On Soft P Beta Alpha Open Set in Soft Topological Space

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Abstract: In this work, a new class of soft open sets, called soft pβα-open sets, is introduced and several fundamental properties and characterizations of this concept are established. The relationships between soft pβα-open sets and other well-known classes of soft open sets are also investigated. In addition, the notion of soft pβα-closed sets is introduced and studied through various classifications and properties. The paper also reviews several basic concepts in soft set theory and soft topological spaces that are required for the development of the proposed results. Definitions and operations related to soft sets, including union, intersection, closure, and interior operators, are considered within the framework of soft topology. Furthermore, the role of dense soft sets and their associated topological properties is examined. The obtained results provide a broader understanding of generalized soft open and closed sets and clarify their relationships with existing concepts in soft topology. It is expected that these findings may contribute to further developments and applications of soft topological structures in mathematical analysis, uncertainty modeling, and decision-making problems.

1 INTRODUCTION

The study of uncertainty has become an essential topic in modern mathematics because many practical problems in engineering, economics, computer science, medicine, and information systems involve incomplete or imprecise information. Several mathematical approaches have been proposed to treat uncertainty, including fuzzy sets, rough sets, intuitionistic fuzzy sets, and soft sets. Among these models, soft set theory introduced by Molodtsov [1] has attracted considerable attention due to its simplicity and flexibility in handling parameterized uncertainty. Later, Molodtsov et al. [2] extended the theory and discussed several applications of soft sets in different scientific areas.

Maji et al. [3] developed the fundamental operations of soft sets and established basic notions such as null soft sets, absolute soft sets, unions, intersections, and complements. Further algebraic and operational properties of soft sets were studied by Ali et al. [18] and Feng et al. [19]. Applications of soft set theory to optimization and decision-making problems were investigated by Agman and Enginoglu [4].

The introduction of soft topological spaces by Shabir and Naz [6] opened a broad area of research connecting soft set theory with general topology. Their work initiated the study of soft open sets, soft closed sets, closure and interior operators, continuity, compactness, and separation axioms in the setting of soft topology. Additional properties and structures of soft topological spaces were later investigated by Hussain and Ahmad [20].

In recent years, many researchers have focused on generalized forms of soft openness and soft closedness. Chen [7] introduced soft semi-open sets and examined their properties. Akdag and Ozkan [8], [10] investigated soft preopen sets, soft b-open sets, and associated continuity concepts. Jamil [9] studied soft preopen sets by using γ-operations and obtained several related results. Moreover, Kharal and Ahmad [11] analyzed mappings on soft classes and discussed image and inverse image operations in soft structures.

Generalized soft closed sets have also been studied extensively. Arockiarani and Arokialancy [21] introduced generalized soft gβ-closed sets and soft gsβ-closed sets, while Khattak et al. [22] investigated soft α-separation and β-separation axioms. Hussain [23] studied properties of soft semi-open and soft semi-closed sets, and Kannan [24] discussed generalized closed sets in soft topological spaces.

Besides soft topology, several related generalized topological structures have been developed in recent years. Tawfeeq [12] studied topological Γ-rings, while Jasem and Tawfeeq [13] introduced a new class of topological manifolds. Further investigations concerning Γ-compact and Γ-closed spaces were presented in [14], and separation axioms in topological gamma-rings were studied in [15]. In fuzzy and nano topological settings, Jamil et al. [16] introduced fuzzy SWT-open sets in double fuzzy topological spaces, whereas Mahdi and Jamil [17] investigated nano contra βpc-continuous functions.

The rapid development of generalized soft topological concepts motivates the search for new classes of soft open sets that may provide stronger connections between existing structures and generate additional topological properties. This motivates the introduction of another generalized class based on the interaction between α-closure and β-interior operators within soft topological spaces.

2 PRELIMINARIES

This section presents the fundamental definitions and theoretical concepts related to soft set theory and soft topological spaces that are used throughout the paper.

Definition 2.1 [1]: Assume Q initial universal, V be the parameters. Assume $P_w(Q)$ labeled as the power of Q, let $D \subseteq V$. A pair (L, D) is named soft of Q where L is function given by $L: D \to P_w(Q)$.

