Binary Relation Fuzzy Soft Points for Decision-Making Applications

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Abstract: This paper introduces a new class of fuzzy soft structures termed *binary relation–fuzzy soft points* (BRFS-points), constructed through the composition of binary relation soft points and fuzzy points defined with respect to a power set. The proposed framework integrates two distinct mathematical concepts within a unified structure, combining the parameterization of binary relation soft sets with graded membership characteristics of fuzzy points. As a result, nine different types of BRFS-points are systematically derived based on different combinations of underlying fuzzy and binary relation soft point types, and each case is illustrated through representative examples. The introduced BRFS-points generalize existing notions of fuzzy soft points by providing a richer and more flexible representation of uncertainty in multi-parameter environments. In contrast to classical fuzzy soft point constructions, the proposed model captures additional structural information arising from binary relations on parameter sets, thereby enhancing the expressive power of the framework. The developed structure is applicable to binary relation–fuzzy soft topological spaces and related separation axioms, where the interaction between soft parameters and fuzzy membership degrees is essential. In particular, BRFS-points provide a mathematically consistent tool for analyzing and representing uncertainty in systems involving complex relationships between elements and parameters. Furthermore, the proposed framework is suitable for decision-making problems under uncertainty, where multiple criteria and interdependent parameters must be processed simultaneously. The model enables a structured aggregation of fuzzy and soft information, making it relevant for applied fields that require formal handling of imprecise or incomplete data.

1 INTRODUCTION

Molodtsov [1] was the first one who started the main idea of soft-set theory which is the useful Mathematical tool. The researcher was talked about the soft-set functionality and applications in somewhat mutual and different methods. In the reference [2] Haci A. et.al imported the certain soft-set components also make a comparison of soft-set across the thought related with rough, fuzzy sets, etc. Ahmad & Kharal in [3] was introduced the certain notions on soft classes functions, and they proposed the certain properties of images of soft sets which would be found in medical diagnosis matters as important applications. Fuzzy-set definition was proposed and introduced by Zadeh in [4] 1965. After fuzzy-set appearance, the scientists offered very many attentions that to develop the theory of fuzzy-set. Das & Borgahain in reference [5] proposed the good view around fuzzy-soft-set and they applied the fuzzy soft set on some problems in the field of decision-making addition to some main difficulties. The researchers of the theory of soft-set had been hardworking, and rapidly development has been achieved, see [6] - [11]. It is noticeable that all those scientists are based on the classical (ordinary) soft set. Aso, Maji et al. [12] basically proposed the notion of fuzzy-soft-set. After that, many scientists wrote on this field. Roy & Maji [13] introduced some main results and application of the fuzzy-soft-set in decision-making field. Also, F. Feng et al. [14] gave some useful application in decision-making field, Z. F. Abd alhussain and A. F. Hassan, introduced the concept of N-Ary fuzzy soft set, see [15], [16], Luay A. Al-Swidi, Auda S. Saeed, Types of fuzzy points w.r.to. power set in [18].

Simply we will denote to the binary relation-fuzzy soft set by (BR-FS-set).

The research contains 3 sections: section one the introduction, section two presents the definitions of binary soft set, fuzzy w.r.to power set, binary soft point and fuzzy point w.r.to power set, section three presents our new types of fuzzy soft points via composite fuzzy point w.r.to power set with binary soft point, with explanation examples.

2 PRELIMINARIES

This section recalls the basic notions of binary relation soft sets, fuzzy points with respect to a power set, and related structures that are required in the sequel.

Definition 2.1 [15]: A pair $(\mathbb{S}, \mathbb{E}_1 \times \mathbb{E}_2)$ is said to be a binary relation-soft set over $\mathbb{X}$, where $\mathbb{S}$ is a mapping given by $\mathbb{S}: \mathbb{E}_1 \times \mathbb{E}_2 \to \mathcal{P}(\mathbb{X})$.
i.e.,$(\mathbb{S}, \mathbb{E}_1 \times \mathbb{E}_2) = \{((\varrho_i, \varrho_k), \mathscr{s}(\varrho_i, \varrho_k)): \varrho_i \in \mathbb{E}_1 \text{ and } \varrho_k \in \mathbb{E}_2, \mathbb{S}: \mathbb{E}_1 \times \mathbb{E}_2 \to \mathcal{P}(\mathbb{X}); i = 1,2, \dots, n, k = 1,2 \dots, m\}$.

