New Convergence and Stability Results of Fixed-Point Iteration in Partial Metric Spaces with an Application

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Abstract: This paper investigates the convergence properties of a new $(w\, u_0)$-three-step iterative algorithm for weakly $(\mu\, k)$-nonexpansive mappings in convex partial metric spaces. The proposed algorithm is developed to improve convergence behavior and achieve faster approximation of fixed points compared with several classical iterative methods. First, examples are provided to demonstrate the efficiency of the proposed three-step algorithm. A comparative analysis is conducted with several well-known iterative schemes, including Mann, Ishikawa, S-Picard, JF, Agarwal, Thakur, M, M^*, and S-iteration methods. The results show improved convergence performance of the proposed method. Second, the stability of the new iterative process is established. Furthermore, several convergence results are obtained in uniformly convex complete partial metric spaces. As an application, the proposed algorithm is employed to obtain approximate solutions for a delay differential equation, supported by a numerical example. The obtained results extend existing fixed-point theory and provide new insights into the development of efficient iterative methods for nonlinear problems in partial metric spaces.

1 INTRODUCTION

The most famous fixed-point iterative procedure is the Picard sequence. The weakness of this iteration is that whenever it is defined for a nonexpansive mapping, even when a unique fixed point exists, it may fail to converge to it [1]. This problem was addressed by mean iteration, then by generalizing the mean iteration, which is the Mann iteration. [2]. Ishikawa introduced an iteration sequence to prove an independent theorem for an approximating fixed point of a continuous pseudo-contraction, see [3]. Many convergence results are proved using some iterative sequences to approximate the fixed points of various generalizations of nonexpansive maps. Consequently, a broad literature has developed on establishing equivalence between the convergence of several well-known iterative schemes and data dependence under these mappings. Many authors have contributed to these studies in complete metric spaces and Banach spaces, including Nahi [4], Albundi [5], [6], Puiwong and Saejung [7], Jachymski et al. [8], as well as several references [9], [10].

A partial metric space (shortly, PMS) provides a powerful and flexible framework for studying spaces with self-referential distances, making them valuable in theoretical computer science and fixed-point analysis (Matthews [11] and Jabar et. al. [12]. For a fixed point, the results of Gopal and Jain [13], Saluja [14], and Karapınar [15], as well as Dakjum et al. [16], etc. In previous work (Under publication), the authors explain that a PMS is a T_0-topology, and Hausdorff status may not be achieved in a PMS. This implies the non-uniqueness of the limit of a convergent sequence. To make metric spaces suitable for generalizing fixed-point results, Takahashi [17] added the concept of convexity to the structure of metrics. Next, many results on fixed points in convex metric spaces appeared, such as in Sen [18], Rahul Shukla [19], and Adian [20]. In a convex partial metric space (shortly, CPMS), Taki and Aliouche [21] proved fixed point theorems for contraction maps as a generalization of previously obtained results in convex metric spaces. Recently, the convergence of kinds of iterative algorithms has been studied on types of non-expansive maps, μ-nonexpansive, weakly k-nonexpansive, and

Table 1: Explains (u, u_0)-three-step algorithm (1).

n	0	1	2	3	4	5	6	7	8	9
u_n	0.1	0.5758	0.6555	0.5326	0.409	0.4193	0.2042	0.12	0.1392	0.1214

weakly μ-k-nonexpansive maps. Some of them may be stronger than quasi-nonexpansiveness or weaker than non-expansiveness, see [22]-[25]. Here, a kind of iteration is introduced to study its convergence, stability, and application.

This study examines the convergence of the following novel iteration to a fixed point Next; a kind of iterative algorithm is introduced:

Definition 1. Let G be a CPMS, $\emptyset \neq C \subset G$, C be closed and convex and Γ: $C \to C$ be a map where $\mathrm{Fix}(\Gamma) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\alpha_n{}^*\}, \{\alpha_n{}^{**}\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$. For fix u and $u_0 \in C, n \geq 0$, define

$$u_{n+1} = \alpha_n a + \alpha_n{}^* u_n + \alpha_n{}^{**} v_n,$$
$$v_n = \mathcal{M}(u_n, \Gamma w_n, \beta_n),$$
$$w_n = \mathcal{M}(u_n, \Gamma u_n, \gamma_n). \qquad (1)$$

Then Algorithm 1 is called (u, u_0)-three-step.

The following example explains Definition 1:

Example 1. Consider (G, ρ) a PMS $G = [0, \infty)$ and $\rho: G \times G \to \mathfrak{R}^+$ such that

$$\rho(g,s) = \begin{Bmatrix} 0 \text{ if } g = s \\ max\,\{g,s\} \text{ if } g \neq s \end{Bmatrix} \text{ and } \Gamma g = g^2,$$

we apply the notion of a (u, u_0)-three-step algorithm in (1) by the following choice. $\alpha_n, \alpha_n{}^*, \alpha_n{}^{**} = \frac{1}{n+2}$ and $\beta_n = \gamma_n = 0.5$, for all $n \geq 0$ and assume that $a = 1 \in G$ with the initial point $u_0 = 0.1$. Then, $\{u_n\}_{n=0}^{\infty}$ is displayed as shown in Figure 1, Table 1. Using MATLAB. The values of $\{u_n\}_{n=0}^{\infty}$ appear to be approaching zero.

The three-step design of this iteration encouraged us to study it, along with its relative independence and speed compared to known iterations in previous literature [26]-[28]. The two examples below support this idea. Here Γ is a generalized non-expansive map due to Hardy and Rogers, but not a non-expansive map, and Γ has a unique fixed point 0. The sequence $\{u_n\}_{n=0}^{\infty}$ generated by the scheme (1) converges faster than some known schemes to the fixed point 0 of Γ. We apply the notion of (u, u_0)-three-step algorithm (1) and other known algorithms, by the following choice. $\alpha_n = 0.7$, $\beta_n = \gamma_n = 0.625$, $\alpha_n{}^* = 0.2$, $\alpha_n{}^{**} = 0.1$ such that $a = 1$ with $u_0 =$ 1.5. which is shown in Table 2.

Example 2. Let $G = \mathfrak{R}$ with the usual norm and $S = \mathfrak{R}^+ \subset \mathfrak{R}$. Define a map $\Gamma: \mathfrak{R}^+ \to \mathfrak{R}^+$ by

$$\Gamma(g) = \begin{Bmatrix} \frac{1}{2}\sin g, \text{if } g \in [0, \frac{\pi}{2}] \\ \frac{1}{4}\sin g, \text{if } g \in (\frac{\pi}{2}, \infty] \end{Bmatrix}.$$

Example 3. Let $G = \mathfrak{R}$ be equipped with the usual norm and $S = [1, \infty^+)$. Define a map Γ $: [1, \infty^+) \to [1, \infty^+), \Gamma(g) = \begin{Bmatrix} \frac{g+2}{3}, \text{if } g \in [1,3] \\ \frac{g}{g+1}, \text{if } g \in (3, \infty^+) \end{Bmatrix}.$

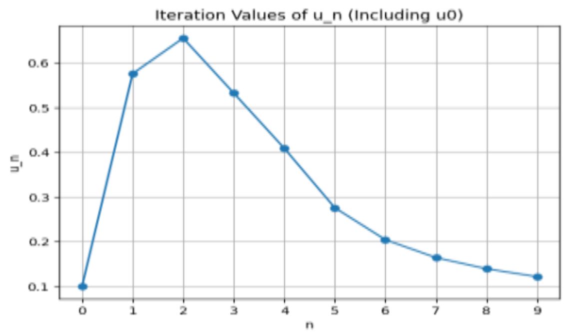

Figure 1: Application of the (u, u_0)-three-step algorithm (1) in Example 1.

Table 2: (u, u_0)-three-step algorithm (1) converges faster than the other iterative schemes.

No. iterate	Iteration
7	Algorithm (1)
7	JF
11	Thakur
11	Picard-S
20	S
21	Noor
22	Ishikawa
24	Mann

Table 3: (u, u_0)-three-step algorithm (1) converges faster than the other iterative schemes.