Remark 2.2: Let Q be a universe set with a parameter set D, let $|Q| = n$; $|D| = m$ then $|(L_i, D)| = (2^n)^m$.

Definition 2.3 [3]: A soft (L, D) over universe set Q is named a null soft is expressed as $\tilde{\phi}$, if $r \in D$,then $L(r) = \phi$.

Definition 2.4 [3]: A soft (L, D) over universe set Q is named an absolute soft is labeled as $\tilde{Q}$, if $r \in D, L(r) = Q$.

Definition 2.5 [3]: The union of soft (L, D), (H, N) over universe set Q is soft (G, K) where $K = D \cup N$,

$$G(r) = \begin{cases} L(r) & if\ r \in D - N \\ H(r) & if\ r \in N - D \\ L(r) \cup H(r) & if\ r \in D \cap N \end{cases}.$$

That is $(L, D) \cup (H, N) = (G, K)$.

Definition 2.6 [3]: The union of soft set (L, D), (H, D) for S/T/S $(Q, \tilde{\tilde{\tau}}, D)$ is soft (G, D) for which $G(r) = L(r) \tilde{\cup} H(r), r \in D$.

Definition 2.7 [18]: The intersection of soft set (L, D), (H, N) for S/T/S$(Q, \tilde{\tilde{\tau}}, D)$ symbolled by $(L, D) \tilde{\cap} (H, N)$ which is represented by $K = D \cap N$,and $G(r) = L(r) \cap H(r), r \in D$.

Definition 2.8 [18]: The complement of soft set (L, D) is symbolled by $(L, D)^c$ and is defined by $(L, D)^c = (L^c, D)$ where $L^c: D \to P_w(Q)$ is mapping given $L^c(e) = Q - L(e)$ for each $e \in D$.

Definition 2.9 [19]**:** Let (L, D) and (H, N) be soft sets over universe set Q, $(L, D) \tilde{\subseteq} (H, N)$ if $D \subseteq N$ and $L(r) \subseteq H(r), r \in D$.

Definition 2.10 [19]: Let (L, D) and (H, D) be two soft sets over universe set Q, $(L, D) \subseteq (H, D), if\ L(r) \subseteq H(r), r \in D$.

Definition 2.11 [6]: Let $\tilde{\tilde{\tau}}$ be the collection of soft over Q, $\tilde{\tilde{\tau}}$ is named by soft topology if satisfying the preceding conditions:

1) $\tilde{\phi}, \tilde{X} \in \tilde{\tilde{\tau}}$
2) The union of any number of soft in $\tilde{\tilde{\tau}}$ belongs to $\tilde{\tilde{\tau}}$.
3) The intersection of any two soft in $\tilde{\tilde{\tau}}$ belongs to $\tilde{\tilde{\tau}}$.

Definition 2.12 [6]**:** Let $(Q, \tilde{\tilde{\tau}}, D)$ be S/T/S, then each element of $\tilde{\tilde{\tau}}$ is known as soft open. A soft set (L, D) over S/T/S$(Q, \tilde{\tilde{\tau}}, D)$ is labeled as soft closed set if $(L, D)^c = (L^c, D) \in \tilde{\tilde{\tau}}$.

Definition 2.13: Let (L, D) be soft set in S/T/S$(Q, \tilde{\tilde{\tau}}, D)$ then:

1) [6]$int(L, D) = \tilde{\cup} \{(H, D): (H, D) \in \tilde{\tilde{\tau}}\ ; (H, D) \subseteq (L, D)\}$;
2) [6] $cl(L, D) = \tilde{\cap} \{(H, D) \in cl(\tilde{Q}); (L, D) \subseteq (H, D)\}$.

Proposition 2.14 [20]**:** Let (L, D) and (H, D) be two soft sets over S/T/S$(Q, \tilde{\tilde{\tau}}, D)$, then:

1) $sint(sint(L, D)) = (L, D)$.
2) If $(L, D) \subseteq (H, D), sint(L, D) \subseteq sint(H, D)$.
3) $sint(L, D) \cap sint(H, D) = sint((L, D) \cap (H, D)$.
4) $sint(L, D) \cup sint(H, D) \subseteq sint[(L, D) \cup (H, D)]$.