Definition 2.2 [15]: Let the set $\mathbb{X}$ be an initial universe, and $\mathbb{E}_1, \mathbb{E}_2$ be sets of parameters, $P(\mathbb{X})$ represents a power set and $I = [0,1]$. The composition of fuzzy set w.r.to power set with binary relation-soft set is called binary relation-fuzzy soft set with function $\#\mathbb{FS}: \mathbb{E}_1 \times \mathbb{E}_2 \to I$, such that $\#\mathbb{FS}(\varrho_i, \varrho_k) = (\mathbb{F} \circ \mathbb{S})(\varrho_i, \varrho_k), \forall \varrho_i \in \mathbb{E}_1$ and $\varrho_k \in \mathbb{E}_2; i = 1,2, \dots, n, k = 1,2 \dots, m\}$,
$\# \mathbb{F}: P(\mathbb{X}) \to I$ and $\mathbb{S}: \mathbb{E}_1 \times \mathbb{E}_2 \to P(\mathbb{X})$. So $\mathbb{FS} = \{((\varrho_i, \varrho_k), \mathbb{F}(\mathbb{S}(\varrho_i, \varrho_k))); \forall \varrho_i \in \mathbb{E}_1 \text{ and } \varrho_k \in \mathbb{E}_2; i = 1,2, \dots, n, k = 1,2 \dots, m\}$

Definition 2.3 [15]: A BR-FS-set $\mathfrak{A}$ is called a null BR-FS-set iff $f_{\mathfrak{A}}(\varrho_i, \varrho_k) = 0$, i.e. $0(\varrho_i, \varrho_k) = \{((\varrho_i, \varrho_k), 0), \forall (\varrho_i, \varrho_k) \in \mathbb{E}_1 \times \mathbb{E}_2 ; i = 1,.,n ; j = 1, \dots, m\}$

Definition 2.4 [15]: A BR-FS-set $\mathfrak{A}$ is said to be a universal BR-FS-set if $f_{\mathfrak{A}}(\varrho_i, \varrho_k) = 1$, i.e. $1(\varrho_i, \varrho_k) = \{((\varrho_i, \varrho_k), 1), \forall (\varrho_i, \varrho_k) \in \mathbb{E}_1 \times \mathbb{E}_2 ; i = 1,.,n ; j = 1, \dots, m\}$

Definition 2.5 [17]: Types of Binary Relation soft points:

1) The first type (I-type) of binary soft point, denoted by $\mathbb{S}^{\mathbb{x}}_{(\varrho_a, \varrho_b)}$, where:

$$\mathbb{S}^{\mathbb{x}}_{(\varrho_a, \varrho_b)}((\varrho_i, \varrho_k)) = \begin{cases} \{\mathbb{x}\} & \text{if } (\varrho_i, \varrho_k) = (\varrho_a, \varrho_b) \\ \emptyset & \text{if } (\varrho_i, \varrho_k) \neq (\varrho_a, \varrho_b) \end{cases}$$

2) The second type (II-type) of binary soft point, denoted by $\mathbb{S}^{\mathbb{x}}_{(\varrho_i, \varrho_k)}$, where:

$$\mathbb{S}^{\mathbb{x}}_{(\varrho_i, \varrho_k)} = \{((\varrho_i, \varrho_k), \{\mathbb{x}\}), \text{for all } (\varrho_i, \varrho_k) \in \mathbb{E}_1 \times \mathbb{E}_2\}$$

3) The third type (III-type) of binary soft point, denoted by $\mathbb{S}_{(\varrho_a, \varrho_b)}$, where:

$$\mathbb{S}_{(\varrho_a, \varrho_b)}((\varrho_i, \varrho_k)) = \begin{cases} \neq \emptyset & \text{if } (\varrho_i, \varrho_k) = (\varrho_a, \varrho_b) \\ \emptyset & \text{if } (\varrho_i, \varrho_k) \neq (\varrho_a, \varrho_b) \end{cases}$$

Definition 2.6 [18]: Types of fuzzy points w.r.to. power set:

1) The first type (I-type): Let $0 < \alpha \leq 1$, For any $A, B \in P(\mathbb{X})$ we say that is classical fuzzy point w.r.to power set of type-I where:

$$\mathbb{F}^A_\alpha(B) = \begin{cases} \alpha & \text{if } B = A \\ 0 & \text{if } B \neq A \end{cases}$$

2) The second type (II-type): Let $0 < \alpha \leq 1$ and $\emptyset \notin A \in P(\mathbb{X})$.The formula $\mathbb{F}^A_\alpha$ is called fuzzy point w.r.to. power set of Type- II where:

$$\mathbb{F}^A_\alpha(a) = \begin{cases} \mathbb{F}^a_\alpha & \text{if } a \in A \\ 0 & \text{if } a \notin A \end{cases}$$

3) The third type (III-type): Let $0 < \alpha < 1$, the formula $\mathbb{F}_\alpha$ is called a fuzzy point w.r.to. power set of type-III such that $\mathbb{F}_\alpha(A) = \alpha$, $\forall A \in P(\mathbb{X})$.

Note that $\mathbb{F}_\alpha = \cup_{A \in P(\mathbb{X})} \mathbb{F}^A_\alpha$.

3 THE COMPOSITION OF FUZZY POINTS W.R.TO POWER SET WITH BINARY SOFT POINTS

Now, we will introduce the following definitions with examples to explain the composition of fuzzy points w.r.to. power set in 2.6 with binary relation soft points in 2.5 above, and finally we will list them in Table 1.

