The Influence of Material Properties, the Geometric Shape of the Section and Temperature on the Stressed State of the Working Bodies

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Abstract: In the work on stretching a matrix with a rigid inclusion - an ideal-plastic cylindrical waveguide, a technique was developed for determining the thickness of the boundary layer, which can be generalized and extended to all problems. This research investigates the singular problem of stretching a matrix with a rigid inclusion (ideal-plastic cylindrical waveguide) and proposes a generalized methodology for determining the thickness of the boundary layer under varying loading and geometric conditions. The methodology integrates analytical derivations with computational algorithms, enabling accurate evaluation of boundary layer thickness. The approach is designed for seamless implementation in modern computational mechanics platforms, such as finite element analysis (FEA) [20], digital twin systems, and automated engineering design workflows. Numerical simulations validate the method's accuracy, while potential industrial applications include optimization of composite material reinforcement strategies in fibrous, woven, and laminated systems [23]–[27] and pipeline structural integrity assessment. The work also identifies pathways for experimental validation and integration into digital modelling environments, bridging classical mechanics with contemporary IT, automation, and computational engineering trends.

1 INTRODUCTION

In this paper, we consider the influence of edge effects arising when pipes and pipelines of a finite length are immersed in another medium. It is assumed that one end or both ends of a pipe or pipeline (hereinafter inclusions) having a finite length is immersed in some foreign medium (hereinafter matrix) in the well and boundary value problems are considered about the current stress state of pipes, pipelines and rods [1]–[3].

Furthermore, the proposed approach is adaptable for integration into automated structural analysis pipelines, digital modeling platforms, and Industry 4.0 environments. By leveraging CAD/CAE integration and digital twins, the model can be applied for real-time structural health monitoring and predictive maintenance in industrial systems [4].

In the following, we continue to study the boundary value problems considered in the previous work, taking into account the following additions [5], [6]:

a) changes in the shape of the cross-section of the pipe - inclusions;
b) changes in the properties of the inclusion material and matrix - rock;
c) the influence of the thermal effects of the matrix.

2 RESEARCH METHODS

$\lambda = \mathrm{E}^{+}/\mathrm{E}^{-} \gg 1, \delta/e \ll 1 \mathrm{E}^{+}\ \mathrm{E}^{-} \tau_{12}$Consider the singular problem of stretching an elastic plane along a thin rigid inclusion of high thermal conductivity and finite width 2l, coupled to the rest of the body by the matrix at all points of the lateral surface, i.e. conjugate to the matrix [2]. In this case, when the Young's modulus of the inclusion 2l is much greater than its thickness , i.e. , where and -are Young's moduli of the inclusion and the matrix, respectively, near the inclusion in the matrix there is a boundary layer in which only the shear stress is significant, and all other stresss are negligible [7].

Let us assume that the inclusion is located along the segment $x_2 = 0, |x_1| < l$ in the matrix stretched at infinity by the stress $\tau_{11} = P_{11}$ along the inclusion (Fig. 1).

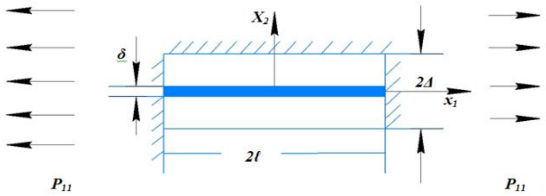

Figure 1: Schematic of a boundary layer in an elastic plane with solid inclusions.

The general solution of the boundary layer equations in the matrix is as follows:

$$u_1 = V + \frac{\tau_{12}}{\mu^-} x_2, \quad \tau_{12} = \tau = const. \qquad (1)$$

Here υ and τ are arbitrary functions x_1, μ is the matrix shift modulus.

τ_{12} As can be observed, the stress is constant in the cross-section of the boundary layer and is equal to the shear stress (x1) on the banks of the inclusion, and the displacement u1 is linear (function) x2.

At $x_2 = 0$, the boundary layer is associated with an inclusion. Due to rigid adhesion to the matrix, the displacement u_1 of the matrix at this point should be the same as the displacement u_1 of the inclusion.

