Spectral Solution of Fractional Delay Equations via Fermat Polynomials

Abdullah Hussein[1] and Ali Khalaf Hussain Al-Hachami[2]
Department of Mathematics, College of Education for Pure Science, Wasit University, 52001 Wasit, Iraq
std.2024205.a.khalee@uowasit.edu.iq, alhachamia@uowasit.edu.iq

Keywords: Fermat Polynomials, Fractional a Delay Equation, Fermat Operational Matrix, Galerkin Method.

Abstract: This paper presents an efficient and accurate numerical approach for solving fractional differential equations (FDEs), particularly those with delay terms, using a truncated series of Fermat polynomials. The developed method converts the fractional differential equation and its associated initial conditions into a system of algebraic equations by employing the Galerkin spectral technique combined with the operational matrix of fractional-order derivatives in the Caputo sense. Fermat polynomials are adopted as basis functions due to their advantageous analytical properties and recursive structure, making them suitable for constructing spectral approximations. The operational matrix formulation significantly reduces computational complexity and facilitates efficient implementation. The approximate solution is obtained with high accuracy using only a limited number of basis functions, ensuring rapid convergence. To demonstrate the effectiveness of this approach, several numerical examples, including fractional delay differential equations, are investigated. The results show excellent agreement with exact solutions, even for non-polynomial problems, and comparisons with existing methods confirm the superiority, stability, and computational efficiency of the proposed technique.

1 INTRODUCTION

The capacity of fractional differential equations (FDEs) to simulate complicated processes that display memory effects and cumulative behaviors in a variety of disciplines, including as physics, engineering, biological systems, and control theory, has drawn a lot of interest in recent years. A particularly important category within this field is fractional differential equations with delay, when a system's development is not only dependent on its present condition, but also on its previous states. The presence of fractional derivatives with delay increases the complexity of these equations, making obtaining analytical solutions difficult, and therefore numerical methods are often used. Spectral methods are very effective numerical techniques for solving FDEs, offering quick convergence and great accuracy in comparison to conventional techniques. These methods approximate the solution using a series of basis functions, with the expansion coefficients determined by spectroscopic techniques such as (Collocation), (Tau), and (Galerkin) methods. Among them, Galerkin's method is widely used, approximating the solution using a set of base functions while imposing the condition of orthogonality to the remainder with respect to the chosen domain of functions. For more details about spectral methods, you can refer to [1]-[4]. Fermat polynomials have also received attention in numerical analysis because of their mathematical properties and iterative relationships, which make them suitable for constructing spectral approximations and operational matrices of fractional derivatives. It is a special case of generalized Fibonacci polynomials of the type (p, q) [5]. Several studies [6]-[8] have shown that the use of Fermat polynomials within spectral methods leads to accurate and efficient numerical solutions to a variety of fractional equations. In order to solve FDEs with delay, we provide a spectral approach based on Fermat polynomials in this paper. Fermat polynomials in a finite series are used to approximate the answer, a proposed Galerkin-type formulation is employed to convert the equation and its associated conditions into a set of algebraic equations that can be used to compute the expansion coefficients. Several numerical examples were given to illustrate the effectiveness of the suggested approach, comparing the results with other existing methods, showing high accuracy and rapid convergence using relatively few basis functions. The structure of this research is as the paper is organized as follows. First, the basic concepts of fractional differentiation are reviewed. Next, the definitions of Fermat polynomials are introduced. Then, the operational matrices for fractional derivatives in the Caputo sense are developed. After that, the proposed numerical method is described, followed by various numerical examples. Finally, concluding remarks are presented.

2 PRELIMINARIES

The principles of fractional calculus theory, which will be applied in this investigation, are reviewed in this part.

Definition 1: [9] The following is the definition of the Riemann-Liouville (R._L.) fractional integrals of order β operator I^{β}:

$$I^{\beta}\varphi(x)=\begin{cases}\frac{1}{\Gamma(\beta)}\int_a^x\frac{\varphi(t)}{(x-t)^{1-\beta}}\,dt, & x>a,\\ \varphi(x) & \beta=0\end{cases}. \quad (1)$$

These are the characteristics of the (R._L.) fractional integral [10]:

1) if $\beta > 0$ & $\delta > 0$ $I^{\beta}I^{\delta}\varphi(x)=I^{\beta+\delta}\varphi(x)$;
2) let $n,m\in R$: $I^{\beta}\big(n\varphi(x)+m\psi(x)\big)=nI^{\beta}\varphi(x)+mI^{\beta}\psi(x)$;
3) for $k>0$: $I^{\beta}x^{k}=\frac{\Gamma(k+1)}{\Gamma(k+1+\beta)}x^{k+\beta}$;
4) C is constant: $I^{\beta}C=\frac{C}{\Gamma(\beta+1)}x^{\beta}$.