No. iterate	Iteration
5	Algorithm (1)
5	JF
8	M
8	M*
8	Thakur
13	Agarwal
24	Ishikawa
30	Mann

Here Γ is a weakly k-nonexpansive map $k = 2$ and Γ has a unique fixed point $g = 1$. The sequence $\{u_n\}_{n=0}^{\infty}$ generated by the (u, u_0)-three-step algorithm in (1) converges faster than some known schemes to a fixed point 1 of Γ, if choose $\alpha_n = 0.6$, $\beta_n = 0.7$, $\gamma_n = 0.6$, $\alpha_n{}^* = \alpha_n{}^{**} = 0.2$ and $a = 1$ with $u_0 = 1.5$. which is shown in Table 3.

2 PRELIMINARIES

In the following, some basic notions are recalled in PMS.

Definition 2 [11]. Let G be a nonempty set and $\rho: G \times G \to \Re^+$ be a function where $\Re^+ = [0, \infty)$ such that for all $g, s, h \in G$,

- $\rho(g, g) \le \rho(g, s)$;
- if $\rho(g, g) = \rho(g, s) = \rho(s, s)$ then, $g = s$;
- $\rho(g, s) = \rho(s, g)$;
- $\rho(g, h) + \rho(s, s) \le \rho(g, s) + \rho(s, h)$.

Then the pair (G, ρ) is called a partial metric space.

Lemma 1 [29], [30]: Let $\{g_n\}$ and $\{s_n\}$ be two sequences in a PMS (G, ρ) such that
$\rho(g, g) = \lim_{n\to\infty} \rho(g_n, g) = \lim_{n\to\infty} \rho(g_n, g_n)$
and $\rho(s, s) = \lim_{n\to\infty} \rho(s_n, s) = \lim_{n\to\infty} \rho(s_n, s_n)$.
Then, $\rho(g, s) = \lim_{n\to\infty} \rho(g_n, s_n)$.
In particular, $\rho(g, h) = \lim_{n\to\infty} \rho(g_n, h)$, for every $h \in G$.
For g, s in G

1) If $\rho(g, s) = 0$, then $g = s$,
2) If $g \ne s$, then $\rho(g, s) > 0$.

Taki and Aliouche [21] add a convex structure on a PMS as follows:

Definition 3 [21]: Let (G, ρ) be a PMS.

1) $\mathcal{M}$: $G^2 \times K \to G$ is called a convex structure map on G, if $\rho(u, \mathcal{M}(g, s, \lambda)) \le \lambda\rho(u, g) + (1 - \lambda)\rho(u, s)$, for all $(g, s, \lambda) \in G^2 \times K$ and $u \in G$, such that $\lambda \in K = [0,1]$.
2) A PMS (G, ρ) combined with a convex structure $\mathcal{M}$ is called a CPMS and denoted by $(G, \rho, \mathcal{M})$.
3) In a CPMS $(G, \rho, \mathcal{M})$. A nonempty $C \subset G$ is called convex if $\mathcal{M}(g, s, \lambda) \in C$ whenever $(g, s, \lambda) \in C^2 \times K$.

The following example satisfied the above definition.

Example 4. Let $G = \Re^+$ with $\rho(g, s) = \max\{g, s\}$ for all $g, s \in G$. A map $\mathcal{M}$: $G^2 \times K \to G$. And $\mathcal{M}(g, s, \lambda) = \lambda g + (1 - \lambda)s$ for $\lambda \in [0,1]$. Clearly, (G, ρ) is a PMS [21]. Since, $\rho(u, \mathcal{M}(g, s, \lambda)) = \max\{\lambda g + (1 - \lambda)s\} \le \lambda \max\{u, g\} + (1 - \lambda)\max\{u, s\} = \lambda\rho(u, g) + (1 - \lambda)\rho(u, s)$. Therefore, $(G, \rho, \mathcal{M})$ is CPMS.

Definition 4[21]: A CPMS $(G, \rho, \mathcal{M})$ is called (UCPMS) if for any $\epsilon > 0, \exists\ \alpha \in (0,1]$, for all $r > 0$ and $g, s, h \in G$ with $\rho(g, h) \le r$, $\rho(s, h) \le r$, and $\rho(g, s) \ge r\epsilon$
such that $\rho\left(h, \mathcal{M}\left(g, s, \frac{1}{2}\right)\right) \le r(1 - \alpha) < r$.

A map Ω: $(0, \infty) \times (0,2] \to (0, 1]$ such that, a map $\Omega(r, \epsilon)$ is called a modulus of uniform convexity for any $r > 0$ and $\epsilon \in (0,2]$.

Lemma 2. Let $(G, \rho, \mathcal{M})$ be a UCPMS with a monotone modulus Ω and $\epsilon > 0$. Then for any $r > 0$, $\lambda \in [0,1]$ and $g, s, h \in G$ with
$\rho(g, h) \le r, \rho(s, h) \le r$, and $\rho(g, s) \ge r\epsilon$,
$\rho(h, \mathcal{M}(g, s, \lambda)) \le r(1 - 2\lambda(1 - \lambda)\ \Omega(r, \epsilon))$.

Proof. By the fact that $2\lambda(1 - \lambda) \le 2(1 - \lambda)$ and $2\lambda(1 - \lambda) \le 2\lambda$ when $\lambda \in [0,1]$. Then, by Definition 4 $\rho(h, \mathcal{M}(g, s, \lambda)) \le r(1 - \Upsilon(r, \epsilon, \lambda))$

when $\Upsilon(r, \epsilon, \lambda) = \begin{Bmatrix} 2\lambda\ \Omega(r, \epsilon) \text{ if } \lambda \le \frac{1}{2} \\ 2(1 - \lambda)\Omega(r, \epsilon) \text{ if } \lambda \ge \frac{1}{2} \end{Bmatrix}$.

Definition 5: Let $(G, \rho, \mathcal{M})$ be a CPMS $(G, \rho, \mathcal{M})$ is called a SCPMS if for all $r \in [0,1]$, $\exists! h \in G$ such that $\rho(g,h) = r\rho(g, s)$ and $\rho(h, s) = (1 - r)\rho(g, s)$ for all $g, s, h \in G$.

Proposition 1: Every UCPMS is SCPMS.

proof: By uniformly convex, let $g, s, h \in G$ with $\rho(g, h) \le r$, $\rho(s, h) \le r$, and $\rho(g, s) \ge r\epsilon$ for all $r > 0$. To show $\rho(g, h) = r\rho(g,s)$ and $\rho(h, s) = (1 - r)\rho(g, s)$ for all $g, s, h \in G$. Assume that there exist two different points. $h, w \in G$ such that
$\rho(g, h) = \rho(g, w) = r\rho(g, s)$ and $\rho(s, h) = \rho(s, w) = (1 - r)\rho(g, s)$ such that $g \ne s$.
Let $r_1 = r\rho(g, s)$ and $r_2 = (1 - r)\rho(g, s)$, suppose that $\epsilon_1, \epsilon_2 \in (0,2]$ such that
$\epsilon_1 = \frac{\rho(h,w)}{r_1}, \epsilon_2 = \frac{\rho(h,w)}{r_2}$. Used Lemma 2, when $\lambda \in [0,1]$. Let $\lambda = \frac{1}{2}$ get $\rho\left(\frac{h}{2} + \frac{w}{2}, g\right) \le r_1(1 - \Upsilon(r_1, \epsilon_1))$ and $\rho\left(\frac{h}{2} + \frac{w}{2}, s\right) \le r_2(1 - \Upsilon(r_2, \epsilon_2))$.
Used Lemma 1, if $g \ne s$, then $\rho(g, s) > 0$, get to

$$\rho(g, s) \le \rho\left(\frac{h}{2} + \frac{w}{2}, g\right) + \rho\left(\frac{h}{2} + \frac{w}{2}, s\right),$$

$\le r_1(1 - \Upsilon(r_1, \epsilon_1)) + r_2(1 - \Upsilon(r_2, \epsilon_2))$,
since $\epsilon_1, \epsilon_2 \in (0,2]$ and must be $\Upsilon(r_1, \epsilon_1) < 1$ or $\Upsilon(r_1, \epsilon_1) < 1$. Get
$\rho(g, s) < r_1 + r_2$,

$< r\rho(g,s) + (1-r)\rho(g,s)$,
$< \rho$. This is a contradiction.