Therefore, $V = V(x_1)$ in formulas (1) is the displacement of the inclusion at the corresponding points.

At the boundary of the boundary layer at $x_2 = \pm A$, the field in the boundary layer (1) must be "stitched" with the external, unperturbed field $u_1 = x_1 \rho_{11} E^-$ as a condition for "stitching" it is natural to require the equality of the longitudinal deformation $\varepsilon_{11} = \partial u_1 / \partial x_1$ and another field at the boundary of the boundary layer at $x_2 = \pm\Delta$.. As a result, based on (1), we obtain

$$V'(x_1) + \frac{\Delta}{\mu-} \tau'(x_1) = \frac{P_{11}}{E^-}. \qquad (2)$$

Together with the equations

$$\tau = E^+ V(x_1), \quad \tau = -1/2\partial\dot{\tau}(x_1). \qquad (3)$$

we obtain a closed system of ordinary differential equations for the functions V, τ and τ.

The solution to this system gives the desired solution to the problem posed in the boundary layer approximation [8].

The structure of the field is shown in Figure 1: the inside of the rectangle $|x_1| \le l, |x_2| \le \Delta$ is the boundary layer, the outside of the same rectangle is the unperturbed field.

First, we exclude $V'(x_1)$ from system (2), (3), we find

$$\tau = E^+ \left(\frac{P_{11}}{E^-} - \frac{\Delta}{\mu^-} \tau'(x_1) \right) \qquad \tau = -1/2\partial\dot{\tau}(x_1) (4)$$

Now we substitute τ from the second equation in (4) into the first, we obtain

$$\frac{\tau(x_1)}{E^+} = \frac{P_{11}}{E^-} + \frac{\delta\Delta}{2\mu^-} \tau''(x_1). \qquad (5)$$

Here, the thickness δ is assumed to be independent of x_1

The general solution of this second-order differential equation has the form:

$$\tau = C_1 l^{\gamma x_1} + C_2 l^{-\gamma x_1} + \frac{P_{11}}{\Delta\delta\gamma^2 (1+\nu^-)}. \qquad (6)$$

where

$$\gamma = \sqrt{\frac{2\mu^-}{\Delta\delta E^-}}. \qquad (7)$$

Here C_1 and C_2 are arbitrary constants.

3 RESULTS AND DISCUSSIONS

The inclusion can be subjected to direct external loads applied to it. Consider the following four options loaded (Fig. 2).

From an applied perspective, this methodology can be embedded into computational tools for automated pipeline design, structural optimization, and simulation-based engineering. The model's adaptability makes it suitable for software-based material testing and rapid prototyping in manufacturing [9].

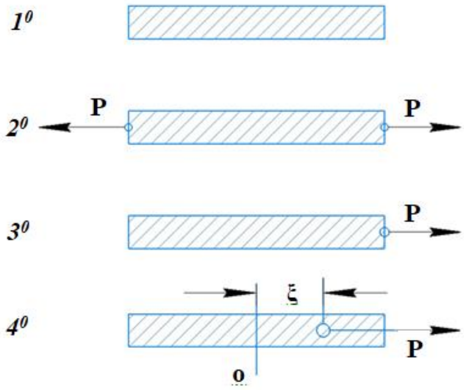

Figure 2: Different loading states of the inclusion.

1) The inclusion is free from external loads. In this case, the free ends of the inclusion can be considered unfinished, so that the stress at the ends is equal to zero

$$\text{at } x_1 = \pm l \quad \tau = 0. \tag{8}$$

2) External forces are applied to both ends of the inclusion. In this case, we have

$$\text{at } x_1 = \pm l \quad \tau = \frac{P}{\delta}. \tag{9}$$

where p is the force per unit of thickness (normal to Fig. 2).

3) Force P is applied to only one end.

In this case, the following boundary conditions will be

$$\text{at } x_1 = -l \quad \tau = 0 \quad \text{at } x_1 = \pm l \quad \tau = \frac{P}{\delta}. \tag{10}$$

4) The force P is applied at some point $x_1 = \xi$ of the inclusion. In this case, the solution has a discontinuity at the point $x_1 = \xi$; find

$$\text{at } x_1 = \pm l \quad \tau = 0$$

$$\text{at } x_1 = \xi \pm 0 \quad \big(\tau(\xi + 0) - \tau(\xi - c)\big) = \frac{P}{\delta}. \tag{11}$$

The linear superposition of these four cases allows us to study the general case of an arbitrary external load on the inclusion [10].