Definition 3.1: A fuzzy soft point is a function $\#\mathbb{FS}: \mathbb{E}_1 \times \mathbb{E}_2 \to I$ which depends on the composition of the membership function to the types of fuzzy points w.r.to power set in 2.6 with the types of binary relation soft points in 2.5, then we get the table of binary relation-fuzzy soft (BR-FS) points types, as the following;

Followings are the Explanations or clarifications for all cases of the table above:

1) The composition of fuzzy point w.r.to power set $\mathbb{F}^A_\alpha$ with binary relation-soft point $\mathbb{S}^{\mathbb{x}}_{(e_a, \varrho_b)}$ produces the BR-FS-point $\mathbb{FS}^{(e_a, e_b)}_\alpha$, as the following;

$$\mathbb{F}^A_\alpha \circ \mathbb{S}^{\mathbb{x}}_{(\varrho_a, \varrho_b)} = \begin{cases} \mathbb{FS}^{(\varrho_a, \varrho_b)}_\alpha & \text{if } \{x\} = A \\ 0(\varrho_i, \varrho_k) & \text{if } x \notin A \end{cases}$$

Table 1: Binary relation fuzzy soft points.

Fuzzy points w.r.to. power set	Binary relation-soft points	Binary relation-fuzzy soft points
$\mathbb{F}_{\alpha}^{A}$	$\mathbb{S}^{x}$ with $A = \{x\}$	$\mathbb{FS}_{\alpha}$
$\mathbb{F}_{\alpha}^{C}$	$\mathbb{S}_{(\varrho_a, \varrho_b)}^{x}$ with $\emptyset, \{x\} \in C \subset P(\mathbb{X})$	
$\mathbb{F}_{\alpha}^{C}$	$\mathbb{S}^{x}$ with $\{x\} \in C$	
$\mathbb{F}_{\alpha}^{C}$	$\mathbb{S}_{(\varrho_a, \varrho_b)}$ with $\emptyset, F((\varrho_a, \varrho_b)) \in C$	
$\mathbb{F}_{\alpha}$	$\mathbb{S}_{(\varrho_a, \varrho_b)}^{x}$	
$\mathbb{F}_{\alpha}$	$\mathbb{S}^{x}$	
$\mathbb{F}_{\alpha}$	$\mathbb{S}_{(\varrho_a, \varrho_b)}$	
$\mathbb{F}_{\alpha}^{A}$	$\mathbb{S}_{(\varrho_a, \varrho_b)}^{x}$ with $A = \{x\}$	$\mathbb{FS}_{\alpha}^{(\varrho_a, \varrho_b)}$
$\mathbb{F}_{\alpha}^{A}$	$\mathbb{S}_{(\varrho_a, \varrho_b)}$ with $A = F(e)$	
$\mathbb{F}_{\alpha}^{C}$	$\mathbb{S}_{(\varrho_a, \varrho_b)}^{x}$ with $\emptyset \notin C, \{x\} \in C$	
$\mathbb{F}_{\alpha}^{C}$	$\mathbb{S}_{(\varrho_a, \varrho_b)}$ with $\emptyset \notin C, \{x\} \in C$	
$\mathbb{F}_{\alpha}^{C}$	$\mathbb{S}_{(\varrho_a, \varrho_b)}^{x}$ with $\emptyset \in C, \{x\} \notin C$	$\mathbb{FS}_{\alpha}^{B}, (\varrho_a, \varrho_b) \notin B \subset \mathbb{E}_1 \times \mathbb{E}_2$
$\mathbb{F}_{\alpha}^{C}$	$\mathbb{S}_{(\varrho_a, \varrho_b)}$ with $\emptyset \in C, \{x\} \notin C$	

Example 3.2:
Let $X = \{x_1, x_2\}$, $\mathbb{E}_1 = \{\varrho_1, \varrho_2\}$, $\mathbb{E}_2 = \{\varrho_3, \varrho_4\}$,
$\mathbb{F}_{0.2}^{A} = \mathbb{F}_{0.2}^{\{x_2\}} = \{(\{x_1\}, 0), (\{x_2\}, 0.2), (\emptyset, 0), (X, 0)\}$,
$\mathbb{F}_{0.2}^{X} = \{(\{x_1\}, 0), (\{x_2\}, 0), (\emptyset, 0), (X, 0.2)\}$,

$$\mathbb{S}_{(\varrho_1, \varrho_3)}^{x_2} = \begin{Bmatrix} ((\varrho_1, \varrho_3), \{x_2\}), ((\varrho_1, \varrho_4), \emptyset), \\ ((\varrho_2, \varrho_3), \{x_1\}), ((\varrho_2, \varrho_4), X) \end{Bmatrix}$$

Then the composite of $\mathbb{F}_{0.2}^{\{x_2\}}$ with $\mathbb{S}_{(\varrho_1, \varrho_3)}^{x_2}$, i.e. $\{x_2\} = A$.

