)} \psi(\ell)d\ell \leq \varphi\left(\int_0^{\mathbb{L}(\ell,\ell,y^*)} \psi(\ell)d\ell\right).$$

Such that

$$\mathbb{L}(\ell,\ell,y^*) = \left\{\begin{array}{l} max\left\{\begin{array}{c}\mathcal{D}_d^*(\mathcal{P}\ell,\mathcal{P}\ell,\mathfrak{F}y^*),\\ \mathcal{D}_d^*(\mathcal{P}\ell,\mathcal{P}\ell,\mathcal{G}y^*),\\ \mathcal{D}_d^*(\mathcal{F}\ell,\mathcal{F}\ell,\mathcal{P}\ell),\\ \mathcal{D}_d^*(\mathcal{G}y^*,\mathcal{G}y^*,\mathfrak{F}y^*),\\ \mathcal{D}_d^*(\mathcal{F}\ell,\mathcal{F}\ell,\mathfrak{F}y^*)\end{array}\right\}\\ -min\left\{\begin{array}{c}\mathcal{D}_d^*(\mathcal{P}\ell,\mathcal{P}\ell,\mathfrak{F}y^*),\\ \mathcal{D}_d^*(\mathcal{P}\ell,\mathcal{P}\ell,\mathcal{G}y^*),\\ \mathcal{D}_d^*(\mathcal{F}\ell,\mathcal{F}\ell,\mathcal{P}\ell),\\ \mathcal{D}_d^*(\mathcal{G}y^*,\mathcal{G}y^*,\mathfrak{F}y^*)\\ ,\mathcal{D}_d^*(\mathcal{F}\ell,\mathcal{F}\ell,\mathfrak{F}y^*)\end{array}\right\}\end{array}\right\}$$

Therefore, $\mathbb{L}(\ell,\ell,y^*) = \mathcal{D}_d^*(\mathcal{F}\ell,\mathcal{F}\ell,\ell)$

$$\int_0^{\mathcal{D}_d^*(\mathcal{F}\ell,\mathcal{F}\ell,\ell)} \psi(\ell)d\ell \leq \varphi\left(\int_0^{\mathcal{D}_d^*(\mathcal{F}\ell,\mathcal{F}\ell,\ell)} \psi(\ell)d\ell\right) < \int_0^{\mathcal{D}_d^*(\mathcal{F}\ell,\mathcal{F}\ell,\ell)} \psi(\ell)d\ell$$

A contradiction. Thus,$\mathcal{F}\ell = \ell = \mathcal{P}\ell$. As a result ℓ is (C.F.P) of $\mathcal{F}$and $\mathcal{P}$.

Likewise, we can establish ℓ is (C.F.P) of $\mathcal{G}$ and $\mathfrak{F}$. So, ℓ is (C. F. P) of $\mathcal{F},\mathcal{P},\mathcal{G}$ & $\mathfrak{F}$.

Theorem3.8: *If* $(\mathcal{F},\mathcal{P})$ and $(\mathcal{G},\mathfrak{F})$ are two pairs of self-mappings on $(\mathcal{X},\mathcal{D}_d^*)$ satisfying $(\mathbb{CLR}_{(\mathcal{P},\mathfrak{F})})$ Property. Assume that $\varphi \in \Phi$ satisfying:

$$\int_0^{\mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}y^*,\mathcal{G}z^*)} \psi(\ell)d\ell \leq \varphi\left(\int_0^{\mathbb{L}(\varkappa^*,y^*,z^*)} \psi(\ell)d\ell\right) \quad (3)$$

$\forall\, \varkappa^*, y^*, z^*$, (s.t) $\psi \in \Psi$ with

$$\mathbb{L}(\varkappa^*,y^*,z^*) = \left\{\begin{array}{l} max\left\{\begin{array}{c}\mathcal{D}_d^*(\mathcal{P}\varkappa^*,\mathcal{P}y^*,\mathfrak{F}z^*),\\ \mathcal{D}_d^*(\mathcal{P}\varkappa^*,\mathcal{P}y^*,\mathcal{G}z^*),\\ \mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}x^*,\mathcal{P}x^*),\\ \mathcal{D}_d^*(\mathcal{G}y^*,\mathcal{G}y^*,\mathfrak{F}z^*),\\ \mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}y^*,\mathfrak{F}z^*)\end{array}\right\}\\ -min\left\{\begin{array}{c}\mathcal{D}_d^*(\mathcal{P}\varkappa^*,\mathcal{P}y^*,\mathfrak{F}z^*),\\ \mathcal{D}_d^*(\mathcal{P}\varkappa^*,\mathcal{P}y^*,\mathcal{G}z^*),\\ \mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}\varkappa^*,\mathcal{P}\varkappa^*),\\ \mathcal{D}_d^*(\mathcal{G}y^*,\mathcal{G}y^*,\mathfrak{F}z^*),\\ \mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}y^*,\mathfrak{F}z^*)\end{array}\right\}\end{array}\right\}$$