We present the values of the constants C_1 and C_2 in the general solution (6) for the first three cases, obtained as a result of simple calculations:

1) The inclusion is free from external loads; only the external p11 field in the matrix acts

$$C_1 = C_2 = \frac{-P_{11}}{2\Delta\delta\gamma^2(1+\nu^-)ch\gamma e}. \tag{12}$$

2) Tensile forces P are applied to both ends of the inclusion, there is no external field in the matrix (p_{11}=0):

$$C_1 = C_2 = \frac{P}{2\delta ch\gamma e}. \tag{13}$$

3) Force P is applied only to the right end of the inclusion, there is no external field in the matrix (p11=0):

$$C_1 = \frac{P}{4\delta}\left(\frac{C_1}{ch\gamma l} + \frac{J_1}{jh\gamma l}\right) \qquad C_2 = \frac{P}{4\delta}\left(\frac{1}{ch\gamma l} + \frac{1}{jh\gamma l}\right) \tag{14}$$

In the case of the action of the internal force P at the point $x_1 = \xi$, the solution consists of two analytical parts (piecewise analytical solution):

$$\tau = \begin{cases} V_1 + \dfrac{1}{\gamma E^+}\left(C_1 e^{\gamma x_1} - C_2 e^{-\gamma x_1}\right) \bullet \mathrm{p}\,\text{Џ} - l < x_1 < \xi \\ V_2 + \dfrac{1}{\gamma E^+}\left(D_1 e^{\gamma x_1} - D_2 e^{-\gamma x_1}\right) \bullet \mathrm{p}\,\text{Џ}\, \xi < x_1 < l \end{cases} \tag{15}$$

Here V1 and V2 are some constants; only one of them can be set equal to zero, since it is determined by the displacement of the body as a whole. Therefore, condition (15) does not provide an additional condition for determining the stress field in the inclusion.

Note that the equilibrium condition for the entire inclusion as a whole must always be satisfied [11]

$$2\int_{-l}^{+l} \tau(x_1)dx_1 = P \tag{16}$$

It is straightforward to verify that in all previously considered cases 1, 2, 3, the equilibrium condition as a whole is satisfied automatically.

Using solution (14) and (15), we present the following convenient formulas:

$$\begin{cases} C_1 = \dfrac{1}{4}\tau(\xi - 0)\left(\dfrac{1}{jh\gamma(l+\xi)} + \dfrac{1}{ch\gamma(l+\xi)}\right) \\ C_2 = \dfrac{1}{4}\tau(\xi - 0)\left(\dfrac{1}{ch\gamma(l+\xi)} - \dfrac{1}{\xi h\gamma(l+\xi)}\right) \end{cases} \tag{17}$$

$$\begin{cases} D_1 = \frac{1}{4}\tau(\xi - 0)\left(\frac{1}{ch\gamma(l-\xi)} - \frac{1}{\xi h\gamma(l-\xi)}\right) \\ D_2 = \frac{1}{4}\tau(\xi - 0)\left(\frac{1}{ch\gamma(l-\xi)} + \frac{1}{\xi h\gamma(l-\xi)}\right) \end{cases} \quad (18)$$

In the considered case of the action of an internal force, on the basis of (2), (3) and (18), one should additionally assume that the function $\tau(x_1)$ is continuous at the point $x_1 = \xi$ of application of an external force P, i.e.

$$\tau(\xi - 0) = \tau(\xi + 0). \quad (19)$$

Using the second relation (3) and (15), from this we find:

$$\tau'(\xi - 0) = \tau'(\xi + 0). \quad (20)$$

and

$$C_1' l^{\gamma\xi} - C_2' l^{-\gamma\xi} = D_1\ l^{\gamma\xi} - D_2\ l^{-\gamma\xi}$$