$(\mathbb{F} \circ \mathbb{S})((\varrho_1, \varrho_3)) = \mathbb{F}(\mathbb{S}((\varrho_1, \varrho_3))) = \mathbb{F}(\{x_2\}) = 0.2$

$(\mathbb{F} \circ \mathbb{S})((\varrho_1, \varrho_4)) = \mathbb{F}(\mathbb{S}((\varrho_1, \varrho_4))) = \mathbb{F}(\emptyset) = 0.$

$(\mathbb{F} \circ \mathbb{S})((\varrho_2, \varrho_3)) = \mathbb{F}(\mathbb{S}((\varrho_2, \varrho_3))) = \mathbb{F}(\{x_1\}) = 0$

$(\mathbb{F} \circ \mathbb{S})((\varrho_2, \varrho_4)) = \mathbb{F}(\mathbb{S}((\varrho_2, \varrho_4))) = \mathbb{F}(X) = 0$

That is, $\mathbb{F}_{0.2}^{\{x_2\}}$ o $\mathbb{S}_{(\varrho_1, \varrho_3)}^{x_2} =$

$$\begin{Bmatrix} ((\varrho_1, \varrho_3), 0.2), ((\varrho_1, \varrho_4), 0), \\ ((\varrho_2, \varrho_3), 0), ((\varrho_2, \varrho_4), 0) \end{Bmatrix} = \mathbb{FS}_{0.2}^{(\varrho_1, \varrho_3)}.$$

The composite of $\mathbb{F}_{0.2}^{X}$ with $\mathbb{S}_{(\varrho_1, \varrho_3)}^{x_2}$,

$(\mathbb{F} \circ \mathbb{S})((\varrho_1, \varrho_3)) = \mathbb{F}\left(\mathbb{S}((\varrho_1, \varrho_3))\right) = \mathbb{F}(\{x_2\}) = 0$

$(\mathbb{F} \circ \mathbb{S})((\varrho_1, \varrho_4)) = \mathbb{F}\left(\mathbb{S}((\varrho_1, \varrho_4))\right) = \mathbb{F}(\emptyset) = 0$

$(\mathbb{F} \circ \mathbb{S})((\varrho_2, \varrho_3)) = \mathbb{F}\left(\mathbb{S}((\varrho_2, \varrho_3))\right) = \mathbb{F}(\{x_1\}) = 0$

$(\mathbb{F} \circ \mathbb{S})((\varrho_2, \varrho_4)) = \mathbb{F}\left(\mathbb{S}((\varrho_2, \varrho_4))\right) = \mathbb{F}(X) = 0$

$\#\mathbb{FS} = \{((\varrho_1, \varrho_3), 0), ((\varrho_1, \varrho_4), 0), ((\varrho_2, \varrho_3), 0), ((\varrho_2, \varrho_4), 0)\} = 0(\varrho_i, \varrho_k)$ the null fuzzy soft set.

2) The composition of fuzzy point w.r.to power set $\mathbb{F}_{\alpha}^{A}$ with binary relation-soft point $\mathbb{S}^{x}$ produces the BR-FS-point $\mathbb{FS}_{\alpha}$, as the following;

$$\mathbb{F}_{\alpha}^{A} \circ \mathbb{S}^{x} = \begin{cases} \mathbb{FS}_{\alpha} & \text{if } \{x\} = A \\ 0(\varrho_i, \varrho_k) & \text{if } x \notin A \end{cases}.$$

Example 3.3:
Let $X = \{x_1, x_2\}$, $\mathbb{E}_1 = \{\varrho_1\}$, $\mathbb{E}_2 = \{\varrho_2, \varrho_3\}$
$\mathbb{F}_{0.2}^{\{x_1\}} = \{(\{x_1\}, 0.2), (\{x_2\}, 0), (\emptyset, 0), (X, 0)\}$,
$\mathbb{S}^{x_1} = \{((\varrho_{1\_}, \varrho_2), \{x_1\}), ((\varrho_{1\_}, \varrho_3), \{x_1\})\}$,
$\mathbb{S}^{x_2} = \{((\varrho_{1\_}, \varrho_2), \{x_2\}), ((\varrho_{1\_}, \varrho_3), \{x_2\})\}$
$\mathbb{F}_{0.2}^{\{x_1\}} \circ \mathbb{S}^{x_1} = \{(\varrho_{1\_}, \varrho_2), 0.2), ((\varrho_{1\_}, \varrho_3), 0.2)\} = \mathbb{F}_{0.2}$ because
$\mathbb{F}\left(\mathbb{S}(\varrho_{1\_}, \varrho_2)\right) = \mathbb{F}(\{x_1\}) = 0.2$ and
$\mathbb{F}\left(\mathbb{S}((\varrho_{1\_}, \varrho_3))\right) = \mathbb{F}(\{x_1\}) = 0.2$,

$$\mathbb{FS}_{0.2} = \{((\varrho_1, \varrho_2), 0.2), ((\varrho_1, \varrho_3), 0.2)\}$$

$\mathbb{F}_{0.2}^{\{x_1\}} \circ \mathbb{S}^{x_2} = 0(\varrho_i, \varrho_k)$ the null fuzzy soft set because $\mathbb{F}(\mathbb{S}(\varrho_1, \varrho_2)) = \mathbb{F}(\{x_2\}) = 0$ and $\mathbb{F}(\mathbb{S}(\varrho_1, \varrho_3)) = \mathbb{F}(\{x_2\}) = 0$.