Example 5: Let $G = \mathfrak{R}^+$ with $\rho(g,s) = \max\{g,s\} + |s-g|$ for all $g, s \in G$. It is evident strictly convex but not uniformly convex.

Reform the following concept in PMS as in [21].

Definition 6: Let (G, ρ) be a PMS. A map $\Gamma: G \to G$ is called a Generalized non-expansive if $\rho(\Gamma(g),\Gamma(s)) \leq \rho(\Gamma(g),s)$, for all $g,s \in G$ with $\mu_i \geq 0$ such that

$$\rho(\Gamma(g),\Gamma(s)) \leq \mu_1\,\rho(g,s) + \mu_2\,\rho(g,\Gamma(g)) + \mu_3\,\rho(s,\Gamma(s)) + \mu_4\,\rho(g,\Gamma(s)) + \mu_5\,\rho(s,\Gamma(g)).$$

If $Fix(\Gamma) \neq \emptyset$, then Γ is a quasi-nonexpansive map.

Remark 1: Every generalized nonexpansive map with a fixed point is a quasi-nonexpansive map.

Proof. Let $d \in Fix(\Gamma)$, if $s = d$. Then

$$\rho(\Gamma(g),\Gamma(d)) \leq \mu_1\,\rho(g,d) + \mu_2\,\rho(g,\Gamma(g)) + \mu_3\,\rho(d,\Gamma(d)) + \mu_4\rho(g,\Gamma(d)) + \mu_5\rho(d,\Gamma(g)).$$

$$\rho(\Gamma(g),\Gamma(d)) \leq (\mu_1+\mu_4)\rho(g,d) + (\mu_2+\mu_5)\,\rho(g,\Gamma(g)) \text{ for all } g,s \in G \text{ with } \mu_i \geq 0.$$

Getting $(\Gamma(g),\Gamma(d)) \leq \rho(g,d)$. That is a quasi-nonexpansive map.

Reform the following concepts in the PMS case as in [22] and [28].

Definition 7: Let (G, ρ) be a PMS. A map $\Gamma: G \to G$ is called

1) μ-nonexpansive if $\rho(\Gamma(g),\Gamma(s))^2 \leq \mu\left(\rho(g,\Gamma(s))^2 + \rho(s,\Gamma(g))^2\right) + (1-2\mu)\rho(g,s)^2$.
2) Generalized μ-nonexpansive if $\frac{1}{2}\rho(g,\Gamma(s)) \leq \rho(g,s)$ implies
$\rho(\Gamma(g),\Gamma(s)) \leq \mu\left(\rho(g,\Gamma(s)) + \rho(s,\Gamma(g))\right) + (1-2\mu)\rho(g,s)$, for all $g,s \in G$, $0 \leq \mu < 1$.
3) C-Condition if $\frac{1}{2}\rho(g,\Gamma(s)) \leq \rho(g,s)$, then $\rho(\Gamma(g),\Gamma(s)) \leq \rho(g,s)$ for all $g,s \in G$.

Definition 8: Let (G, ρ) be a PMS. A map $\Gamma: G \to G$ is called

1) k-nonexpansive if there exists $k \geq 0$ such that $\rho\left(\Gamma(g)^k,\Gamma(s)\right) \leq \rho\left(\Gamma(g)^{k-1},s\right)$ for all $g,s \in G$.

If $k = 1$ then $\rho(\Gamma(g),\Gamma(s)) \leq \rho(g,s)$ is called a nonexpansive map.

If $k = 2$ then $\rho(\Gamma(g)^2,\Gamma(s)) \leq \rho(\Gamma(g),s)$ it is called a fundamentally nonexpansive map.

2) Weakly k −nonexpansive if there exists $k \geq 0$ such that $\rho(\Gamma(g),\Gamma(s)) \leq \rho(g,s) + k[\rho(g,\Gamma(g)).\rho(s,\Gamma(s))]$ for all $g,s \in G$.
3) Weakly μ-k-nonexpansive if there exists $k \geq 0$ such that
$\rho(\Gamma(g),\Gamma(s)) \leq \mu\rho(g,s) + k[\rho(g,\Gamma(g)).\rho(s,\Gamma(s))]$ for all $g,s \in G$, $0 \leq \mu < 1$.

To illustrate the above Definition:

Example 6. Consider (G,ρ) a PMS $G = [0,\infty)$ and $\rho: G \times G \to \mathfrak{R}^+$ such that

$$\rho(g,s) = \begin{Bmatrix} 0 \text{ if } g = s \\ \max\{g,s\} \text{ if } g \neq s \end{Bmatrix} \text{ and } \Gamma g = g^2.$$

It is evident weakly μ-k-nonexpansive map since if $g = 2, s = 3$ and $\mu = \frac{1}{3}, k = \frac{2}{3}$ then $\rho(\Gamma(g),\Gamma(s)) = \max\{4,9\}$, get

$$\rho(\Gamma(g),\Gamma(s)) = 9 \leq \frac{1}{3}.3 + \frac{2}{3}[\max\{4,9\}.\max\{4,9\}] \leq 1 + \frac{2}{3}[4.9], \text{ so } 9 \leq 25.$$

Also, it is weakly k-nonexpansive map. But not a nonexpansive map, since if $g = 2, s = 3$ then $\rho(\Gamma(g),\Gamma(s)) = \max\{4,9\} = 9 \nleq \rho(g,s) = \max\{2,3\} = 3$, get $9 \nleq 3$.

The implication is that all nonexpansive is weakly k-nonexpansive, but in this case, the opposite is obviously false.

Remark 2: Every Weakly k −nonexpansive with $\mu = 1$ is Weakly $\mu - k$ −nonexpansive. Since $\exists\, k \geq 0$ and $\mu = 1$ then $\rho(\Gamma(g),\Gamma(s)) \leq \rho(g,s) + k[\rho(g,\Gamma(g)).\rho(s,\Gamma(s))]$ for all $g,s \in G$.

Remark 3. Every nonexpansive is fundamentally nonexpansive. Since if $(\Gamma(u),\Gamma(v)) \leq \rho(u,v)$, for all $u, v \in G$. for any g and s. Put $u = \Gamma(g)$, $v = s$ then obtain $\rho(\Gamma(\Gamma(g)),\Gamma s) = \rho(\Gamma^2(g),\Gamma(s)) \leq \rho(\Gamma(g),s)$, but the opposite is not true, as follows.

Example 7. Consider a PMS (G,ρ), where $G = \mathfrak{R}^+$ and $\rho: G \times G \to \mathfrak{R}^+$, $\rho(g,s) = |g-s|$.

$$\text{A map } \Gamma: G \to G,\ \Gamma g = \begin{Bmatrix} 1.5 & \text{if } g = 3 \\ 1 & \text{if } g \neq 3 \end{Bmatrix},$$

is a fundamentally nonexpansive map such that

If $g = 3, s = 3, \Gamma^2(3) = 1, \Gamma(3) = 1.5$ then $|1-1.5| = 0.5$, $\Gamma(3) = 1.5$ then $|1.5-3| = 1.5$,

If $g = 3, s \neq 3, \Gamma^2(3) = \Gamma(1.5) = 1$ then $|1-3| = 2$, $\Gamma(3) = 1$ then $|1-1| = 0$.

If $g \neq 3, s \neq 3, \Gamma^2(g) = 1, \Gamma(s) = 1$ then $|1-1| = 0$, $\Gamma(g) = 1$ then $|1-1| = 0$.

Obtain Γg is a fundamentally nonexpansive. But it is not a nonexpansive map.

If $g = 3, s \neq 3$ then $|\Gamma(3) - \Gamma(s)| = |1.5 - 1| = 0.5$, $|3 - 2.8| = 0.2$ then $0.5 \nleq 0.2$.

Proposition 2: Let (G, ρ) be a PMS. A map $\Gamma: G \to G$ be a weakly k-nonexpansive. Then $\rho(\Gamma(g), \Gamma(s)) \leq \rho(g, s) + \rho(g, \Gamma(g))[1 + k\,\rho(s, \Gamma(s))]$ for all $g, s \in G$.