On the other hand, conditions (11) based on (15) give

$$\left(D_1 l^{\gamma\xi} + D_2 l^{-\gamma\xi}\right) - \left(C_1' l^{\gamma\xi} + C_{22}' l^{-\gamma\xi}\right) = \frac{P}{\delta}$$

$$C_1' l^{\gamma\xi} + C_2' l^{-\gamma\xi} = 0$$

$$D_1\ l^{\gamma\xi} + D_2\ l^{-\gamma\xi} = 0 \quad (21)$$

The solution of system (20) and (21), consisting of four linear equations, can be written in the following final form:

4. Force P is applied at the internal point $x_1 = \xi$ of the inclusion, there is no external field in the matrix ($\rho_{11} = 0$)

$$C_1^{(\xi)?} = \frac{P}{2\delta}\left(e^{\gamma\xi}\frac{e^{2\gamma l} - e^{-2\gamma\xi}}{1 - e^{4\gamma l}} - e^{-\gamma\xi}\right) \quad (22)$$

$$D_1^{(\xi)?} = \frac{P}{2\delta} \cdot e^{\gamma\xi} \cdot \frac{e^{2\gamma l} - e^{-2\gamma\xi}}{1 - e^{4\gamma l}}$$

$$C_1^{(\xi)?} = \frac{P}{2\delta} \cdot e^{-2\gamma\xi}\left(e^{-\gamma\xi} - e^{\gamma\xi}\frac{e^{2\gamma l} - e^{-2\gamma\xi}}{1 - e^{4\gamma l}}\right)$$

$$D_1^{(\xi)?} = -\frac{P}{2\delta} \cdot e^{\gamma(2l+\xi)} \cdot \frac{e^{2\gamma l} - e^{-2\gamma\xi}}{1 - e^{4\gamma l}}$$

$$\tau = C_1' e^{\gamma x_1} + C_2' e^{-\gamma x_1} \text{ at } \quad -l < x_1 < \xi$$

$$\tau = D_1\ e^{\gamma x_1} + D_2\ e^{-\gamma x_1} \text{ at } \quad \xi < x_1 < l$$

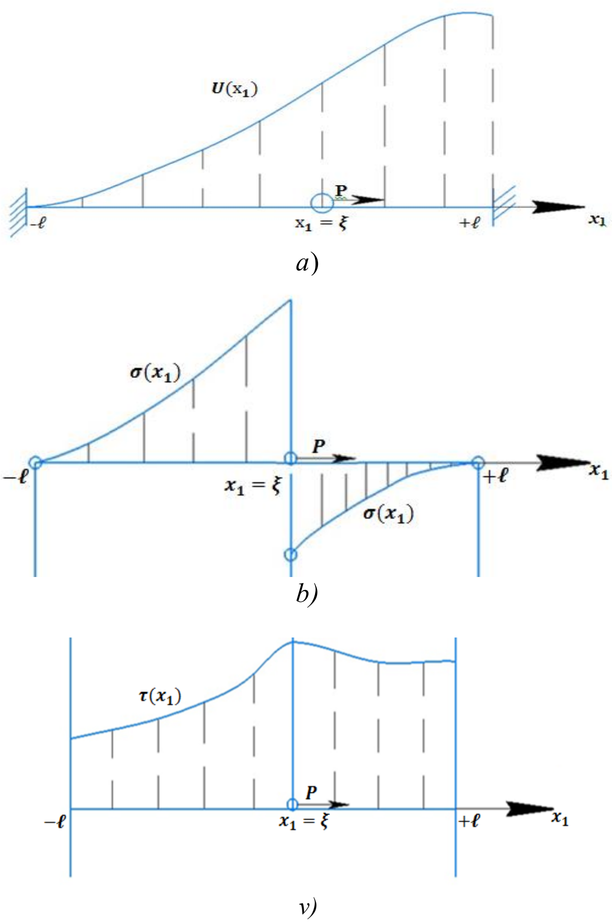

Figure 3: Qualitative change in the stress distribution in the inclusion and matrix.