Definition 2: [11] The following is the definition of the Riemann-Liouville (R._L.) fractional differential of order β operator D^{β}:

$$D^{\beta}\varphi(x)=\frac{1}{\Gamma(n-\beta)}\frac{d^{n}}{dx^{n}}\int_a^x\frac{\varphi(t)}{(x-t)^{\beta+1-n}}\,dt. \quad (2)$$

Where n is an integer and $n-1<\beta\le n$

Definition 3: [12] The operator representing the Caputo fractional derivative of order β can be formulated as

$${}^{C}D^{\beta}\varphi(x)=\frac{1}{\Gamma(n-\beta)}\int_a^x\frac{\varphi^{(n)}(t)}{(x-t)^{\beta+1-n}}\,dt. \quad (3)$$

Where n is an integer and $n-1<\beta\le n$
Some Properties of Fractional Derivative:

1) ${}^{C}D^{\beta}I^{\beta}\varphi(x)=\varphi(x)$.
2) $I^{\beta}\quad{}^{C}D^{\beta}\varphi(x)=\varphi(x)-\sum_{k=0}^{n-1}\frac{\varphi^{k}(0)}{k!}x^{k}, x>0$
3) $D^{\beta}x^{k}=\frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}x^{k-\beta}, k\in\mathbb{N}, k\ge\lceil\beta\rceil$.
4) ${}^{C}D^{\beta}\varphi(x)=D^{\beta}\varphi(x)-\sum_{k=0}^{n-1}\frac{(x-t)^{k-\beta}}{\Gamma(k+1-\beta)}D^{K}\varphi(t)$.

3 A REVIEW OF FERMAT POLYNOMIALS

If we examine the (p,q)-Fibonacci polynomials' recurrence relation

$$u_n(x)=p(x)u_{n-1}(x)+q(x)u_{n-2}(x).$$

Considering the initial circumstances $u_0(x)=0$ and $u_1(x)=1$ by substituting $p(x)=3x$ and $q(x)=-2$, we obtain the Fermat polynomials, which are considered A particular instance of the Fibonacci polynomials [13]. We can define the Fermat polynomials by means of the difference equation [14].

$$F_i(x)=3xF_{i-1}(x)-2F_{i-2}(x), \quad (3)$$
$$F_0(x)=0, F_1(x)=1, \quad i\ge 2,$$

we can write the Binet formula for the Fermat polynomials. First, we find the two roots:

$$\alpha(x)=\frac{3x+\sqrt{9x^2-8}}{2}, \qquad \beta(x)=\frac{3x-\sqrt{9x^2-8}}{2}.$$

$$F_i=\frac{\alpha^{i}(x)-\beta^{i}(x)}{\alpha(x)-\beta(x)}. \quad (4)$$

A Fermat polynomial can be written explicitly over the domain $x\in(0,1)$ as follows:

$$F_k(x)=\begin{cases}\frac{\left(3x+\sqrt{9x^2-8}\right)^{k}-\left(3x-\sqrt{9x^2-8}\right)^{k}}{2^{k}\sqrt{9x^2-8}}, & x\ne\frac{2}{3}\\ 2^{\frac{k}{2}}sin\left(\frac{\pi}{4}k\right), & x=\frac{2}{3}\end{cases} \quad k=1,2,. \quad (5)$$

The polynomial $F_k(x)$, of degree $k-1$, is characterized by integer coefficients, which is an important point

The analytic version of Fermat polynomials can be expressed as.$F_{i+1}(x), i\ge 1$ as follows:

$$F_{i+1}=\sum_{m=0}^{\left\lfloor\frac{i}{2}\right\rfloor}(-2)^{m}\,3^{i-2m}\binom{i-m}{m}x^{i-2m}. \quad (6)$$

Theorem 1.[6] The inversion formula presented below is applicable for every nonnegative integer k

$$x^{k}=\frac{1}{3^{k}}\sum_{i=0}^{\left\lfloor\frac{k}{2}\right\rfloor}\frac{2^{i}(k+1-2i)}{k+1-i}\binom{k}{i}F_{k+1-2i}(x). \quad (7)$$

Theorem 2.[6] The formula below establishes the connection between Fermat polynomials and their first derivatives:

$$DF_{i+1}(x)=3\sum_{k=0}^{\left\lfloor\frac{i-1}{2}\right\rfloor}2^{k}\,(i-2k)F_{i-2k}(x). \quad (8)$$

4 THE CAPUTO FRACTIONAL DERIVATIVE'S FERMAT OPERATIONAL MATRIX

4.1 Preliminary Formulation of Fermat Operational Matrices

Let $\varphi(x)$ denote a function on (0,1) that is square-integrable under the Lebesgue measure. It is assumed that $\varphi(x)$ can be expanded in terms of a collection of linearly independent Fermat polynomials as follows:

$$\varphi(x) = \sum_{m=1}^{\infty} r_j \, F_j(x).$$

The following is applicable if the series is shortened.

$$\varphi(x) \approx \varphi_N(x) = \textstyle\sum_{m=1}^{N+1} r_m \, F_m(x) = \mathrm{R}^T \Phi(x). \quad (9)$$

Where

$$\mathrm{R}^T = [r_1, r_2, r_3, \ldots\ldots, r_{N+1}], \quad (10)$$

and

$$\Phi(x) = [F_1(x), F_2(x), F_3(x), \ldots\ldots, F_{N+1}(x)].^T \quad (11)$$

Assuming $\frac{d\Phi(x)}{dx}$, we may write it as:

$$\frac{d\Phi(x)}{dx} = \mathrm{G}^1 \Phi(x), \quad (12)$$

$\varphi(x) = R^T \mathrm{G}^1 \Phi(x)$ where the non-zero entries of the matrix G^1,can be obtained directly from relation (8). which is the $(N+1) \times (N+1)$ (OMD).