Proof. From (4) and Definition 2, we have $\rho(g, \Gamma(s)) \leq \rho(g, \Gamma(g)) + \rho(\Gamma(g), \Gamma(s)) - \rho(\Gamma(s), \Gamma(s))$, Γ is weakly k-nonexpansive map, get $\rho(g, \Gamma(s)) \leq \rho(g, \Gamma(g)) + \rho(g, s) + k[\rho(g, \Gamma(g)).\rho(s, \Gamma(s))] - \rho(\Gamma(s), \Gamma(s))$. Obtain $\rho(g, \Gamma(s)) \leq \rho(g, s) + \rho(g, \Gamma(g))[1 + k\,\rho(s, \Gamma(s))] - \rho(\Gamma(s), \Gamma(s))$,
Therefore $\rho(g, \Gamma(s)) \leq \rho(g, s) + \rho(g, \Gamma(g))[1 + k\,\rho(s, \Gamma(s))]$ for all $g, s \in G$.

Lemma 3: Let (G, ρ) be a PMS. A map $\Gamma: G \to G$ be a weakly k-nonexpansive where C is a closed $C \subset G$. Then, $Fix(\Gamma)$ is closed, and if G strictly convex with C is convex. Then $Fix(\Gamma)$ is convex.

Proof. First, proof is required. $Fix(\Gamma)$ is closed, suppose that $d \in \overline{Fix(\Gamma)}$. Then $\{d_n\}$ be a sequence subset of $Fix(\Gamma)$, $d_n \to d$. By Proposition 2, we have $\rho(d_n, \Gamma(d)) \leq \rho(d_n, d) + \rho(d_n, \Gamma(d_n))[1 + k\,\rho(d, \Gamma(d))]$ get $\rho(d_n, \Gamma(d)) \leq \rho(d_n, d)$ this by $d_n = \Gamma(d_n)$ and $d_n \to d$ for all $n \to \infty$.

Take $\lim_{n\to\infty}$ for above then $\lim_{n\to\infty}\rho(d_n, \Gamma(d)) \leq \lim_{n\to\infty}\rho(d_n, d) = 0$ which implies $d \in Fix(\Gamma)$. With these steps, obtain $Fix(\Gamma)$ is closed.

Now, assume that G strictly convex with C is convex. To show that $Fix(\Gamma)$ is convex. Suppose let $d_1, d_2 \in Fix(\Gamma)$ and $d_1 \neq d_2$ such that $d := \delta d_1 + (1 - \delta)d_2$ for all $\in (0,1)$, used Proposition 2, we have $\rho(d_1, \Gamma(d)) \leq \rho(d_1, d) + \rho(d_1, \Gamma(d_1))[1 + k\,\rho(d, \Gamma(d))]$ getting

$$\rho(d_1, \Gamma(d)) \leq \rho(d_1, d) \text{ since } d_1 = \Gamma(d_1).$$

Similarly,
$\rho(d_2, \Gamma(d)) \leq \rho(d_2, d) + \rho(d_2, \Gamma(d_2))[1 + k\,\rho(d, \Gamma(d))]$, so

$$\rho(d_2, \Gamma(d)) \leq \rho(d_2, d) \text{ since } d_2 = \Gamma(d_2).$$

Therefore $\rho(d_1, d_2) \leq \rho(d_1, \Gamma(d)) + \rho(\Gamma(d), d_2) - \rho(\Gamma(d), \Gamma(d))$,
$\rho(d_1, d_2) \leq \rho(d_1, d) + \rho(d, d_2) - \rho(d, d)$. So,
$\rho(d_1, \Gamma(d)) + \rho(\Gamma(d), d_2) = \rho(d_1, d_2)$.
Since
G strictly convex, $\exists k > 0, \Gamma(d) = rd_1 + (1 - r)d_2$ where $r = \frac{1}{1+k}$ get

$$(1 - r)\rho(d_1, d_2) = \rho(d_1, \Gamma(d)) \leq \rho(d_1, d) = (1 - \delta)\rho(d_1, d_2).$$

And

$$r\rho(d_1, d_2) = \rho(d_2, \Gamma(d)) \leq \rho(d_2, d) = \delta\rho(d_1, d_2).$$

Obtain, $1 - r \leq 1 - \delta$ then $\delta \leq r$ getting $\delta = r$. Then $\Gamma(d) = d$ which implies $d \in Fix(\Gamma)$.

Thus, $Fix(\Gamma)$ is convex.

3 MAIN RESULTS

Here, the concept of stability in the PMS is given, presenting a new iteration that takes into account a new kind of contraction, using a weakly. μ-k-nonexpansive map.

Reform the following concept in PMS as in [31]:

Definition 9: Let $(G\ \rho)$ be a PMS. A map $\Gamma: G \to G$. Let is $u_1 \in G$ and $u_{n+1} = \wp(\Gamma, u_n)$ be an iterative process involving Γ, with $d \in Fix(\Gamma)$ assume that $\{u_n\}_n$ converges to d. Let $\{g_n\}_n$ be any sequence in G and $\theta_n := \rho(g_{n+1}, \wp(\Gamma, g_n))$ for all $n \in \mathbb{N}$. Then $u_{n+1} = \wp(\Gamma, u_n)$ is called stable w.r.t Γ if and only if $\lim_{n\to\infty}\theta_n = 0$ implies $\lim_{n\to\infty}g_n = d$.

The following example is on Definition 9:
Example 8: Let $G = \Re^+$ with $\rho(g,s) = \max\{g, s\}$ for all $g, s \in G$. A map $\Gamma: G \to G$ where $\Gamma g = \frac{g}{2}$ then $d = 0 \in Fix(\Gamma)$ and $\theta_n := \rho(g_{n+1}, \wp(\Gamma, g_n)) \to 0$, thus $\lim_{n\to\infty}g_n \to d$. Obtain, $\lim_{n\to\infty}g_n = 0$. Then Γ is stable.

Lemma 4. Let $\delta \in [0,1)$ and $\{\theta_n\} \subset (0, \infty)$ with $\lim_{n\to\infty}\theta_n = 0$. Then, for any sequence $\{g_n\}$ in $(0, \infty)$ such that $g_{n+1} \leq \theta_n + \delta g_n$. Then $\lim_{n\to\infty}g_n = 0$.

Proof. By the way, for any sequence $\{g_n\}$ in $(0, \infty)$ such that $g_{n+1} \leq \sum_{k=0}^{n}\delta^{n-k}\theta_k + \delta^{n+1}g_n$, therefore $\delta^{n+1}g_0$ converge to 0 since $\delta \in [0,1)$.
And $\sum_{k=0}^{n}\delta^{n-k}\theta_k = \sum_{k=0}^{N-1}\delta^{n-k}\theta_k + \sum_{k=N}^{n}\delta^{n-k}\theta_k$ such that $n \geq N$.
Obtain $\sum_{k=0}^{N-1}\delta^{n-k}\theta_k$ converge to 0 since $\in [0,1)$.
Let $\sigma > 0$ and $\theta_n < \sigma$ then $\sum_{k=N}^{n}\delta^{n-k}\theta_k \leq \sigma\sum_{k=N}^{n}\delta^{n-k}$.

Therefore $\sum_{k=0}^{n}\delta^{n-k}\theta_k = \sum_{k=0}^{N-1}\delta^{n-k}\theta_k + \sigma\sum_{k=N}^{n}\delta^{n-k}$.
Then g_{n+1} converge to 0, getting $\lim_{n\to\infty}g_n = 0$

Start with the following.

Theorem 1. Let $(G, \rho, \mathcal{M})$ be a PMS and $\Gamma: G \to G$ be a weakly $\mu - k -$nonexpansive mapping. Assume that Γ has a fixed point $d \in G$. Suppose $u_1 \in G$ and $u_{n+1} = \wp(\Gamma, u_n)$ defined by (1). Then $\{u_{n+1}\}$ is stable.