Proposition 2.15 [6]: Let $(Q, \tilde{\tilde{\tau}}, D)$ be S/T/S and $(L, D), (H, D)$ be a soft subset of $\tilde{Q}$, then:

1) $cl(cl$L, D=$cl(L$,$D)$.
2) $If L$, D⊆H,D, then $cl(L$,$D)$⊆$cl(H$,$D)$.
3) $cl((L$,$D)$∩$(H$,$D)$⊆$cl(L$,$D)$∩$cl(H$,$D)$
4) clL,D∪clH,D=$cl[L$,D∪H,$D]$.

Definition 2.16: A soft set (L, D) of S/T/S $(Q, \tilde{\tilde{\tau}}, D)$ is named:

1) Soft semi open [7], if $(L, D) \subseteq cl\ int\ (L, D)$.
2) Soft preopen [21], if $(L, D) \subseteq int\ cl(L, D)$.
3) Soft α-open [22], if $(L, D) \subseteq int\ cl\ int\ (L, D)$.
4) Soft β-open [22], if $(L, D) \subseteq cl\ int\ cl(L, D)$.
5) Soft R.O [22], if $(L, D) = int\ cl\ (L, D)$.

Definition 2.17: A soft set (L, D) of S/T/S$(Q, \tilde{\tilde{\tau}}, D)$ is named soft semi-closed [23], soft pre closed [3], soft α-closed [6], soft β-closed [10], and regular closed [3], if its complement is soft –semi open, soft preopen, soft α-open, soft β-open and soft regular-open [24]. Figure 1 represented the relations among these notions.

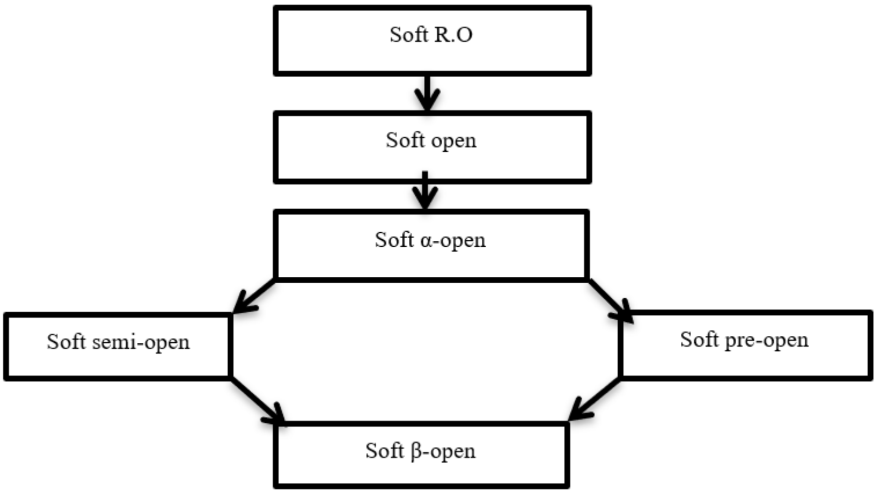

Figure 1: Conceptual model illustrating the relationships among these notions.

Definition 2.18: Let (L, D) be soft set in S/T/S$(Q, \tilde{\tilde{\tau}}, D)$ then,

1) $s\beta int(L, D) = \tilde{\cup}\{(H, D) \in s\beta o(Q, \tilde{\tilde{\tau}}, D); (H, D) \subseteq (L, D)\}$.
   $s\beta cl(L, D) = \tilde{\cap}\{(H, D) \in s\beta c(Q, \tilde{\tilde{\tau}}, D); (L, D) \subseteq (H, D)\}$.
2) $s\alpha int(L, D) = \tilde{\cup}\{(H, D) \in s\alpha o(Q, \tilde{\tilde{\tau}}, D); (H, D) \subseteq (L, D)\}$.
3) $s\alpha cl(L, D) = \tilde{\cap}\{(H, D) \in s\alpha c(Q, \tilde{\tilde{\tau}}, D); (L, D) \subseteq (H, D)\}$.