If $\mathcal{P}(\mathcal{X})$ and $\mathfrak{F}(\mathcal{X})$ closed, so $(\mathcal{F},\mathcal{P})$ & $(\mathcal{G},\mathfrak{F})$ have a unique coincidence point.

Moreover, *if* $(\mathcal{F},\mathcal{P})$ & $(\mathcal{G},\mathfrak{F})$ weakly compatible, so $(\mathcal{F},\mathcal{P})$ and $(\mathcal{G},\mathfrak{F})$ have a unique (C.F.P).

Proof: Presume that $(\mathcal{F},\mathcal{P})$ with $(\mathcal{G},\mathfrak{F})$ gratifies $(\mathbb{CLR}_{(\mathcal{P},\mathfrak{F})})$ Property. thus, $\exists\, \{\varkappa_s^*\}$ & $\{y_s^*\}$ where:

$\lim_{s\to\infty} \mathcal{F}\varkappa_s^* = \lim_{s\to\infty} \mathcal{P}\varkappa_s^* = \lim_{s\to\infty} \mathcal{G}y_s^* = \lim_{s\to\infty} \mathfrak{F}y_s^* = \ell$ & $\ell = \mathcal{P}\varkappa^* = \mathfrak{F}y^*$ for $\varkappa^*, y^* \in \mathcal{X}$.

Following, the same steps as the proof of Theorem 3.7, the rest of proof of this theorem follows.

Next, present the following instance which is extending to that in [19], to illustrate validity of Theorem 3.8.

Example 3.9: Assume $\mathcal{X} = [0,4]$ and $\mathcal{D}_d^*: \mathcal{X}\times\mathcal{X}\times\mathcal{X} \longrightarrow [0,1)$ described via $\mathcal{D}_d^*(\varkappa^*,y^*,z^*) = max\{d(\varkappa^*,y^*), d(y^*,z^*), d(z^*,\varkappa^*)\}$, where d is the usual metric on $\mathcal{X}$. In this case $(\mathcal{X},\mathcal{D}_d^*)$ is $\mathcal{D}_d^*$-Symmetric space. Assume, $\mathcal{F},\mathcal{G},\mathcal{P}$&$\mathfrak{F}$ are four self-maps described through $\mathcal{F}\varkappa^* = 2$, $\mathcal{P}\varkappa^* = 4 - \varkappa^*$, $\mathcal{G}\varkappa^* = \frac{5\varkappa^*+12}{11}$ with $\mathfrak{F}\varkappa^* = \frac{x^*}{2} + 1, \forall\, \varkappa^* \in \mathcal{X}$. Describe, $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ through $\varphi(\ell) = \frac{3}{5}\ell$.

Consider, two sequences $\{\varkappa_s^*\} = \left\{\frac{2s+1}{s}\right\}$ and $\{y_s^*\} = \left\{\frac{2s-1}{s}\right\}$, ∀ s. In this case $\mathcal{F}\varkappa_s^* \longrightarrow 2$, $\mathcal{P}\varkappa_s^* \longrightarrow 2$, $\mathcal{G}\varkappa_s^* \longrightarrow 2$ with $\mathfrak{F}y_s^* \longrightarrow 2$.

Consequently, $\lim_{s\to\infty} \mathcal{F}\varkappa_s^* = \lim_{s\to\infty} \mathcal{P}\varkappa_s^* = \lim_{s\to\infty} \mathcal{G}y_s^* = \lim_{s\to\infty} \mathfrak{F}y_s^* = 2$ with $\mathcal{P}2 = \mathfrak{F}2 = 2$.