The general qualitative picture of the behavior of the functions $V(x_1)$, $\tau(x_1)$, and $\tau(x_1)$ in this case, according to (22), is shown in Figure 3. the functions $V(x_1)$ and $\tau(x_1)$ are continuous everywhere, but have a break at the point $x_1 = \xi$ (jump of the derivative). The function $\tau(x_1)$ has a discontinuity of the first kind at the point $x_1 = \xi$.

Solution (22) allows solving the problem of an arbitrarily distributed external load in the inclusion $\rho = \rho(x_1)$. For this, it is necessary to replace p by $p\, d\xi$ in formulas (22) and integrate over ξ from $-l$ to l. We get:

$$\tau = \frac{1}{2\delta}\int_{-1}^{+1} p(\xi)\left(e^{\gamma x_1}\left(e^{\gamma\xi}\frac{e^{2\gamma l}-e^{-2\gamma\xi}}{1-e^{4\gamma l}}-e^{-\gamma\xi}\right)+\right.$$
$$\left.+e^{-\gamma x_1}\cdot e^{-2\gamma\xi}\left(e^{-\gamma\xi}-e^{\gamma\xi}\frac{e^{2\gamma l}-e^{-2\gamma\xi}}{1-e^{4\gamma l}}\right)\right)d\xi$$
$$(-l<x_1<\xi) \qquad (23)$$

$$\tau = \frac{1}{2\delta}\int_{-1}^{+1} p(\xi)\left(e^{\gamma(x_1+\xi)}\cdot\frac{e^{2\gamma l}-e^{-2\gamma\xi}}{1-e^{4\gamma l}}-\right.$$
$$\left.-e^{\gamma(2l-x_1+\xi)x_1}\cdot\frac{e^{2\gamma l}-e^{-2\gamma\xi}}{1-e^{4\gamma l}}\right)d\xi$$
$$(\xi < x_1 < +l)$$

Using formulas (6), (12) - (14), we study the stresses in the connection for the following most interesting cases [12]-[14].

1) The inclusion is free from external loads; the matrix is stretched by stress p11 in an unperturbed field.

In this case:

$$\tau = \frac{P_{11}}{\Delta\delta\gamma^2\left(1+\upsilon^-\right)}\cdot\left(1-\frac{ch\gamma x_1}{ch\gamma l}\right) \qquad (24)$$

$$(|x_1|<l)$$

$$\tau = \frac{P_{11}}{\Delta 2\gamma\left(1+\upsilon^-\right)\Delta}\cdot\frac{jh\gamma x_1}{ch\gamma l}.$$

Diagrams of stresses $\tau(x_1)$ and $\tau(x_1)$ in this case are shown in Figure 4.

The greatest value of the stress in the inclusion = max is achieved in its middle at x1=0.

$$\tau_{\max} = \frac{P_{11}}{\Delta\delta\gamma^2\left(1+\upsilon^-\right)}\cdot\left(1-\frac{1}{ch\gamma l}\right) \qquad (25)$$

The tangential stress $\tau(x_1)$ at this point vanishes. Using (7) we transform the divisor value

$$\Delta\delta\gamma^2\left(1+\upsilon^-\right)=\frac{E^-}{E^+}=\frac{1}{\lambda} \qquad (26)$$

Therefore, we have (λ>>I):

$$\tau = \lambda P_{11}\cdot\left(1-\frac{ch\gamma x_1}{ch\gamma l}\right); \qquad \tau_{\max} = \lambda P_{11}\cdot\left(1-\frac{1}{ch\gamma l}\right) \qquad (27)$$

Let us dwell in more detail on two limiting cases:

- very wide inclusion when γl>>I

$$\tau_{\max} = \lambda p_{11} \qquad (28)$$

- rigid inclusion (ideal-hard case) when l<<I

$$\tau_{\max} = \frac{1}{2}\lambda p_{11}\gamma^2 l^2 = \frac{l^2 P_{11}}{2\Delta\delta\left(1+\upsilon^-\right)} \qquad (29)$$

As can be seen, the wider and stiffer the thin inclusion, the greater the stresses are concentrated in it; in this case, in the disturbed zone (boundary layer), the matrix is completely freed from normal stresses. This reinforcement effect is practically used in composite materials [13]. The effectiveness of such reinforcement can be further influenced by the properties of nanoscale fillers, as demonstrated in studies of composites with nickel [15] and iron [16] nanoparticles. The mechanical behavior of such composites is also governed by fiber and matrix interactions described in the classical framework of composite materials science [25], where the inclusion of nanoparticles significantly altered the thermal and structural characteristics of the base material. Stresses in the inclusion can be thousands of times higher than matrix stresses [17]-[21].