Properties 2.19: Let $(L, D), (H, D)$ be two soft sets over S/T/S$(Q, \tilde{\tilde{\tau}}, D)$, then:

1) $(\alpha int\ (L, D))^c = \alpha cl\ (L, D)^c$.
2) $(\alpha cl\ (L, D))^c = \alpha int\ (L, D)^c$
3) If $(L, D) \subseteq (H, D)$, then $\alpha cl\ (L, D) \subseteq \alpha cl\ (H, D)$ and $\alpha int\ (L, D) \subseteq \alpha int\ (H, D)$.
4) $\alpha cl\ (\tilde{\emptyset}) = \tilde{\emptyset}, \alpha cl\ (\tilde{Q}) = \tilde{Q}$
5) $\alpha cl\ (L \cup H, D) = \alpha cl\ ((L, D) \cup (H, D)) = \alpha cl\ (L, D) \cup \alpha cl\ (H, D)$.
6) $\alpha cl\ ((L, D) \cap (H, D)) \subseteq \alpha cl(L, D) \cap \alpha cl(H, D)$.
7) $\alpha int\ ((L, D) \cap (H, D)) = \alpha int\ (L, D) \cap \alpha int\ (H, D)$
8) $\alpha int\ (L, D) \cup \alpha int(H, D) \subseteq \alpha int\ ((L, D) \cup (H, D))$.
9) $\beta cl\ (L, D) \cup \beta cl(H, D) \subseteq \beta cl\ ((L, D) \cup (H, D))$.
10) $\beta int\ ((L, D) \cap (H, D) \subseteq \beta int\ (L, D) \cap \beta int\ (H, D)$.

Definition 2.20: Let $(Q, \tilde{\tilde{\tau}}, D)$ and $(S, \tilde{\tilde{\tau}}^*, N)$ be S/T/S and let $P: D \to N$; $U: Q \to S$, then the mapping $f_{PU}: (Q, \tilde{\tilde{\tau}}, D) \to (S, \tilde{\tilde{\tau}}^*, N)$. The image of soft set (L, D) of S/T/S$(Q, \tilde{\tilde{\tau}}, D)$ under f_{PU} written as follows $f_{PU}(L, D) = (f_{PU}(L), P(D))$ is soft set of $(S, \tilde{\tilde{\tau}}^*, N)$:

$$f_{PU}(L)(S) = \begin{cases} \bigcup_{Q \in P^{-1}(S) \cap D} u(L(Q)) & if\ \ P^{-1}(S) \cap D \neq \phi \\ \phi & o.w \end{cases}$$

The inverse image of soft set (H, N) is:

$$f^{-1}(H)(Q) = \begin{cases} u^{-1}(H(P(Q))) & if\ \ P(Q) \in N \\ \phi & o.w \end{cases}$$

Definition 2.21: Let $(Q, \tilde{\tilde{\tau}}, D)$ be S/T/S and let S be a non –empty set of Q then the soft subset (L, D) over S denoted by (L_S, D) is defined for which $L_S(e) = S \cap L(e)$, for all $e \in D$.consequently $(L_S, D) = S \cap (L, D)$.

Definition 2.22: Let $(Q, \tilde{\tilde{\tau}}, D)$ be S/T/S and let S be a non – empty subset of Q, then $\tilde{\tilde{\tau}}_S = \{(L_S, D); (L, D) \in \tilde{\tilde{\tau}}\}$ is named soft relative topology,$(S, \tilde{\tilde{\tau}}_S, D)$ is known as soft subspace of $(Q, \tilde{\tilde{\tau}}, D)$.