Remark 1. The overall structure of the (OMD) where $\mathrm{G}^1 = g_{ij}^{(1)}$ then can be written as follows:

$$g_{ij}^{(1)} = \begin{cases} 3(j+1)2^{\frac{i-j-1}{2}} & \text{if } i > j, (i+j) odd \\ 0 & , otherwise \end{cases} \quad (13)$$

4.2 Building of Fermat OMFD

In this section, we aim to formulate the operational matrix of fractional derivatives as an extension of the corresponding matrix for integer-order derivatives. It follows directly from Relation (8) that for any positive integer:

$$\frac{d^s \Phi(x)}{dx^s} = \mathrm{G}^{(s)} \Phi(x) = \left(\mathrm{G}^{(1)}\right)^s \Phi(x). \quad (14)$$

Theorem 3. [6] Let the vector be $\Phi(x)$ of Fermat polynomials described by Equation (11). For $x \in (0,1)$ and any $\beta > 0$, one has

$$D^\beta \Phi(x) = x^{-\beta} \mathrm{G}^{(\beta)} \Phi(x), \quad (15)$$

where $\mathrm{G}^{(\beta)} = g_{i,j}^{(\beta)}$, which is explicitly written as follows, represents the $(N+1) \times (N+1)$ Fermat (OMFE) of order β in the Caputo sense.

$$G^{(\beta)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma_\beta(\lceil\beta\rceil, 1) & \gamma_\beta(\lceil\beta\rceil, \lceil\beta\rceil) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma_\beta(i, 1) & \cdots & \gamma_\beta(i, i) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma_\beta(n+1,1) & \gamma_\beta(n+1,2) & \gamma_\beta(n+1,3) & \cdots & \gamma_\beta(N+1, N+1) \end{pmatrix}. \quad (17)$$

You may write the entries $g_{i,j}^{(\beta)}$ like this:

$$g_{ij}^{(\beta)} = \begin{cases} \gamma_\beta(i,j) & \text{if } i \geq \lceil\beta\rceil, i \geq j \\ 0 & , otherwise \end{cases}. \quad (16)$$

Where:

$$\gamma_\beta(i,j) = j \sum_{\substack{m=\lceil\beta\rceil \\ (i+j) odd, (j+m) odd}}^{i} \frac{m!\,(-1)^{\frac{1}{2}(i-2j+m+1)} 2^{\frac{i-j}{2}} \left(\frac{i+m-1}{2}\right)!}{\left(\frac{i-m-1}{2}\right)! \left(\frac{m-j+1}{2}\right)! \left(\frac{m+j+1}{2}\right)! \, \Gamma(-\beta+m+1)} \quad (19)$$

For instance, when $N = 5$, the operational matrix $\mathrm{G}^{\frac{3}{2}}$ is defined as:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{8}{\sqrt{\pi}} & 0 & \frac{4}{\sqrt{\pi}} & 0 & 0 & 0 \\ 0 & \frac{32}{\sqrt{\pi}} & 0 & \frac{8}{\sqrt{\pi}} & 0 & 0 \\ \frac{272}{5\sqrt{\pi}} & 0 & \frac{264}{5\sqrt{\pi}} & 0 & \frac{64}{5\sqrt{\pi}} & 0 \\ 0 & \frac{768}{7\sqrt{\pi}} & 0 & \frac{576}{7\sqrt{\pi}} & 0 & \frac{128}{7\sqrt{\pi}} \end{pmatrix}.$$

5 METHOD OF SOLUTION

The recommended approach for creating a numerical solution to the Delay equation is described in this section.

Examine the fractional delay differential equation that follows:

$$D^s \varphi(x) + D^\beta \varphi(x) = H[\varphi(x), \varphi(dx - \tau), z(x)] \quad x \in (0,1) \quad (20)$$

with initial conditions

$$\varphi^{(i)}(0) = v_i, \quad i = 0,1, \ldots, m \quad (17)$$

where $m < \beta \leq m+1$, β Largest fractional power $d \in (0,1]$, $\tau \in (0,1]$ $H(\varphi(x), \varphi(dx - \tau), z(x))$ is a polynomial in $\varphi(x), \varphi(dx), z(x)$, $D^s\varphi(x)$ is the derivative of integers,$D^\beta\varphi(x)$ is the Caputo derivative.