3) The composition of fuzzy point w.r.to power set $\mathbb{F}_{\alpha}^{A}$ with binary relation-soft point $\mathbb{S}_{(\varrho_a, \varrho_b)}$ produces the BR-FS-point $\mathbb{FS}_{\alpha}^{(\varrho_a, \varrho_b)}$, as follows;

$$\mathbb{F}_{\alpha}^{A} \circ \mathbb{S}_{(\varrho_a, \varrho_b)} = \begin{cases} \mathbb{FS}_{\alpha}^{(\varrho_a, \varrho_b)} & \text{if } A = \mathbb{S}(\varrho_a, \varrho_b) \\ 0(\varrho_i, \varrho_k) & \text{if } A \neq \mathbb{S}(\varrho_a, \varrho_b) \end{cases}$$

Example 3.4:
Let $X = \{x_1, x_2\}$, $\mathbb{E}_1 = \{\varrho_1, \varrho_2\}$, $\mathbb{E}_2 = \{\varrho_3\}$
$\mathbb{F}_{0.2}^{\{x_2\}} = \{(\{x_1\}, 0), (\{x_2\}, 0.2), (\emptyset, 0), (X, 0)\}$,,
$\mathbb{S}_{(\varrho_1, \varrho_3)} = \{((\varrho_1, \varrho_3), X), ((\varrho_2, \varrho_3), \emptyset)\}$,
$\mathbb{S}_{(\varrho_2, \varrho_3)} = \{((\varrho_1, \varrho_3), \emptyset), ((\varrho_2, \varrho_3), \{x_2\})\}$,

$\mathbb{F}_{0.2}^{\{x_2\}} \circ \mathbb{S}_{(\varrho_1, \varrho_3)} = 0(\varrho_i, \varrho_k)$ because
$\mathbb{F}(\mathbb{S}((\varrho_1, \varrho_3))) = \mathbb{F}(X) = 0$, and
$\mathbb{F}(\mathbb{S}((\varrho_1, \varrho_3))) = \mathbb{F}(\emptyset) = 0$
$\mathbb{F}_{0.2}^{\{x_2\}} \circ \mathbb{S}_{(\varrho_2, \varrho_3)} = \{((\varrho_1, \varrho_3),0), ((\varrho_2, \varrho_3),0.2)\} = \mathbb{FS}_{0.2}^{(\varrho_2, \varrho_3)}$,
$\mathbb{F}(\mathbb{S}((\varrho_1, \varrho_3))) = \mathbb{F}(\emptyset) = 0$ and
$\mathbb{F}(\mathbb{S}((\varrho_2, \varrho_3))) = \mathbb{F}(\{x_2\}) = 0.2$.

4) The composition of fuzzy point w.r. to power set $\mathbb{F}_\alpha^C$ (where $C \subset P(\mathbb{X})$.) with binary relation – soft point $\mathbb{S}_{(\varrho_a, \varrho_b)}^x$ produces the BR-FS-points, as follows:
   - If $\emptyset \in C$ and $\{x\} \in C$, then $\mathbb{F}_\alpha^C \circ \mathbb{S}_{(\varrho_a, \varrho_b)}^x = \mathbb{FS}_\alpha$.
   - If $\emptyset \in C$ and $\{x\} \notin C$, then $\mathbb{F}_\alpha^C \circ \mathbb{S}_{(\varrho_a, \varrho_b)}^x = \mathbb{FS}_\alpha^B$, for $(\varrho_a, \varrho_b) \in B \subset \mathbb{E}_1 \times \mathbb{E}_2$.
   - If $\emptyset \notin C$ and $\{x\} \in C$, then $\mathbb{F}_\alpha^C \circ \mathbb{S}_{(\varrho_a, \varrho_b)}^x = \mathbb{FS}_\alpha^{(\varrho_a, \varrho_b)}$.
   - If $\emptyset \notin C$ and $\{x\} \notin C$, then $\mathbb{F}_\alpha^C \circ \mathbb{S}_{(\varrho_a, \varrho_b)}^x = 0(\varrho_i, \varrho_k)$.