Proposition 2.23: Let $(S, \tilde{\tilde{\tau}}_S, D)$ be a soft subspace of S/T/S$(Q, \tilde{\tilde{\tau}}, D)$ and let (L, D) be soft for $\tilde{Q}$,

1) $(L, D) \in \tilde{\tilde{\tau}}_S$ iff $(L, D) = S \cap (H, D)$, for some $(H, D) \in \tilde{\tilde{\tau}}$.
2) $(L, D) \in sc(S, \tilde{\tilde{\tau}}_S, D)$ iff $(L, D) = S \cap (H, D)$,for some $(H, D) \in sc(Q, \tilde{\tilde{\tau}}, D)$

Proposition 2.24: Let $(Q, \tilde{\tilde{\tau}}, D)$ be S/T/S and assume (L, D) be soft for Q,

1) $int(L, D) \subseteq \alpha int\ (L, D) \subseteq pint\ (L, D) \subseteq \beta int\ (L, D)$.

2) $\beta cl\ (L,D) \subseteq pcl\ (L,D) \subseteq \alpha cl\ (L,D) \subseteq cl(L,D)$.

Proposition 2.25: Let $(S,\tilde{\tilde{\tau}}_S,D)$ be a soft subspace of S/T/S$(Q,\tilde{\tilde{\tau}},D)$ and let (H,D) be soft set over S, then $\beta cl_S(H,D) = \beta cl_Q(H,D) \cap (S,D)$ and $cl_S(H,D) = cl_Q(H,D) \cap (S,D)$.

Definition 2.26: A S/T/S$(Q,\tilde{\tilde{\tau}},D)$ is extremely disconnected if soft closure of every soft open is soft open set.

Definition 2.27: Let $(Q,\tilde{\tilde{\tau}},D)$ be S/T/S and S be a non- empty subset of Q,then soft subspace $(S,\tilde{\tilde{\tau}}_S,D)$ represented as following $(H,D)_S \in \tilde{\tilde{\tau}}_S$ if and only if there is $(H,D) \in \tilde{\tilde{\tau}}$ for which $(H,D)_S = (H,D) \cap S$.

Proposition 2.28: Let $(L,D) \subseteq (S,\tilde{\tilde{\tau}}_S,D)$ and $(S,\tilde{\tilde{\tau}}_S,D)$ is soft subspace of $(Q,\tilde{\tilde{\tau}},D)$, then $(cl_\beta)_S(L,D) = (cl_\beta)_Q(L,D) \cap S$.

Proposition 2.29: In soft extremely disconnected, every $s\beta o -$ set is soft open set.

Proposition 2.30: Let $(Q,\tilde{\tilde{\tau}},D)$ be S/T/S and let (L,D) and (H,D) are any two soft sets in S/T/S$(Q,\tilde{\tilde{\tau}},D)$,if $(L,D) \in s\alpha o(Q,\tilde{\tilde{\tau}},D)$, then $(L,D) \cap \alpha cl(H,D) \subseteq \alpha cl((L,D) \cap (H,D))$.

Proposition 2.31: If $(L,D) \in \tilde{\tilde{\tau}}$ and $(H,D) \in s\beta o(Q,\tilde{\tilde{\tau}},D)$, then $(L,D) \cap (H,D) \in s\beta o(Q,\tilde{\tilde{\tau}},D)$.

3 SOFT $p\beta\alpha$ - OPEN SETS

In this section, a new class of generalized soft open sets, called soft pβα-open sets, is introduced in a soft topological space (S/T/S). This notion is defined by combining the concepts of α-closure and β-interior operators within the framework of soft topology. The proposed class extends several previously studied types of soft open sets, including soft β-open and soft α-open sets, and establishes new relationships among them.

The fundamental properties of soft pβα-open sets are investigated, including characterizations and counterexamples. Their behavior under standard set operations such as arbitrary unions and finite intersections is also studied. In addition, relationships with soft β-open sets, soft preopen sets, and soft α-open sets are analyzed, as well as their role in special types of soft topological spaces such as sβα-hyperconnected and extremely disconnected soft spaces.

Finally, the dual notion of soft pβα-closed sets is introduced and their structural properties are examined in relation to soft interior and closure operators.

Definition 3.1: A soft (L,D) over S/T/S$(Q,\tilde{\tilde{\tau}},D)$ is labeled as $sp\beta\alpha o -$ set if $(L,D) \subseteq \beta int(\alpha cl(L,D)$. The family soft $p\beta\alpha -$ open sets is symbolled by $sp\beta\alpha o(Q,\tilde{\tilde{\tau}},D)$.