Therefore, $(\mathcal{F},\mathcal{P})$ and $(\mathcal{G},\mathfrak{F})$ satisfies $(\mathbb{CLR}_{(\mathcal{P},\mathfrak{F})})$ Property. Additional, $(\mathcal{F},\mathcal{P})$ & $(\mathcal{G},\mathfrak{F})$ are (W. C) maps and $\mathfrak{F}$ satisfies each the conditions of Theorem-3.8 and $\varkappa^* = 2$ is unique (C.F.P) of $\mathcal{F},\mathcal{G},\mathcal{P}$ & $\mathfrak{F}$.

Corollary3.10: Assume that $(\mathcal{F},\mathcal{P})$ and $(\mathcal{G},\mathfrak{F})$ are pairs of self-maps on $(\mathcal{X},\mathcal{D}_d^*)$ satisfying Property of $(\mathbb{CLR}_{(\mathcal{P},\mathfrak{F})})$. Assume that $\Omega \in [0,1)$ with:

$$\int_0^{\mathcal{D}_d^*(\mathcal{F}\varkappa^*,\mathcal{F}y^*,\mathcal{G}z^*)} \psi(\ell)d\ell \leq \varphi\left(\int_0^{\mathbb{L}(\varkappa^*,y^*,z^*)} \psi(\ell)d\ell\right)$$

$\forall\, \varkappa^*, y^*, z^* \in \mathcal{X}$, (s. t) $\psi \in \Psi$ with:

$$\left\{\begin{array}{c} \mathbb{L}(\varkappa^*, y^*, z^*) = \\ max\left\{\begin{array}{l} \mathcal{D}_d^*(\mathcal{P}\varkappa^*, \mathcal{P}y^*, \mathfrak{F}z^*), \\ \mathcal{D}_d^*(\mathcal{P}\varkappa^*, \mathcal{P}y^*, \mathcal{G}z^*), \\ \mathcal{D}_d^*(\mathcal{F}\varkappa^*, \mathcal{F}\varkappa^*, \mathcal{P}\varkappa^*), \\ \mathcal{D}_d^*(\mathcal{G}y^*, \mathcal{G}y^*, \mathfrak{F}z^*), \\ \mathcal{D}_d^*(\mathcal{F}\varkappa^*, \mathcal{F}y^*, \mathfrak{F}z^*) \end{array}\right\} \\ -min\left\{\begin{array}{l} \mathcal{D}_d^*(\mathcal{P}\varkappa^*, \mathcal{P}y^*, \mathfrak{F}z^*), \\ \mathcal{D}_d^*(\mathcal{P}\varkappa^*, \mathcal{P}y^*, \mathcal{G}z^*), \\ \mathcal{D}_d^*(\mathcal{F}\varkappa^*, \mathcal{F}\varkappa^*, \mathcal{P}\varkappa^*), \\ \mathcal{D}_d^*(\mathcal{G}y^*, \mathcal{G}y^*, \mathfrak{F}z^*), \\ \mathcal{D}_d^*(\mathcal{F}\varkappa^*, \mathcal{F}y^*, \mathfrak{F}z^*) \end{array}\right\} \end{array}\right\}$$

If $\mathcal{P}(\mathcal{X})$ and $\mathfrak{F}(\mathcal{X})$ are closed, in this case $(\mathcal{F}, \mathcal{P})$ and $(\mathcal{G}, \mathfrak{F})$ have unique coincidence point.

Moreover, if $(\mathcal{F}, \mathcal{P})\&(\mathcal{G}, \mathfrak{F})$ weakly compatible, then $(\mathcal{F}, \mathcal{P})$ & $(\mathcal{G}, \mathfrak{F})$ contain unique (C.F.P).

Proof: instantly through setting $\varphi(\ell) = \Omega(\ell)$ (s. t) $\Omega \in [0,1)$ in Theo-3.8.

4 CONCLUSIONS

Common fixed point theory under the influence of different superior categories of expanded contraction conditions of integral type has progressed over the last years, to comprise different fruitful applications in numerous areas in mathematics and several other sciences. Subsequently, this article devoted to verify uniqueness of some sorts of (C. F. P) outcomes involving other enhanced sorts of extended integral type conditions have been discussed and proved in $\mathcal{D}_d^*$-Symmetric spaces. Furthermore, our second major outcomes have been obtained utilizing in range of $\mathcal{G}$ the property of common limit$\left(\mathbb{CLR}_{\mathcal{G}}\right)$. The proposed results are enhancing and expanding of different recognized analogous numerous current outcomes relative to these classes of common fixed points. We expect that the innovations will assist workers in enhancing the researchers on expanded metric spaces in order to elevate general structures for their specially applications in each progress parts of mathematics and other fields.

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