Example 3.5:
Let $X = \{x_1, x_2\}$, $\mathbb{E}_1 = \{\varrho_1\}$, $\mathbb{E}_2 = \{\varrho_2, \varrho_3\}$, $C_1 = \{(\{x_1\}, \emptyset)\}$, $C_2 = \{(\{x_2\}, X)\}$,
$\mathbb{F}_{0.2}^{C_1} = \{(\{x_1\}, 0.2), (\{x_2\}, 0), (\emptyset, 0.2), (X, 0)\}$,
$\mathbb{F}_{0.2}^{C_2} = \{(\{x_1\}, 0), (\{x_2\}, 0.2), (\emptyset, 0), (X, 0.2)\}$.
$\mathbb{S}_{(\varrho_1, \varrho_2)}^{\{x_1\}} = \{((\varrho_1, \varrho_2), \{x_1\}), ((\varrho_1, \varrho_3), \emptyset)\}$
$\mathbb{S}_{(\varrho_2, \varrho_3)}^{\{x_2\}} = \{((\varrho_1, \varrho_2), \emptyset), ((\varrho_2, \varrho_3), \{x_2\})\}$
$\mathbb{F}_{0.2}^{C_1} \circ \mathbb{S}_{(\varrho_1, \varrho_2)}^{\{x_1\}} = \{((\varrho_1, \varrho_2),0.2), ((\varrho_1, \varrho_3),0.2)\} = \mathbb{FS}_{0.2}$ because
$\mathbb{F}(\mathbb{S}(\varrho_1, \varrho_2)) = \mathbb{F}(\{x_1\}) = 0.2$ and
$\mathbb{F}(\mathbb{S}(\varrho_1, \varrho_3)) = \mathbb{F}(\emptyset) = 0.2$
$\mathbb{F}_{0.2}^{C_2} \circ \mathbb{S}_{(\varrho_1, \varrho_2)}^{\{x_2\}} = \{((\varrho_1, \varrho_2),0.2), ((\varrho_1, \varrho_3),0)\} = \mathbb{FS}_{0.2}^{(\varrho_1, \varrho_2)}$, because
$\mathbb{F}(\mathbb{S}((\varrho_1, \varrho_2))) = \mathbb{F}(\{x_2\}) = 0.2$
and $\mathbb{F}(\mathbb{S}((\varrho_1, \varrho_3))) = \mathbb{F}(\emptyset) = 0$
Also $\mathbb{F}_{0.2}^{C_1} \circ \mathbb{S}_{(\varrho_1, \varrho_2)}^{\{x_2\}} = \{((\varrho_1, \varrho_2),0), ((\varrho_1, \varrho_3),0.2)\} = \mathbb{FS}_{0.2}^B$, $(\varrho_i, \varrho_k) \in B \subset \mathbb{E}_1 \times \mathbb{E}_2$., $B = \{\varrho_1, \varrho_3\}$ because
$\mathbb{F}(\mathbb{S}(\varrho_1, \varrho_2)) = \mathbb{F}(\{x_2\}) = 0$ and $\mathbb{F}(\mathbb{S}(\varrho_1, \varrho_3)) = \mathbb{F}(\emptyset) = 0.2$
$\mathbb{F}_{0.2}^{C_2} \circ \mathbb{S}_{\varrho_1}^{x_1} = \{((\varrho_1, \varrho_2),0), ((\varrho_1, \varrho_3),0)\} = 0(\varrho_i, \varrho_k)$, because
$\mathbb{F}(\mathbb{S}((\varrho_1, \varrho_2))) = \mathbb{F}(\{x_1\}) = 0$ and
$\mathbb{F}(\mathbb{S}(\varrho_1, \varrho_3)) = \mathbb{F}(\emptyset) = 0$.

5) The composition of fuzzy point w.r. to power set $\mathbb{F}_\alpha^C$ (where $C \subset P(\mathbb{X})$) with binary relation – soft point $\mathbb{S}^x$ produces the BR-FS-points $\mathbb{FS}_\alpha^{(\varrho_a, \varrho_b)}$, as follows;

$$\circ\, \mathbb{S}^x = \begin{cases} \mathbb{FS}_\alpha & \text{if } \{x\} \in C \\ 0(\varrho_i, \varrho_k) & \text{if } \{x\} \notin C \end{cases}$$

Example 3.6:
Let $\mathbb{S}^{x_1} = \{((\varrho_1, \varrho_2), \{x_1\}), ((\varrho_1, \varrho_3), \{x_1\})\}$ and $\mathbb{F}_{0.2}^{C_1}, \mathbb{F}_{0.2}^{C_2}$ from example 3.5 we have
$\mathbb{F}_{0.2}^{C_1} \circ \mathbb{S}_{e_1}^{x_2} = \{((\varrho_1, \varrho_2),0.2), ((\varrho_1, \varrho_3),0.2)\} = \mathbb{FS}_{0.2}$, because
$\mathbb{F}(\mathbb{S}(\varrho_1, \varrho_2)) = \mathbb{F}(\{x_1\}) = 0.2$ and
$\mathbb{F}(\mathbb{S}(\varrho_1, \varrho_3)) = \mathbb{F}(\{x_1\}) = 0.2$
$\mathbb{F}_{0.2}^{C_2} \circ \mathbb{S}^{x_1} = \{((\varrho_1, \varrho_2),0), ((\varrho_1, \varrho_3),0)\} = 0(\varrho_i, \varrho_k)$ because
$\mathbb{F}(\mathbb{S}(\varrho_1, \varrho_2)) = \mathbb{F}(\mathbb{S}((\varrho_1, \varrho_3))) = \mathbb{F}(\{x_1\}) = 0$.

6) The composition of fuzzy point w.r. to power set $\mathbb{F}_\alpha^C$ (where $C \subset P(\mathbb{X})$.) with binary relation – soft point $\mathbb{S}_{(\varrho_a, \varrho_b)}$ produces the BR-FS-points, as follows;
   1) If $\emptyset \in C$ and $\mathbb{S}(\varrho_a, \varrho_b) \in C$, then $\mathbb{F}_\alpha^C \circ \mathbb{S}_{(\varrho_a, \varrho_b)} = \mathbb{FS}_\alpha$.
   2) If $\emptyset \in C$ and $\mathbb{S}(\varrho_a, \varrho_b) \notin C$, then $\mathbb{F}_\alpha^C \circ \mathbb{S}_{(\varrho_a, \varrho_b)} = \mathbb{FS}_\alpha^B$, for $(\varrho_a, \varrho_b) \in B \subset \mathbb{E}_1 \times \mathbb{E}_2$.
   3) If $\emptyset \notin C$ and $\mathbb{S}(\varrho_a, \varrho_b) \in C$, then $\mathbb{F}_\alpha^C \circ \mathbb{S}_{(\varrho_a, \varrho_b)} = \mathbb{FS}_\alpha^{(\varrho_a, \varrho_b)}$.
   4) If $\emptyset \notin C$ and $\mathbb{S}(\varrho_a, \varrho_b) \notin C$, then $\mathbb{F}_\alpha^C \circ \mathbb{S}_{(\varrho_a, \varrho_b)} = 0(\varrho_i, \varrho_k)$.