Proof. According to the hypothesis $u_{n+1} = \wp(\Gamma, u_n)$ defined by (1). Suppose that $\{\alpha_n\}, \{\alpha_n{}^*\}, \{\alpha_n{}^{**}\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$, let $\{g_n\}_n$ be any sequence in G and $\theta_n := \rho(g_{n+1},\wp(\Gamma,g_n))$ for all $n \in \mathbb{N}$ with $d \in Fix(\Gamma) \neq \emptyset$. To show $\lim_{n\to\infty} g_n = d$, suppose that $\lim_{n\to\infty}\theta_n = 0$. Then, from (4), Definition 2, get

$\rho(g_{n+1},d) \leq \rho(g_{n+1},\wp(\Gamma,g_n)) + \rho(\wp(\Gamma,g_n),d) - \rho(\wp(\Gamma,g_n),\wp(\Gamma,g_n))$,

$\rho(g_{n+1},d) \leq \theta_n + \rho(u_{n+1},d) - \rho(u_{n+1},u_{n+1}))$, by the hypothesis above,

$\rho(g_{n+1},d) \leq \theta_n + \rho(\alpha_n a + \alpha_n{}^{*}u_n + \alpha_n{}^{**}v_n,d)$, by Definition 1,

$$\rho(g_{n+1},d) \leq \theta_n + \alpha_n\rho(a,d) + \alpha_n{}^{*}\rho(u_n,d) + \alpha_n{}^{**}\rho(v_n,d),$$
$$\leq \theta_n + \alpha_n\rho(a,d) + \alpha_n{}^{*}\rho(u_n,d) + \alpha_n{}^{**}\rho(\mathcal{M}(u_n,\Gamma w_n,\beta_n),d)$$
$$\leq \theta_n + \alpha_n\rho(a,d) + \alpha_n{}^{*}\rho(u_n,d) + \alpha_n{}^{**}[\beta_n\rho(u_n,d)+(1-\beta_n)\rho(\Gamma w_n,d),$$
$$\rho(g_{n+1},d) \leq \theta_n + \alpha_n\rho(a,d) + \alpha_n{}^{*}\rho(u_n,d) + \alpha_n{}^{**}[\beta_n\rho(u_n,d) + (1-\beta_n)[\gamma_n\rho(u_n,d) + (1-\gamma_n)\rho(\Gamma u_n,d)]],$$
$$\rho(g_{n+1},d) \leq \theta_n + \alpha_n\rho(a,d) + \alpha_n{}^{*}\rho(u_n,d) + \alpha_n{}^{**}\beta_n\rho(u_n,d) + (1-\beta_n)\gamma_n\rho(u_n,d) + (1-\beta_n)(1-\gamma_n)\rho(u_n,d)$$

by Definition 5,

$$\rho(g_{n+1},d) \leq \theta_n + \alpha_n\rho(a,d) + \rho(u_n,d)[\alpha_n{}^{*} + \alpha_n{}^{**}\beta_n + \gamma_n(1-\beta_n) + (1-\beta_n)(1-\gamma_n)],$$
$$\leq \theta_n + \alpha_n\rho(a,d) + \rho(u_n,d)[\alpha_n{}^{*} + \alpha_n{}^{**}\beta_n + (1-\beta_n)],$$
$$\leq \theta_n + \alpha_n\rho(a,d) + \rho(u_n,d)[(1-\beta_n) + \alpha_n{}^{*} + \alpha_n{}^{**}\beta_n],$$
$$\leq \theta_n + \alpha_n\rho(a,d) + (1-\beta_n)\rho(u_n,d) + \alpha_n{}^{*}\rho(u_n,d) + \alpha_n{}^{**}\beta_n\rho(u_n,d),$$

Suppose that $\alpha_n\rho(a,d) + (1-\beta_n)\rho(u_n,d) = \max\{\rho(a,d).\rho(u_n,d)\}$.

Hence, $\rho(g_{n+1},d) \leq \theta_n + \max\{\rho(a,d).\rho(u_n,d)\} + \rho(u_n,d)[\alpha_n{}^{*} + \alpha_n{}^{**}\beta_n]$,

since $\{\alpha_n\},\{\alpha_n{}^{*}\},\{\alpha_n{}^{**}\},\{\beta_n\},\{\gamma_n\} \subset (0,1)$ and $\alpha_n{}^{*} + \alpha_n{}^{**}\beta_n < 1$. Get

$$\rho(g_{n+1},d) \leq \theta_n + \max\{\rho(a,d).\rho(u_n,d)\} + \rho(u_n,d),$$

by Lemma 4, then $g_{n+1} \leq \theta_n + \delta g_n$ we have $\lim_{n\to\infty}\rho(g_n,d) = 0$. Getting is stable w.r.t Γ.

Now, the results of the convergence have been proven.

Theorem 2: Let $(G,\ \rho,\mathcal{M})$ be a complete UCPMS with a monotone modulus Ω and let $\{\alpha_n\}$ be a sequence in $[u,v]$ for some $u,v \in (0,1)$ and the sequences $\{g_n\}$, $\{s_n\}$ in G and for some $r \geq 0$ such that $\lim_{n\to\infty}\sup\rho(g,g_n) \leq r$, $\lim_{n\to\infty}\sup\rho(g,s_n) \leq r$ and $\lim_{n\to\infty}\rho(g,\mathcal{M}(g_n,s_n,\alpha_n)) = r$ for $g \in G$. Then $\lim_{n\to\infty}\rho(g_n,s_n) = 0$.

Proof. For some $r \geq 0$, there are two cases:

The first one: If $r = 0$ then $\lim_{n\to\infty}\sup\rho(g,g_n) \leq 0$, $\lim_{n\to\infty}\sup\rho(g,s_n) \leq 0$ and $\lim_{n\to\infty}\rho(g,\mathcal{M}(g_n,s_n,\alpha_n)) = 0$ getting $\lim_{n\to\infty}\rho(g_n,s_n) = 0$.

The second: If $r > 0$, and suppose that $\lim_{n\to\infty}\rho(g_n,s_n) \neq 0$, $\rho(g_n,s_n) > 0$

Let $\rho(g_n,s_n) \geq \frac{\lambda}{2} > 0$ such that $\lambda > 0$ with $\lim_{n\to\infty}\sup\rho(g,g_n) \leq r$, $\lim_{n\to\infty}\sup\rho(g,s_n) \leq r$. Then

$\rho(g,g_n) < r \leq r + \frac{1}{n}$, thus $\rho(g,g_n) < r + \frac{1}{n}$, $n \in N$.

$\rho(g,s_n) < r \leq r + \frac{1}{n}$, thus $\rho(g,g_n) < r + \frac{1}{n}$, for all $n \in \mathbb{Z}^{+}$

And $\rho(g_n,s_n) \geq \frac{\lambda}{2} > 0$. where $\lambda > 0$, $r > 0$.

Also, $(g_n,s_n) \geq \frac{\lambda}{2} \geq (r + \frac{1}{n})\frac{\lambda}{2(r+1)}$, thus $\rho(g_n,s_n) \geq (r + \frac{1}{n})\frac{\lambda}{2(r+1)}$, such that $\frac{\lambda}{2(r+1)} < 1$.

By Lemma 2, organize

$$\rho(g,\mathcal{M}(g_n,s_n,\alpha_n)) \leq r(1 - 2\alpha_n(1-\alpha_n)\Upsilon(r,\epsilon)),$$
$$\leq (r + \frac{1}{n})(1 - 2\alpha_n(1-\alpha_n)\Upsilon\left(\left(r + \frac{1}{n}\right),\frac{\lambda}{2(r+1)}\right),$$
$$\leq (r + \frac{1}{n})(1 - 2\alpha_n(1-\alpha_n)\Upsilon\left((r+1),\frac{\lambda}{2(r+1)}\right),$$
$$\leq (r + \frac{1}{n})(1 - 2v(1-u)\Upsilon\left((r+1),\frac{\lambda}{2(r+1)}\right),$$

since $\{\alpha_n\} \subset [u,v] \subset (0,1)$. Take $\lim_{n\to\infty}$ for both sides to get

$$\lim_{n\to\infty}\rho(g,\mathcal{M}(g_n,s_n,\alpha_n)) \leq \lim_{n\to\infty}(r + \frac{1}{n})(1 - 2v(1-u)\Upsilon\left((r+1),\frac{\lambda}{2(r+1)}\right),$$

so

$$\lim_{n\to\infty}\rho(g,\mathcal{M}(g_n,s_n,\alpha_n)) \leq r\,(1 - 2v(1-u)\Upsilon\left((r+1),\frac{\lambda}{2(r+1)}\right).$$

Then

$$\lim_{n\to\infty}\rho(g,\mathcal{M}(g_n,s_n,\alpha_n)) < r.$$

But $\lim_{n\to\infty}\rho(g,\mathcal{M}(g_n,s_n,\alpha_n)) = 0$ for some $r \geq 0$. This is a contraction. Getting $\lim_{n\to\infty}\rho(g_n,s_n) = 0$.