Now we can approximate $\varphi(x)$ and $\varphi(dx-\tau)$

$$\varphi(x) \approx \varphi_N(x) = \mathrm{R}^T\Phi(x),$$

$$\varphi(dx-\tau) \approx \varphi_N(dx-\tau) = \mathrm{R}^T\Phi(dx-\tau), \quad (18)$$

$$D^s\varphi(x) \approx \mathrm{R}^T\mathrm{G}^{(s)}\Phi(x). \quad (19)$$

The following approximation can be produced by virtue of Theorem 3:

$$D^\beta\Phi(x) \approx x^{-\beta}\mathrm{R}^T\mathrm{G}^{(\beta)}\Phi(x). \quad (20)$$

The formula for the residual of **Помилка! Джерело посилання не знайдено.** is obtained by applying the approximations in (19) and (20). (25)

$$x^\beta R(x) = x^\beta\mathrm{R}^T\mathrm{G}^{(s)}\Phi(x) + \mathrm{R}^T\mathrm{G}^{(\beta)}\Phi(x) - x^\beta H[\varphi(x), \varphi(dx), z(x)]$$

and hence the application of Our proposed Galerkin-type approach:

$$\langle R(x), F_j(x)\rangle = \int_0^1 x^\beta\, R(x)F_j(x)dx = 0 \quad , \quad j = 1,2,3,\ldots, N-m. \quad (26)$$

Also, by inserting (18) into (17),we get:

$$\begin{aligned} \varphi(0) \ &=\ R{}^\wedge T\ \Phi(0)\ =\ v\_0 \\ \varphi^1(0)\ &=\ R^T G^1\ \Phi(0) = \ v\_1 \\ &\vdots \\ \varphi^m(0)\ &=\ R^T G^m\ \Phi(0)\ =\ v_m. \end{aligned} \quad (27)$$

Finally, using (26) and (27) of dimension $(N+1)$, a system of algebraic equations is obtained. It is possible to solve these linear eq. given the unknown expansion factors.$\mathrm{r_i}$. As a result, (22) can estimate $\varphi(x)$.

6 NUMERICAL EXAMPLES

In this part, we use the Fermat Galerkin operational matrix (FGM) method to numerically solve fractional differential equations.

Example 1: Consider the following FDD equation [15]-[17]:

$$D^\beta\varphi(x) + \varphi(x) - \varphi(x-\tau) = \frac{\Gamma(3)}{\Gamma(3-\beta)}x^{2-\beta} - \frac{\Gamma(2)}{\Gamma(2-\beta)}x^{1-\beta} - \tau^2 + 2\tau x - \tau \quad (28)$$

$$\varphi(0) = 0, \quad x \in (0,1), \quad \tau \in (0,1], \quad \beta \in (0,1]$$

The precise solution to (28) is $\varphi(x) = x^2 - x$ at $\beta = 1$ by applying the method from Section 5, Figure 1 shows the estimated answer and the corresponding precise real solution. The findings of the Krawtchouk wavelet technique (KWM) described in reference [15] and the spectral approach with non-integer order Taylor basis reported in reference [16] are contrasted with the achievable absolute error produced by the provided FGM for $N = 2$ in Table 1. It is evident that the introduced method is more accurate because we obtain the exact solution for $\beta = 1$ and for any value of τ. As an example, we used three terms for FGM when $\tau = 0.01$ in Table 1, whereas they only obtained approximate solutions using Taylor functions with eight terms of fractional order and nine terms of the Krawtchouk wavelet method. and also, better than the methods used in [18]-[20].

Table 1: The absolute errors at $\tau = 0.01$ and $\beta = 1$ are compared for Example 1.

x	Results of [16]	KWM	P. method
0.2	3.47×10^{-16}	0	0
0.4	2.36×10^{-16}	5.55×10^{-17}	0
0.6	3.47×10^{-16}	5.55×10^{-17}	0
0.8	9.69×10^{-16}	5.55×10^{-17}	0
1	0	7.81×10^{-17}	0

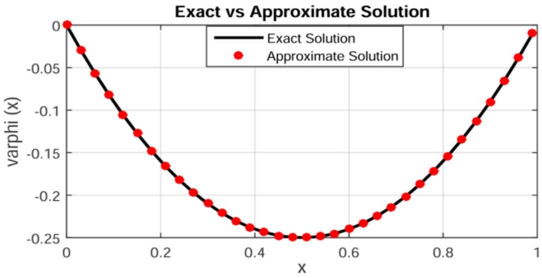

Figure 1: Graphical representations of the exact and approximate solutions for Example 1

Example 2: Consider the following FDD equation [21], [22]:

$$D^{0.2}\varphi(x) + \varphi(x-1) = \frac{\Gamma(3)}{\Gamma(0.2)}x^{1.8} - \frac{\Gamma(2)}{\Gamma(1.8)}x^{0.8} + x^2 - 3x + 1 \quad (29)$$

$$\varphi(0) = -1 \qquad x \in (0,1).$$

The precise solution to equation **Помилка! Джерело посилання не знайдено.** is $\varphi(x) = x^2 - x - 1$ by applying the method from Section 5, The estimated solution is displayed beside the corresponding precise real solution and the absolute error in Figure 2, Figure 3. In Table 2, the results of the fractional-order Chelyshkov functions (FCHFs) given in reference [21] and the results of the Legendre collocation technique reported in reference

[22] have been compared with the achievable absolute error acquired by the provided FGM for $N = 3$.