Example 3.7:
Let $\mathbb{S}_{(\varrho_1, \varrho_2)} = \{((\varrho_1, \varrho_2), X), ((\varrho_1, \varrho_3), \emptyset)\}$ and $\mathbb{S}_{(\varrho_1, \varrho_3)} = \{((\varrho_1, \varrho_2), \emptyset), ((\varrho_1, \varrho_3), X)\}$ and take $\mathbb{F}_{0.2}^{C_1}$ and $\mathbb{F}_{0.2}^{C_2}$ from example 3.5 so we have
$\mathbb{F}_{0.2}^{C_1} \circ \mathbb{S}_{(\varrho_1, \varrho_2)} = \{((\varrho_1, \varrho_2), 0), ((\varrho_1, \varrho_3), 0.2)\} = \mathbb{FS}_{0.2}^B$, $B = \{\varrho_1, \varrho_3\}$, because
$\mathbb{F}(\mathbb{S}((\varrho_1, \varrho_2))) = \mathbb{F}(X) = 0$ and $\mathbb{F}(\mathbb{S}((\varrho_1, \varrho_3))) = \mathbb{F}(\emptyset) = 0.2$
$\mathbb{F}_{0.2}^{C_2} \circ \mathbb{S}_{(\varrho_1, \varrho_2)} = \{((\varrho_1, \varrho_2),0.2), ((\varrho_1, \varrho_3),0)\} = \mathbb{FS}_{0.2}^{C_1}$ because
$\mathbb{F}(\mathbb{S}((\varrho_1, \varrho_2))) = \mathbb{F}(X) = 0.2$ and
$\mathbb{F}(\mathbb{S}((\varrho_1, \varrho_3))) = \mathbb{F}(\emptyset) = 0.2$
$\mathbb{F}_{0.2}^{C_1} \circ \mathbb{S}_{(\varrho_1, \varrho_3)} = \{((\varrho_1, \varrho_2),0.2), ((\varrho_1, \varrho_3),0)\} = \mathbb{FS}_{0.2}^B$, $B = \{\varrho_1, \varrho_2\}$ because
$\mathbb{F}(\mathbb{S}((\varrho_1, \varrho_2))) = \mathbb{F}(\emptyset) = 0.2$ and
$\mathbb{F}(\mathbb{S}((\varrho_1, \varrho_3))) = \mathbb{F}(X) = 0$
$\mathbb{F}_{0.2}^{C_2} \circ \mathbb{S}_{(\varrho_1, \varrho_3)} = \{((\varrho_1, \varrho_2),0), ((\varrho_1, \varrho_3),0.2)\} = \mathbb{FS}_{0.2}^{(\varrho_1, \varrho_3)}$ because

$\mathbb{F}(\mathbb{S}((\varrho_1, \varrho_2))) = \mathbb{F}(\emptyset) = 0$ and $\mathbb{F}(\mathbb{S}((\varrho_1, \varrho_3))) = \mathbb{F}(\{x_2\}) = 0.2$.

7) The composition of fuzzy point w.r. to power set $\mathbb{F}_\alpha$ with binary relation – soft point $\mathbb{S}^{x}_{(\varrho_a, \varrho_b)}$ produces the BR-FS-points $\mathbb{FS}_\alpha$, as follows;

Example 3.8:
Let $X = \{x_1, x_2, x_3\}$, $\mathbb{E}_1 = \{\varrho_1, \varrho_2\}, \mathbb{E}_2 = \{\varrho_3, \varrho_4\}$, and let:

$$\mathbb{F}_{0.1} = \begin{Bmatrix} (\{x_1\}, 0.1), (\{x_2\}, 0.1), (\{x_3\}, 0.1), (\{x_1, x_2\}, 0.1), \\ (\{x_1, x_3\}, 0.1), (\{x_2, x_3\}, 0.1), (\emptyset, 0.1), (X, 0.1) \end{Bmatrix}$$

$$\mathbb{S}^{x_2}_{(\varrho_1, \varrho_4)} = \begin{Bmatrix} ((\varrho_1, \varrho_3), \emptyset), ((\varrho_1, \varrho_4), \{x_2\}), \\ ((\varrho_2, \varrho_3), \emptyset), ((\varrho_2, \varrho_4), \emptyset) \end{Bmatrix}.$$