The following proposition represents a first step on the convergence of the (u,u_0)-three-step algorithm (1) for weakly. k-nonexpansive map Γ.

Proposition 3: Let $(G,\ \rho,\mathcal{M})$ be a complete UCPMS with $\emptyset \neq C \subset G$, a map $\Gamma:\ C \to C$ be a weakly $k-$nonexpansive with $Fix(\Gamma) \neq \emptyset$. Let $\{u_n\}_{n=0}^{\infty}$ be a sequence generated in (1), then $\lim_{n\to\infty}\rho(u_n,d)$ exist for $d \in Fix(\Gamma)$.

Proof. Assume that $d \in Fix(\Gamma)$, and Γ is a weakly $k-$nonexpansive map. Using algorithm (1), then

$$\rho(w_n,d) = \rho(\mathcal{M}(u_n,\Gamma u_n,\gamma_n),d) \leq \gamma_n\rho(u_n,d) + (1-\gamma_n)\rho(\Gamma u_n,d),$$
$$\rho(w_n,d) \leq \gamma_n\rho(u_n,d) + (1-\gamma_n)\rho(u_n,d).$$

Since Γ is a weakly $k-$nonexpansive map, get $\rho(w_n,d) \leq \rho(u_n,d)$.

Now, $\rho(v_n, d) = \rho(\mathcal{M}(u_n, \Gamma w_n, \beta_n), d) \leq \beta_n \rho(u_n, d) + (1-\beta_n)\rho(\Gamma w_n, d)$ then

$$\rho(v_n, d) \leq \beta_n \rho(u_n, d) + (1-\beta_n)\rho(\Gamma w_n, d), \leq \beta_n \rho(u_n, d) + (1-\beta_n)\rho(u_n, d).$$

Since Γ is a weakly $k-$nonexpansive map, get $\rho(v_n, d) \leq \rho(u_n, d)$.

And $\rho(u_{n+1}, d) = \rho(\alpha_n a + \alpha_n^* u_n + \alpha_n^{**} v_n, d) \leq \alpha_n \rho(a, d) + \alpha_n^* \rho(u_n, d) + \alpha_n^{**} \rho(v_n, d)$,

$\rho(u_{n+1}, d) \leq \alpha_n \rho(a, d) + \alpha_n^* \rho(u_n, d) + \alpha_n^{**} \rho(\mathcal{M}(u_n, \Gamma w_n, \beta_n), d)$,

by (1), $\rho(u_{n+1}, d) \leq \alpha_n \rho(a, d) + \alpha_n^* \rho(u_n, d) + \alpha_n^{**}[\beta_n \rho(u_n, d) + (1-\beta_n)\rho(\Gamma w_n, d)$,

$\leq \alpha_n \rho(a, d) + \alpha_n^* \rho(u_n, d) + \alpha_n^{**}[\beta_n \rho(u_n, d) + (1-\beta_n)[\gamma_n \rho(u_n, d) + (1-\gamma_n)\rho(\Gamma u_n, d)]]$,

$\rho(u_{n+1}, d) \leq \alpha_n \rho(a, d) + \alpha_n^* \rho(u_n, d) + \alpha_n^{**}\beta_n \rho(u_n, d) + (1-\beta_n)\gamma_n \rho(u_n, d) + (1-\beta_n)(1-\gamma_n)\rho(u_n, d)$,

$\leq \alpha_n \rho(a, d) + \rho(u_n, d)[\alpha_n^* + \alpha_n^{**}\beta_n + \gamma_n(1-\beta_n) + (1-\beta_n)(1-\gamma_n)]$, so

$\rho(u_{n+1}, d) \leq \alpha_n \rho(a, d) + \rho(u_n, d)[\alpha_n^* + \alpha_n^{**}\beta_n + (1-\beta_n)]$,

$\leq \alpha_n \rho(a, d) + \rho(u_n, d)[(1-\beta_n) + \alpha_n^* + \alpha_n^{**}\beta_n]$,

Then $\rho(u_{n+1}, d) \leq \alpha_n \rho(a, d) + (1-\beta_n)\rho(u_n, d) + \alpha_n^* \rho(u_n, d) + \alpha_n^{**}\beta_n \rho(u_n, d)$.

Suppose that $\alpha_n \rho(a, d) + (1-\beta_n)\rho(u_n, d) = \max\{\rho(a, d). \rho(u_n, d)\}$

Hence, $\rho(u_{n+1}, d) \leq \max\{\rho(a, d). \rho(u_n, d)\} + \rho(u_n, d)[\alpha_n^* + \alpha_n^{**}\beta_n]$, since $\{\alpha_n\}, \{\alpha_n^*\}, \{\alpha_n^{**}\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ and $\alpha_n^* + \alpha_n^{**}\beta_n < 1$. Then

$\rho(u_{n+1}, d) \leq \max\{\rho(a, d). \rho(u_n, d)\} + \rho(u_n, d)$.

Take the limit for both sides, obtain $\{\rho(u_n, d)\}_n$ is decreasing in $[0, \infty)$. Getting $\lim_{n\to\infty}\rho(u_n, d)$ exists for all $d \in Fix(\Gamma)$.

Lemma 5: Let $(G, \rho, \mathcal{M})$ be a complete UCPMS with $\emptyset \neq C \subset G$, a map $\Gamma: C \to C$ be a weakly $k-$nonexpansive with $Fix(\Gamma) \neq \emptyset$. Let $\{u_n\}_{n=0}^{\infty}$ be a sequence generated in (1). Then, $\{u_n\}_{n=0}^{\infty}$ be a bounded sequence and $\lim_{n\to\infty}\rho(u_n, \Gamma u_n) = 0$.

Proof. Assume that $Fix(\Gamma) \neq \emptyset$, let $\{u_n\}_{n=0}^{\infty}$ be a sequence generated in (1) with $d \in Fix(\Gamma)$, by using

Proposition 3, then $\lim_{n\to\infty}\rho(u_n, d)$ exists for all $d \in Fix(\Gamma)$. Thus $\{u_n\}_{n=0}^{\infty}$ be a bounded sequence. Suppose $\lim_{n\to\infty}\rho(u_n, d) = r$, from Proposition 3, so $\lim_{n\to\infty}\sup\rho(w_n, d) \leq r$ and $\lim_{n\to\infty}\sup\rho(v_n, d) \leq r$.

Γ is a weakly $k-$nonexpansive map, getting $\rho(\Gamma u_n, d) = \rho(\Gamma u_n, \Gamma d) \leq \rho(u_n, d)$ thus $\lim_{n\to\infty}\sup\rho(\Gamma u_n, d) \leq r$. And $\rho(u_{n+1}, d) \leq \rho(v_n, d) \leq \rho(w_n, d) \leq \rho(u_n, d)$, take $\lim_{n\to\infty}\inf$ for both sides, thus

$$r = \lim_{n\to\infty}\inf\rho(u_{n+1}, d) \leq \lim_{n\to\infty}\inf\rho(v_n, d) \leq \lim_{n\to\infty}\inf\rho(w_n, d) \leq r.$$

Obtain $\lim_{n\to\infty}\rho(v_n, d) = r$.