However, the possible effect of the inclusion width is limited by the tensile strength of the inclusion [18], [19], since an inclusion that is too wide breaks in half in the middle. Let us find the maximum width of the inclusion l=lmax, at which in its middle, at the point x1=0, the limiting tensile strength of the inclusion is achieved =B; using (29), we find

$$l_{\max} = \frac{1}{\gamma}arch\left(1-\frac{\tau_B}{\lambda P_{11}}\right)^{-1} \qquad (30)$$

$$0<l<l_{max}$$

Thus, the greater the Young's modulus of the inclusion and its tensile strength, the greater the reinforcing effect.

2) Tensile forces p are applied to both ends of the inclusion, there is no external field in the matrix ($\rho_{11} = 0$).

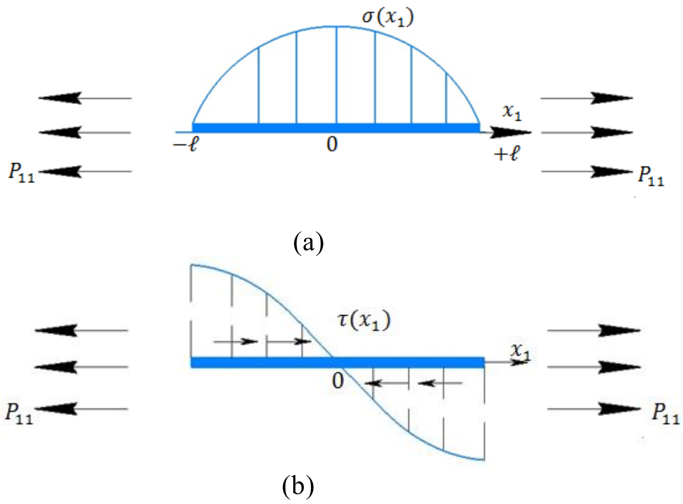

Figure 4: Stress distribution in the inclusion when the matrix is stretched

P

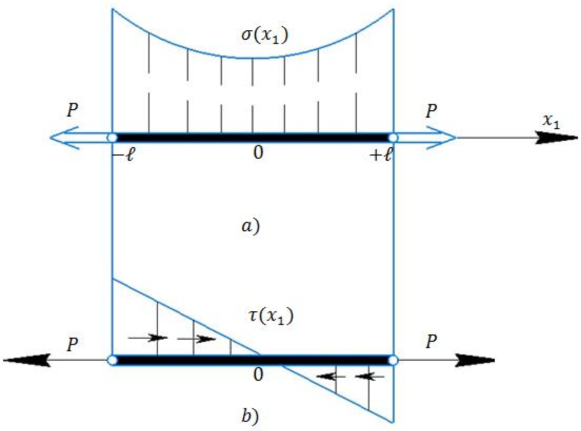

Figure 5: Stress distribution when force P is applied to the ends of the inclusion