Table 2. The findings concerning the approximation solutions absolute errors Example 2.

x	FCHFs	Results of [16]	P. method
0.2	5.2×10^{-10}	8.9×10^{-16}	2.6×10^{-17}
0.4	2.4×10^{-10}	1.6×10^{-15}	5.8×10^{-17}
0.6	1.1×10^{-10}	1.6×10^{-15}	9.5×10^{-17}
0.8	9.7×10^{-12}	1.3×10^{-15}	1.4×10^{-16}
1	7.3×10^{-11}	7.8×10^{-16}	1.8×10^{-16}

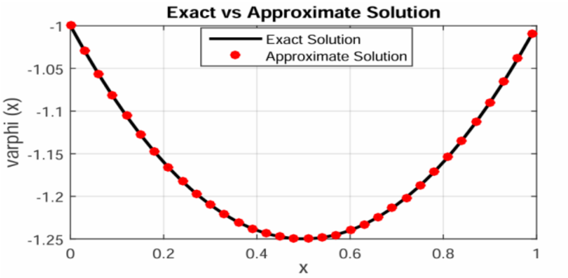

Figure 2: Graphical representations of the exact and approximate solutions for Example 2.

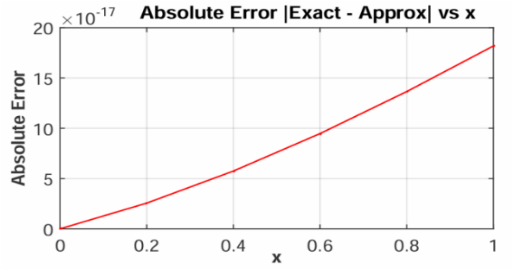

Figure 3: Absolute errors of Example 2.

Example 3: Consider the following FDD equation [20], [21], [23]:

$$D^{\beta}\varphi(x)+\varphi(x)=\frac{\tau}{2}\varphi(\tau x)-\frac{\tau}{2}e^{-\tau x}. \qquad (21)$$

$$\varphi(0)=1, \quad x\in(0,1), \quad \tau\in(0,1], \quad \beta\in(0,1]$$

The precise solution to equation (21) is $\varphi(x) = e^{-x}$, at $\beta = 1$ by applying the method from Section 5,In Figure 4 We show how the approximate solution behaves, and Figure 5 shows the absolute errors obtained using the current approach. In Table 3, We have compared the findings of the fractional-order Chelyshkov functions (FCHFs) published in reference [21] with the attained absolute error acquired by the provided FGM for $N = 10$,The comparison indicates that the maximum absolute error produced by the proposed FGM is of order (4.57×10^{-14}), while the FCHFs method yields an error of order (1.49×10^{-12}), highlighting the effectiveness and high precision of the proposed method.

Table 3: The absolute errors at $\tau = 0.01$ and $\beta = 1$ are compared for Example 3.

x	FCHFs	P. method
0	2.16×10^{-14}	0
0.2	4.66×10^{-15}	4.57×10^{-14}
0.4	6.22×10^{-15}	3.11×10^{-14}
0.6	8.09×10^{-14}	1.84×10^{-14}
0.8	3.84×10^{-13}	2.10×10^{-14}
1	1.49×10^{-12}	1.80×10^{-14}

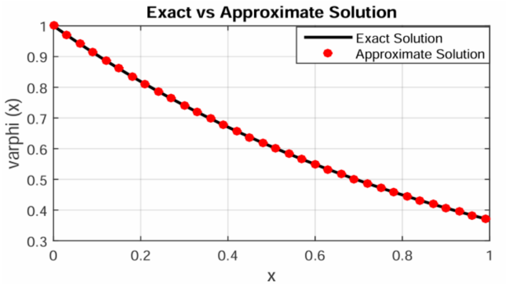

Figure 4: The visual representations of the exact and approximate solutions for Example 3.

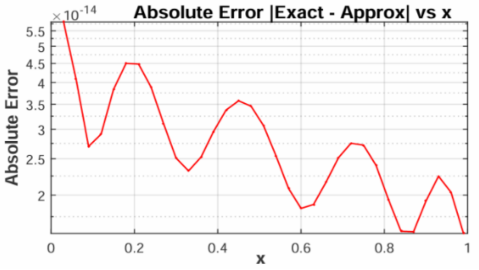

Figure 5: Absolute errors of Example 3.