Again, $r = \lim_{n\to\infty}\inf\rho(v_n, d) \leq \lim_{n\to\infty}\inf\rho(w_n, d) \leq \lim_{n\to\infty}\sup\rho(w_n, d) \leq r$

implying that $\lim_{n\to\infty}\rho(w_n, d) = r$. Get

$r = \lim_{n\to\infty}\rho(w_n, d) = \lim_{n\to\infty}\rho(\mathcal{M}(u_n, \Gamma u_n, \gamma_n), d)$,

$\leq \lim_{n\to\infty}\{\gamma_n \rho(u_n, d) + (1-\gamma_n)\rho(\Gamma u_n, d)\}$,

$r = \lim_{n\to\infty}\rho(w_n, d) \leq \lim_{n\to\infty}\{\gamma_n \rho(u_n, d) + (1-\gamma_n)\rho(u_n, d)\}$, since Γ be a weakly $k-$nonexpansive map, get $\lim_{n\to\infty}\rho(w_n, d) \leq \lim_{n\to\infty}\rho(u_n, d) = r$.

Then

$\lim_{n\to\infty}\rho(\mathcal{M}(u_n, \Gamma u_n, \gamma_n), d) = \lim_{n\to\infty}\{\gamma_n \rho(u_n, d) + (1-\gamma_n)\rho(\Gamma u_n, d)\} = r$ and $\lim_{n\to\infty}\sup\rho(w_n, d) \leq r$, $\lim_{n\to\infty}\sup\rho(v_n, d) \leq r$.

Next, from Theorem 2, yields $\lim_{n\to\infty}\rho(u_n, \Gamma u_n) = 0$.

As in [32], the condition $(\mathcal{H})$ is introduced to use it in the next theorem.

Definition 10: Let (G, ρ) be a PMS, and $\emptyset \neq C \subset G$ be a convex set, the maps $\Gamma: C \to C$, with $Fix(\Gamma) \neq \emptyset$, satisfies condition $(\mathcal{H})$: if a non-decreasing function $\mho: \Re^+ \to \Re^+$ exist as $\mho(0) = 0$, $\mho(a) > 0$ for $0 < a < \infty$ and $\rho(g, \Gamma g)) \geq \mho(\rho(g, Fix(\Gamma))$, for $g \in C$. Where, $\rho(g, Fix(\Gamma)) = \inf\{\rho(g, d), \ d \in Fix(\Gamma)\}$.

Theorem 3: Let $(G, \rho, \mathcal{M})$ be a complete UCPMS with $\emptyset \neq C \subset G$, a map $\Gamma: C \to C$ be a weakly $k-$nonexpansive with $Fix(\Gamma) \neq \emptyset$. Let $\{u_n\}_{n=0}^{\infty}$ be a sequence generated in (1). Then, $\{u_n\}_{n=0}^{\infty}$ converges to an element in $Fix(\Gamma)$. If one holds the following.

- $\lim_{n\to\infty}\inf\rho(u_n, Fix(\Gamma)) = 0$,
- Γ satisfies condition $(\mathcal{H})$.

Proof. Assume that $\lim_{n\to\infty}\inf\rho(u_n, Fix(\Gamma)) = 0$, let $d \in Fix(\Gamma)$ such that $Fix(\Gamma) \neq \emptyset$. Using Lemma 5, then $\rho(u_{n+1}, d) \leq \rho(u_n, d)$ thus $\rho(u_{n+1}, Fix(\Gamma)) \leq \rho(u_n, Fix(\Gamma))$. So, $\{\rho(u_n, Fix(\Gamma))\}_{n=0}^{\infty}$ is decreasing. sequence and bounded below. Obtains $\{\rho(u_n, Fix(\Gamma))\}_{n=0}^{\infty}$ exists and by (1) $\lim_{n\to\infty}\rho(u_n, Fix(\Gamma)) = 0$, Suppose that there exists $\{u_{nk}\}_k \subset \{u_n\}$ with $\{d_k\}$ be a sequence of $Fix(\Gamma)$ then $\rho(u_{nk}, d_k) < \frac{1}{2^k}$ for all k positive number, getting

$$\rho(u_{nk+1}, d_k) < \rho(u_{nk}, d_k) < \frac{1}{2^k}.$$

Then

$$\rho(u_{nk+1}, d_k) < \frac{1}{2^k}.$$

We have $\rho(d_{k+1}, d_k) < \rho(d_{k+1}, u_{nk+1}) + \rho(u_{nk+1}, d_k) - \rho(u_{nk+1}, u_{nk+1})$ get

$\rho(d_{k+1}, d_k) < \frac{1}{2^k}$

By Definition 2, $\rho(u_{nk+1}, u_{nk+1}) \geq 0$, then $\rho(d_{k+1}, d_k) < \frac{1}{2^{k+1}} + \frac{1}{2^k}$. Obtains $\rho(d_{k+1}, d_k) < \frac{1}{2^{k-1}}$, and since $Fix(\Gamma)$ is closed with $\{d_k\}$ be a Cauchy sequence of $Fix(\Gamma)$. Then d_k converge to d^* such that $d^* \in Fix(\Gamma)$.

Also, $\rho(u_{nk}, d^*) < \rho(u_{nk}, d_k) + \rho(d_k, d^*) - \rho(d_k, d_k)$

$\rho(u_{nk}, d^*) < \frac{1}{2^k}$ for all k positive number.

By Definition 2, $\rho(d^*, d^*) \geq 0$, then $\rho(u_{nk}, d^*) < \rho(u_{nk}, d_k) + \rho(d_k, d^*)$ converge to 0 for all k. Obtains u_{nk} converge to d^*

And by Proposition 3, $\lim_{n\to\infty} \rho(u_n, d^*)$ exist, then u_n converge to d^*. This is proof (1).

To Proof (2) Γ satisfies condition $(\mathcal{H})$, by Lemma 5, $\lim_{n\to\infty} \rho(u_n, \Gamma u_n) = 0$, we have

$0 \leq \lim_{n\to\infty} \mho(\rho(u_n, Fix(\Gamma)) \leq \lim_{n\to\infty} \rho(u_n, \Gamma u_n) = 0.$

Since Γ is a weakly $k-$nonexpansive map, getting $\lim_{n\to\infty} \mho(\rho(u_n, Fix(\Gamma)) = 0$, thus $\lim_{n\to\infty} \rho(u_n, Fix(\Gamma)) = 0$. Then, $\{u_n\}_{n=0}^{\infty}$ converges to the fixed point of Γ.

4 APPLICATIONS TO A NON-LINEAR FRACTIONAL DIFFERENTIAL EQUATION

Fractional differential calculus has emerged as a fascinating and productive field of study within science and engineering. Its importance has become apparent due to its numerous applications, including signal processing, fluid flow, diffusive transport, electrical networks, electronics, robotics, and telecommunication, among others. Sometimes, no explicit analytical solution is available; this section focuses on applying a (u, u_0)-three-step algorithm in (1) to an approximate solution of a nonlinear fractional differential equation. In such instances, it becomes crucial to determine an approximate solution to address the challenges posed by these complex equations. For more details, see [33]-[35]. A particular nonlinear fractional differential equation may have no analytic solution. In this case, we need to find an approximate solution. In this section, we will estimate an approximate solution of a nonlinear fractional differential equation using the iterative (u, u_0)-three-step algorithm (1).

Consider the complete UCPMS. $(G, \mathcal{M}, \rho)$ where $C = c[0,1]$ and $\rho_\infty(s, g) = \|s - g\|_\infty + \|s\|_\infty + \|g\|_\infty$ and $\|s\|_\infty = \max_{r\in(0,1]} |s(r)|$

Theorem 4. Let $\{u_n\}_{n=0}^{\infty}$ be a sequence in C associated with the operator $\Gamma: C \to C$ defined by $(\Gamma g)(a) = \int_0^1 D(a,t) h(t, g(t)) dt \ \forall a \in [0,1], g \in C$, where

$$D(a,t) = \begin{Bmatrix} \frac{1}{f(g)}[a(1-t)^{g-1} - (a-t)^{g-1}, \text{if } t \in [0,a] \\ \frac{1}{f(g)} a(1-t)^{g-1}, \text{if } t \in [1,a] \end{Bmatrix},$$

the function $h: [0,1] \times \Re \to \Re$ is continuous. Assume that $h(t, g(t))$ is a Lipschitz function with respect to the second variable such that $|h(a, g_1) - h(a, g_2)| \leq |g_1 - g_2| \ \forall a \in [0,1], g_1, g_2 \in C$. Then, the sequence $\{u_n\}_{n=0}^{\infty}$ in (1) converges to a solution of the

$$B^n g(a) + h(a, g(a)) = 0, \ n \in (1,2), \qquad (2)$$

and $B^n g(a)$ denotes the fractional differential of order g.