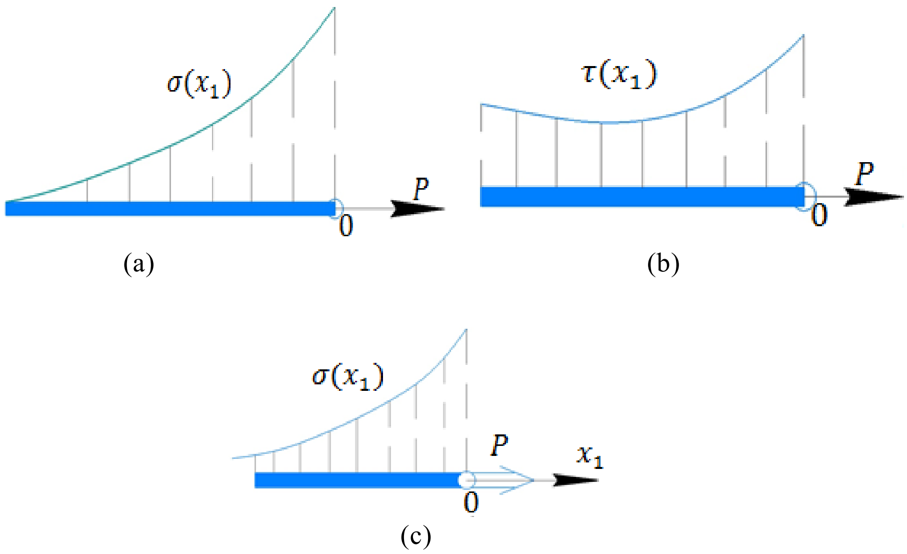

Figure 6: Stress distribution when a force P is applied to one end of the inclusion

In this case, we have

$$\tau = \frac{P_{11}}{\delta} \cdot \frac{ch\gamma x_1}{ch\gamma l} \quad (31)$$

$$\tau = \frac{-P\gamma}{2} \cdot \frac{jh\gamma x_1}{ch\gamma l} \qquad |x_1|<l$$

The stress diagrams τ and τ in this case are shown in Figure 5.

The greatest value of stress τ is achieved at the ends of the inclusion, therefore, the possible magnitude of the force p is limited by the strength of the inclusion at break $p \leq \delta\tau_B$.

3) Force p is applied only to the right end of the inclusion; there is no external field in the matrix (p_{11}=0).

In this case, we have

$$\tau = \frac{P}{\delta} \cdot \left(\frac{ch\gamma x_1}{ch\gamma l} + \frac{jh\gamma x_1}{jh\gamma l} \right) \quad (32)$$

$$\tau = \frac{-P\gamma}{\delta} \cdot \left(\frac{jh\gamma x_1}{ch\gamma l} + \frac{ch\gamma x_1}{jh\gamma l} \right) \qquad (|x_1|<l)$$

Diagrams of stresses τ and τ in the inclusion are shown in Figure 6.

Consider one interesting limiting case of this problem, when the width of the inclusion tends to infinity ($\gamma l \rightarrow \infty$).

First, in formula (32), we transfer the origin of coordinates to the right end (Fig. 6), we obtain

$$\tau = \frac{P}{2\delta} \cdot \left(\frac{ch\gamma(l+x_1)}{ch\gamma l} + \frac{jh\gamma(l+x_1)}{jh\gamma l} \right) \quad (x_1<0) \quad (33)$$

For a very wide inclusion, when ($\gamma l \rightarrow \infty$), from this we find

$$\tau = \frac{P}{\delta} e^{\gamma x_1}; \qquad \tau = -\frac{1}{2} p\gamma e^{\gamma x_1} \quad (34)$$

$$(\gamma l >> 1, \qquad x_1<0)$$

As can be seen, the perturbation from the external force p decays at a distance of the order of ($3/\gamma$) from the end of the inclusion [22]-[28].

4 CONCLUSIONS

The developed methodology not only offers a robust analytical framework for determining boundary layer thickness in elastic-plastic systems with rigid inclusions but also demonstrates strong potential for integration with modern computational mechanics platforms and Industry 4.0 technologies. The approach enables predictive modeling, optimization, and real-time monitoring when combined with finite element analysis, digital twins, and automated design systems. In industrial applications, it can be leveraged for structural integrity assessment of pipelines, reinforcement of composite materials, and optimization of engineering components under complex loading conditions. Future work should focus on experimental validation, calibration of the model with real-world data, and deployment within automated design workflows to enhance its practical impact. [24]–[26]

To summarize the scope of the analysis: the singular problem addresses stretching of an elastic plane along a rigid inclusion of high thermal conductivity and finite length, conjugate to the matrix at all points of the lateral surface. A general technique for solving such problems was developed, yielding qualitative and quantitative estimates of the stress states of the inclusion and matrix under the following loading scenarios:

- switching on is free from external loads;
- external loads are applied to both ends of the connection;
- an external force is applied only to one of the ends of the inclusion;
- external force is applied to an arbitrary switching point.

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