Example 4: Consider the following FDD equation [24]-[26]

$$D^{0.3}\varphi(x)+\varphi(x)=\varphi(x-1)+3x^2-3x+1 + \frac{6}{\Gamma(4-0.3)}x^{2.7} \qquad (31)$$

$$\varphi(0)=0 \qquad x\in(0,1).$$

Table 4: The findings concerning the approximation solutions' absolute errors Example 4.

x	$N = 6$
0	0
0.2	1.33×10^{-18}
0.4	9.01×10^{-18}
0.6	2.70×10^{-17}
0.8	5.98×10^{-17}
1	1.12×10^{-16}

The precise solution to **Помилка! Джерело посилання не знайдено.** is $\varphi(x) = x^3$, by applying the method from Section 5.In Table 4 presents the absolute errors at $N = 6$, The greatest absolute error of the suggested technique (FGM) and the collocation method that uses the Haar wavelet collocation method [25] and the Chebyshev cardinal functions (CCFs) mentioned in reference [24] are compared in Table 5 for different values of N. Figure 6, where the approximate answer is given in close agreement with the precise solution, demonstrates the efficacy and precision of the suggested strategy. Additionally, Figure 7 shows the method's absolute accuracy. Notably, the proposed technique attains high accuracy despite employing fewer terms compared to Methods [24] and [25].

Table 5: The largest absolute error calculated for Example 4.

N	FCHFs	N	Results of [16]	N	P.method
4	5.2×10^{-10}	4	8.9×10^{-16}	4	2.6×10^{-17}
8	2.4×10^{-10}	6	1.6×10^{-15}	5	5.8×10^{-17}
16	1.1×10^{-10}	8	1.6×10^{-15}	6	9.5×10^{-17}
32	9.7×10^{-12}	10	1.3×10^{-15}	7	1.4×10^{-16}

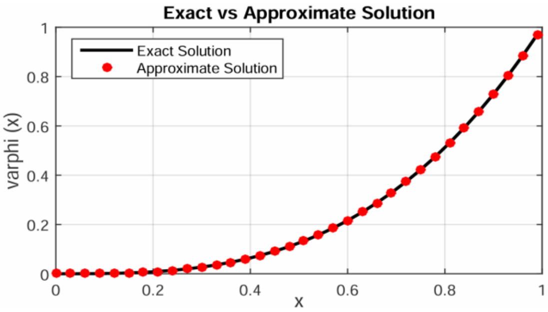

Figure 6: Graphical representations of the exact and approximate solutions for Example 4.

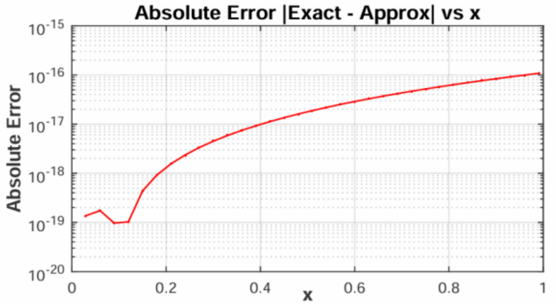

Figure 7: Absolute errors of Example 4.

Example 5: Consider the following FDD equation [27], [28].

$$D^{\beta}\varphi(x) + a\varphi(x) + b\varphi(\tau x) = (a-1)sinx + bsin(\tau x) \quad (32)$$

$$\varphi(0) = 1, \quad \varphi'^{(0)} = 1,$$

$$x \in (0,1), \quad \tau \in (0,1] \quad \beta \in (1,2]$$

The precise solution to **Помилка! Джерело посилання не знайдено.** is $\varphi(x) = sinx$, at $\beta = 2$ for any $a, b \in R$ by applying the method from Section 5.We solve this problem when $\tau = 0.25$. The greatest absolute error of the suggested technique (FGM) and the explicit Chebyshev tau (ECT) approach described in reference [27] are compared in Table 6 for a range of N values. In Table 7 presents the absolute errors for different values of N. Figure 8 and Figure 9 show the approximate answer and the absolute accuracy that our technique produced, respectively.

Table 6: The largest absolute error calculated for Example 5.

x	ECT	P. method
4	2.2×10^{-3}	2.1×10^{-3}
8	4.6×10^{-7}	1.3×10^{-8}
12	2.8×10^{-10}	7.4×10^{-15}
16	4.4×10^{-16}	0

Table 7: The estimated solutions' absolute errors for Example 5 at various N values.

x	$N = 8$	$N = 10$	$N = 12$
0.2	2.8×10^{-9}	2.7×10^{-12}	1.7×10^{-15}
0.4	5.6×10^{-9}	5.4×10^{-12}	3.3×10^{-15}
0.6	8.3×10^{-9}	7.9×10^{-12}	4.9×10^{-15}
0.8	1.1×10^{-8}	1.0×10^{-11}	6.2×10^{-15}
1	1.3×10^{-8}	1.2×10^{-11}	7.4×10^{-15}

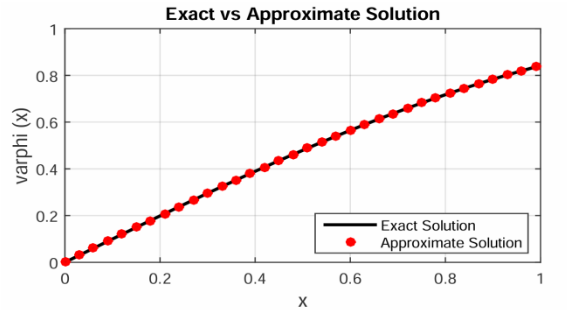

Figure 8: Graphical representations of the exact and approximate solutions for Example 5.