Proof. As we know, Green's function's fractional difference equation (2) solution is

$g(a) = \int_0^1 D(a,t) h(t, g(t)) dt \ \forall a \in [0,1], g \in C.$

Getting

$$\begin{aligned}
&|(\Gamma g_1)(a) - (\Gamma g_2)(a)| \\
&= \left| \int_0^1 D(a,t) h(t, g_1(t)) dt - \int_0^1 D(a,t) h(t, g_2(t)) dt \right| \\
&= \left| \int_0^1 D(a,t) [h(t, g_1(t)) dt - h(t, g_2(t))] dt \right| \\
&\leq \int_0^1 D(a,t) \ |h(t, g_1(t)) dt - h(t, g_2(t))| dt \\
&\leq \int_0^1 D(a,t) \ |g_1(t) - g_2(t)| dt
\end{aligned}$$

$\leq |g_1 - g_2| \ .\sup \int_0^1 G(a,t) \ dt$ then

$|(\Gamma g_1)(a) - (\Gamma g_2)(a)| \leq |g_1 - g_2|$, for all $g_1, g_2 \in C$

Hence, Γ is a weakly $k-$nonexpansive map and non-expansive map with the convergence of the (u, u_0)-three-step algorithm (1) to the solution (2).

Now, in the following, using the fractional differential of order g.

$$B^n g(a) + a^2(a+2) = 0, \ n \in (1,2), \qquad (3)$$

With condition $g(0) = 0.5, g(1) = 1$.

Example 9: Let $\Gamma: C \to C$ where $C = c[0,1] = \{ f: [0,1] \to \Re\}$ be a continuous function, with $\rho: C \times C \to \Re^+, (C, \rho)$ be a PMS. The map Γ defined by

$\Gamma g(a) = \frac{1}{\Gamma(g)} \int_0^1 [a(1-t)^{g-1} - (a-t)^{g-1}].t^2(t+2)dt + \frac{a}{\Gamma(g)} \int_0^1 (1-t)^{g-1}.t^2(t+2)dt \ , t \in [0,a],$

Used $g = \frac{3}{2}$ such that $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$ with initial value $u_1(g) = g^2(1-g)^2$ and $g \in [0,1]$, $\beta_n = 0.1, \gamma_n = 0.7$ with $\cap_i^4 Fix(\Gamma_i) \neq \emptyset$, for all $n \in N$. Then, $\{u_n\}_{n=0}^{\infty}$ converges to the solution in (3). Using MATLAB, See Figure 2, Table 4.

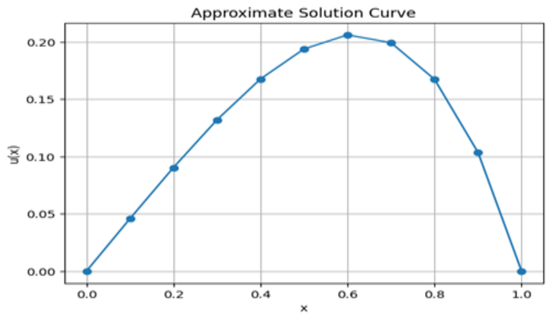

Figure 2: Application of the (u, u_0)-three-step algorithm (1) in Example 9.

Now, in the following, using the condition such that the function $\varphi: [0,1] \times \Re \to \Re$ is continuous,

$$|\varphi(a, g_1(a)) - \varphi(a, g_2(a))| \leq g|g_1(t) - g_2(t)|, \quad (4)$$

$\forall a \in [0,1], g \in (1,2)$.

Theorem 5: Let $\{u_n\}_{n=0}^{\infty}$ be a sequence generated in (1) for the map Γ: $C \to C$ define by $(\Gamma g)(a) = \int_0^a D(a-s)\varphi(s, g(s))ds \;\; \forall a \in [0,1], g \in C$. Moreover satisfies (4). Then,

- Γ is a weakly k −nonexpansive map for $k = 0$ with $Fix(\Gamma) \neq \emptyset$.
- the sequence $\{u_n\}_{n=0}^{\infty}$ in (1) converges to a solution of the nonlinear fractional differential of order g such that

$$B^n g(a) + B^m g(a) + \varphi(a, g(a)) = 0, \quad (5)$$

$\forall a \in [0,1], m, n \in (0,1)$, and $m < n$

Proof. As we know, Green's functions with (5) getting $D(t) = t^{n-1}L_{n-m,n}(-t^{n-m})$ such that $L_{m,n}(z)$ denotes the Mittag-Leffler function, see [36], we have

$$|(\Gamma g_1)(a) - (\Gamma g_2)(a)| = \left| \int_0^a D(a-s)\varphi(s, g_1(s))ds - \int_0^a D(a-s)\varphi(s, g_2(s))ds \right|$$

since $a \in [0,1]$ then

$$= \left| \int_0^1 D(a, s)[\varphi(s, g_1(s)) - \varphi(s, g_2(s))] \right| ds \text{ by (4)}$$

get

$$\leq \int_0^1 |D(a, s)|.g|g_1(s) - g_2(s)| \; ds$$

$$\leq g|g_1 - g_2|.\sup \int_0^1 |D(a, s)| \; ds$$

$$\forall a \in [0,1], \; g_1, g_2 \in C. \quad (6)$$

Now, see [36] the properties Mittag Leffler function, hence $D(t) = t^{n-1}L_{n-m,n}(-t^{n-m}) \leq t^{n-1}, \forall t \in [0,1]$

Then, $\int_0^1 |D(a, s)| \; dt = \frac{a^g}{g}$.

Take two sides of supermoms, so

$$\sup \int_0^1 |D(a, s)| \; ds \leq \sup \frac{a^g}{g} = \frac{1}{g} \quad (7)$$

Put (7) in (6), get

$|(\Gamma g_1)(a) - (\Gamma g_2)(a)| \leq g|g_1 - g_2|. \; \frac{1}{g} \leq |g_1 - g_2| \; \forall a \in [0,1], g_1, g_2 \in C$.

That is proof (1).

Hence, the convergence of the (u, u_0)-three-step algorithm (1) to a fixed point of Γ. And it converges to a solution of the nonlinear fractional differential of order g by $(\Gamma g)(a) = \int_0^a D(a - s)\varphi(s, g(s))ds \;\; \forall a \in [0,1], g \in C$ to solution (5).

5 CONCLUSIONS

This paper primarily studies a class of non-expansive maps, specifically weak nonexpansive maps of type (μ, k), which differ from other types of maps previously studied. We used a novel a (u, u_0)-three-step algorithm in (1) to approximate the fixed points of these maps, demonstrating its stability and some more generalized convergence results than previous findings.

Table 4: Application of the fractional differential in Example 9.

no.	(g)	(u_2)	(u_4)	(u_7)	(u_9)	(u_{10})
1	0.0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
2	0.1	0.04571783	0.04571783	0.04571783	0.04571783	0.04571783
3	0.2	0.09035545	0.09035545	0.09035545	0.09035545	0.09035545
4	0.3	0.13191918	0.13191918	0.13191918	0.13191918	0.13191918
5	0.4	0.16759217	0.16759217	0.16759217	0.16759217	0.16759217
6	0.5	0.19377061	0.19377061	0.19377061	0.19377061	0.19377061
7	0.6	0.20606421	0.20606421	0.20606421	0.20606421	0.20606421
8	0.7	0.19928410	0.19928410	0.19928410	0.19928410	0.19928410
9	0.8	0.16742508	0.16742508	0.16742508	0.16742508	0.16742508
10	0.9	0.10364401	0.10364401	0.10364401	0.10364401	0.10364401
11	1.0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000

We have proved a stability result for the (u, u_0)-three-step algorithm, which is more general than other previous stability results. Furthermore, we presented numerical examples of nonexpansive maps and compared the convergence behavior of different iterative algorithms in light of these examples. We also showed that μ −type nonexpansive maps and generalized μ −type nonexpansive maps are independent. In addition, we applied existing fixed-point theorems to nonlinear fractional differential equations.

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