Then $\mathbb{F}_{0.1} \circ \mathbb{S}^{x_2}_{(\varrho_1, \varrho_4)} = \mathbb{FS}_{0.1}$, because:

$$\mathbb{F}\left(\mathbb{S}((\varrho_1, \varrho_3))\right) = \mathbb{F}(\emptyset) = \mathbb{F}\left(\mathbb{S}((\varrho_1, \varrho_4))\right)$$
$$= \mathbb{F}(\{x_2\}) = \mathbb{F}\left(\mathbb{S}((\varrho_2, \varrho_3))\right)$$
$$= \mathbb{F}(\emptyset) = \mathbb{F}(\mathbb{S}(\varrho_2, \varrho_4))$$
$$= \mathbb{F}(\{x_2, x_3\}) = 0.1$$

Thus, $\mathbb{F}_{0.1} \circ \mathbb{S}^{x_2}_{(\varrho_1, \varrho_4)} =$
$\{((\varrho_1, \varrho_3), 0.1), ((\varrho_1, \varrho_4), 0.1),$
$((\varrho_2, \varrho_3), 0.1), ((\varrho_2, \varrho_4), 0.1)\} = \mathbb{FS}_{0.1}$

8) The composition of fuzzy point w.r. to power set $\mathbb{F}_\alpha$ with binary relation – soft point $\mathbb{S}^{x}$ produces the BR-FS-points $\mathbb{FS}_\alpha$, as follows;

Example 3.9:
Let $\mathbb{S}^{\{x_1, x_3\}} =$
$\{((\varrho_1, \varrho_3), \{x_1, x_3\}), ((\varrho_1, \varrho_4), \{x_1, x_3\}),$
$((\varrho_2, \varrho_3), \{x_1, x_3\}), ((\varrho_2, \varrho_4), \{x_1, x_3\})\}$ and $\mathbb{F}_{0.1}$ from example 3.8 we have
$\mathbb{F}_{0.1} \circ \mathbb{S}^{\{x_1, x_3\}} = \mathbb{FS}_{0.1}$, because $\mathbb{F}(\mathbb{S}(\varrho_1, \varrho_3)) = \mathbb{F}(\mathbb{S}(\varrho_1, \varrho_4)) = \mathbb{F}(\mathbb{S}(\varrho_2, \varrho_3)) = \mathbb{F}(\mathbb{S}(\varrho_2, \varrho_4)) = \mathbb{F}(\{x_1, x_3\}) = 0.1$.

9) The composition of fuzzy point w.r. to power set $\mathbb{F}_\alpha$ with binary relation – soft point $\mathbb{S}^{x}_{(\varrho_a, \varrho_b)}$ produces the BR-FS-points $\mathbb{FS}_\alpha$, as follows;

Example 3.10:
Let:

$$\mathbb{S}_{(\varrho_2, \varrho_3)} = \begin{Bmatrix} ((\varrho_1, \varrho_3), \emptyset), ((\varrho_1, \varrho_4), \emptyset), \\ ((\varrho_2, \varrho_3), \{x_2, x_3\}), ((\varrho_2, \varrho_4), \emptyset) \end{Bmatrix},$$

and $\mathbb{F}_{0.1}$ take from example 3.8, then the composition as follows:
$\mathbb{F}_{0.1} \circ \mathbb{S}_{(\varrho_2, \varrho_4)} = \mathbb{FS}_{0.1}$,

because $\mathbb{F}\left(\mathbb{S}((\varrho_1, \varrho_3))\right) = \mathbb{F}\left(\mathbb{S}((\varrho_1, \varrho_4))\right) = \mathbb{F}\left(\mathbb{S}((\varrho_2, \varrho_4))\right) = \mathbb{F}(\emptyset) = 0.1$ and $\mathbb{F}(\{x_2, x_3\}) = 0.1$, thus,

$$\mathbb{F}_{0.1} \circ \mathbb{S}_{(\varrho_2, \varrho_3)} = \begin{Bmatrix} ((\varrho_1, \varrho_3), 0.1), ((\varrho_1, \varrho_4), 0.1), \\ ((\varrho_2, \varrho_3), 0.1), ((\varrho_2, \varrho_4), 0.1) \end{Bmatrix}$$

$= \mathbb{FS}_{0.1}$

4 APPLICATION

It is possible to apply our new work in the field of decision-making by considering Let's assume that of binary relation-fuzzy soft (BR-FS) points types, as the following; represent the characteristics of a group of employees. A manager with the required qualities should be chosen for those employees, and the examples above illustrate this.

This method facilitates the selection and decision-making process when choosing the required specifications for anything, such as buying a car or a house, etc.

5 CONCLUSIONS

In this paper, we introduced the concept of binary relation fuzzy soft points (BR-FS-points) by combining binary relation soft points with fuzzy points with respect to a power set. The proposed construction generates several new types of BR-FS-points, illustrated through representative examples, and provides a unified framework that links fuzzy, soft, and binary relation structures.

The obtained results show that BR-FS-points can effectively represent complex uncertainty and can be used in fuzzy soft topological structures, including separation axioms. Their structure makes them suitable for practical decision-making problems, where multi-parameter uncertainty must be handled in a systematic way.

Future work will focus on studying continuity in BR-FS topological spaces and extending the proposed framework to applications in image processing, particularly for medical diagnostics such as brain tumor detection and segmentation.

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