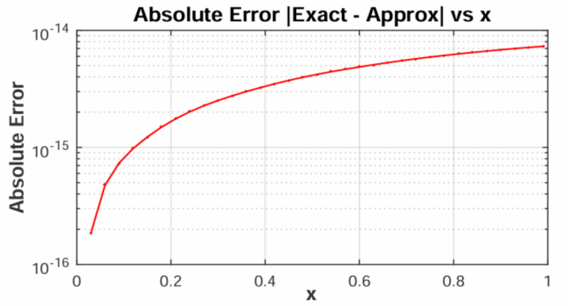

Figure 9: Absolute errors of Example 5.

7 CONCLUSIONS

In this paper, an efficient spectral Galerkin method based on Fermat polynomials and the operational matrix of fractional derivatives in the Caputo sense was proposed for solving fractional delay differential equations. The developed approach transforms the original problem into a system of algebraic equations, which can be solved efficiently with reduced computational effort.

The proposed Fermat Galerkin method demonstrated high accuracy and rapid convergence while requiring only a small number of basis functions. Several numerical examples were investigated to validate the effectiveness of the method. The obtained numerical solutions showed excellent agreement with the corresponding exact solutions and produced smaller absolute errors compared with several existing numerical techniques reported in the literature.

The results confirm that the proposed method is stable, reliable, and computationally efficient for solving fractional delay differential equations. Therefore, the presented approach can serve as an effective numerical tool for a broad class of fractional problems arising in applied mathematics, engineering, and physical sciences. Future work may focus on extending the method to nonlinear systems, multi-dimensional fractional models, and variable-order fractional differential equations.

REFERENCES

[1] J. S. Hesthaven, S. Gottlieb, and D. Gottlieb, Spectral Methods for Time-Dependent Problems, vol. 21, Cambridge Monographs on Applied and Computational Mathematics. Cambridge, U.K.: Cambridge University Press, 2007.

[2] J. P. Boyd, Chebyshev and Fourier Spectral Methods. Mineola, NY, USA: Dover Publications, 2001.

[3] L. N. Trefethen, Spectral Methods in MATLAB, vol. 10, Software, Environments, and Tools. Philadelphia, PA, USA: SIAM, 2000.

[4] H. Bhrawy, T. M. Taha, and J. A. T. Machado, "A review of operational matrices and spectral techniques for fractional calculus," Nonlinear Dynamics, vol. 81, pp. 1023-1052, 2015, [Online]. Available: https://doi.org/10.1007/s11071-015-2087-0.

[5] G. Lee and M. Asci, "Some properties of the (p, q)-Fibonacci and (p, q)-Lucas polynomials," Journal of Applied Mathematics, vol. 2012, Art. no. 264842, 18 pages, 2012, [Online]. Available: https://doi.org/10.1155/2012/264842.

[6] Y. H. Youssri, "A new operational matrix of Caputo fractional derivatives of Fermat polynomials: An application for solving the Bagley-Torvik equation," Advances in Difference Equations, vol. 2017, Art. no. 73, 2017, [Online]. Available: https://doi.org/10.1186/s13662-017-1123-4.

[7] W. M. Abd-Elhameed, O. M. Alqubori, N. M. A. Alsafri, A. K. Amin, and A. G. Atta, "A matrix approach by convolved Fermat polynomials for solving the fractional Burgers' equation," Mathematics, vol. 13, Art. no. 1135, 2025, [Online]. Available: https://doi.org/10.3390/math13071135.

[8] M. A. Taema, M. D. Dagher, and Y. H. Youssri, "Spectral collocation method via Fermat polynomials for Fredholm-Volterra integral equations with singular kernels and fractional differential equations," Palestine Journal of Mathematics, vol. 14, no. 2, pp. 481-492, 2025.

[9] R. Gorenflo and F. Mainardi, "Fractional Calculus," in A. Carpinteri and F. Mainardi (eds.), Fractals and Fractional Calculus in Continuum Mechanics, International Centre for Mechanical Sciences, vol. 378, Springer, Vienna, 1997, [Online]. Available: https://doi.org/10.1007/978-3-7091-2664-6_5.

[10] A. Kilbas, H. M. Srivastava, and J. J. Trujillo, "Theory and Applications of Fractional Differential Equations," North-Holland Mathematics Studies, vol. 204, pp. 1-523, Elsevier, 2006.

[11] M. Abu-Shady and M. K. A. Kaabar, "A generalized definition of the fractional derivative with applications," Mathematical Problems in Engineering, vol. 2021, Art. no. 9444803, 9 pages, 2021, [Online]. Available: https://doi.org/10.1155/2021/9444803.

[12] Li, F. Zeng, and F. Liu, "Spectral approximations to the fractional integral and derivative," Fractional Calculus and Applied Analysis, vol. 15, no. 3, pp. 383-

406, 2012, [Online]. Available: https://doi.org/10.2478/s13540-012-0028-x.

[13] J. Wang, "Some new results for the (p, q)-Fibonacci and Lucas polynomials," Advances in Difference Equations, vol. 2014, Art. no. 64, 2014, [Online]. Available: https://doi.org/10.1186/1687-1847-2014-64.

[14] W. Wang and H. Wang, "Some results on convolved (p, q)-Fibonacci polynomials," Integral Transforms and Special Functions, vol. 26, no. 5, pp. 340-356, 2015, [Online]. Available: https://doi.org/10.1080/10652469.2015.1007502.

[15] H. Mohammadi, H. Saeedi, and M. Izadi, "A novel design of Krawtchouk wavelet based projection approach to solve the nonlinear fractional delay differential equations," Journal of Nonlinear Mathematical Physics, vol. 32, no. 1, 2025, [Online]. Available: https://doi.org/10.1007/s44198-024-00256-3.

[16] İ. Avcı, "Numerical simulation of fractional delay differential equations using the operational matrix of fractional integration for fractional-order Taylor basis," Fractal and Fractional, vol. 6, no. 1, Art. no. 10, 2022, [Online]. Available: https://doi.org/10.3390/fractalfract6010010.

[17] M. Kumar, "An efficient numerical scheme for solving a fractional-order system of delay differential equations," International Journal of Applied and Computational Mathematics, vol. 8, Art. no. 262, 2022, [Online]. Available: https://doi.org/10.1007/s40819-022-01466-3.

[18] H. Singh, "Numerical simulation for fractional delay differential equations," International Journal of Dynamics and Control, vol. 9, pp. 463-474, 2021, [Online]. Available: https://doi.org/10.1007/s40435-020-00671-6.

[19] K. K. Ali, M. A. A. El-Salam, and E. M. Mohamed, "Chebyshev operational matrix for solving fractional order delay-differential equations using spectral collocation method," Arab Journal of Basic and Applied Sciences, vol. 26, no. 1, pp. 342-353, 2019, [Online]. Available: https://doi.org/10.1080/25765299.2019.1629543.

[20] P. Rahimkhani, Y. Ordokhani, and E. Babolian, "A new operational matrix based on Bernoulli wavelets for solving fractional delay differential equations," Numerical Algorithms, vol. 74, pp. 223-245, 2017, [Online]. Available: https://doi.org/10.1007/s11075-016-0146-3.

[21] I. Ahmed and M. S. Al-Sharif, "An effective numerical method for solving fractional delay differential equations using fractional-order Chelyshkov functions," Boundary Value Problems, vol. 2024, Art. no. 107, 2024, [Online]. Available: https://doi.org/10.1186/s13661-024-01913-8.

[22] P. K. Abbassi, M. Fathy, R. A. Elbarkoki, et al., "Semi-analytical solution of linear and nonlinear fractional delay differential equations via the Legendre collocation method," Journal of Nonlinear Mathematical Physics, vol. 32, no. 46, 2025, [Online]. Available: https://doi.org/10.1007/s44198-025-00285-6.

[23] Y. Muroya, E. Ishiwata, and H. Brunner, "On the attainable order of collocation methods for pantograph integro-differential equations," Journal of Computational and Applied Mathematics, vol. 152, nos. 1-2, pp. 347-366, 2003, [Online]. Available: https://doi.org/10.1016/S0377-0427(02)00716-1.

[24] H. B. Jebreen and I. Dassios, "The collocation method based on the new Chebyshev cardinal functions for solving fractional delay differential equations," Mathematics, vol. 12, Art. no. 3388, 2024, [Online]. Available: https://doi.org/10.3390/math12213388.

[25] R. Amin, K. Shah, M. Asif, and I. Khan, "A computational algorithm for the numerical solution of fractional order delay differential equations," Applied Mathematics and Computation, vol. 402, no. 125863, 2021, [Online]. Available: https://doi.org/10.1016/j.amc.2020.125863.

[26] M. L. Morgado, N. J. Ford, and P. M. Lima, "Analysis and numerical methods for fractional differential equations with delay," Journal of Computational and Applied Mathematics, vol. 252, pp. 159-168, 2013, [Online]. Available: https://doi.org/10.1016/j.cam.2012.06.034.

[27] W. M. Abd-Elhameed, J. A. T. Machado, and Y. H. Youssri, "Hypergeometric fractional derivatives formula of shifted Chebyshev polynomials: tau algorithm for a type of fractional delay differential equations," International Journal of Nonlinear Sciences and Numerical Simulation, vol. 23, nos. 7-8, pp. 1253-1268, 2022, [Online]. Available: https://doi.org/10.1515/ijnsns-2020-0124.

[28] H. Brunner, Q. Huang, and H. Xie, "Discontinuous Galerkin methods for delay differential equations of pantograph type," SIAM Journal on Numerical Analysis, vol. 48, no. 5, pp. 1944-1967, 2010, [Online]. Available: https://doi.org/10.1137/090771922.