Proposition 3.2: Let (L,D) be soft set over S/T/S$(Q,\tilde{\tilde{\tau}},D)$. If $(L,D) \in s\beta o(Q,\tilde{\tilde{\tau}},D)$, then $(L,D) \in sp\beta\alpha o(Q,\tilde{\tilde{\tau}},D)$.

Proof: Since $(L,D) \subseteq \alpha cl\ (L,D)$, then $\beta int\ (L,D) \subseteq \beta int[\alpha cl(L,D)]$, by assumption $(L,D) \in s\beta o(Q,\tilde{\tilde{\tau}},D)$ then $(L,D) \subseteq \beta int\ (L,D) \subseteq \beta int[\alpha cl(L,D)]$.

Corollary 3.3: Let (L,D) be soft set over S/T/S$(Q,\tilde{\tilde{\tau}},D)$ if $(L,D) \in \tilde{\tilde{\tau}}$, $spo(Q,\tilde{\tilde{\tau}},D)$, then $(L,D) \in sp\beta\alpha o(Q,\tilde{\tilde{\tau}},D)$.

Proof: since $(L,D) \subseteq \alpha cl(L,D)$, then $\beta int(L,D) \subseteq \beta int[\alpha cl(L,D)]$,by assumption $(L,D) \in spo(Q,\tilde{\tilde{\tau}},D)$ and every spo −set is $s\beta o$ − set → $(L,D) \in s\beta o(Q,\tilde{\tilde{\tau}},D)$,by proposition 3.2, then $(L,D) \subseteq \beta int(L,D) \subseteq \beta int[\alpha cl(L,D)]$.Hence $(L,D) \in sp\beta\alpha o(Q,\tilde{\tilde{\tau}},D)$.

The opposing of proposition 3.2, may not be valid. As seeing in the preceding example.

Example 3.4: Let $Q = \mathbb{R}$ and $\tilde{\tilde{\tau}} = \{(L,D): L(e) \in \tau_u$ for every $e \in D\}$
Consider(H,D),

$$H(e) = The\ set\ of\ rational\ numbers.$$

Clearly, $\beta int\ (H,D) = \widetilde{\emptyset}$ and since $\alpha cl\ (H,D) = cl\ int\ cl\ (H,D) = cl\ int\ (\tilde{R}) = cl(\tilde{R}) = R$.
Thus $(H,D) \subseteq \beta int[\alpha cl(H,D)$.

Consequently, $(H,D) \in sp\beta\alpha o(Q,\tilde{\tilde{\tau}},D)$ is $sp\beta\alpha o -$ set, but $(H,D) \notin s\beta o(Q,\tilde{\tilde{\tau}},D)$. [Because not every $s\alpha o$ −set is a so −set and is $s\beta o$ −set but it is not soft open].

Proposition 3.5: A soft set (L,D) over S/T/S$(Q,\tilde{\tilde{\tau}},D)$ is $sp\beta\alpha o -$ set if and only if there is $(H,D) \in s\beta o(Q,\tilde{\tilde{\tau}},D)$ for which $(L,D) \subseteq (H,D) \subseteq \alpha cl(L,D)$.

Proof: Suppose that $(L,D) \in sp\beta\alpha o(Q,\tilde{\tilde{\tau}},D)$, then $(L,D) \subseteq \beta int[\alpha cl(L,D)]$.

Assume that $(H,D) = \beta int[\alpha cl(L,D)] \in s\beta o(Q,\tilde{\tilde{\tau}},D)$ then $(L,D) \subseteq (H,D) = \beta int[\ \alpha cl(L,D)] \subseteq \alpha cl\ (L,D)$.

Conversely, suppose that there is $(H,D) \in s\beta o(Q,\tilde{\tilde{\tau}},D)$ for which $(L,D) \subseteq (H,D) \subseteq \alpha cl\ (L,D)$

Since $(H,D) \subseteq \alpha cl\ (L,D)$,then $(H,D) = \beta int\ (H,D) \subseteq \beta int[\alpha cl(L,D)]$

Consequently, $(L,D) \subseteq (H,D) \subseteq \beta int\ [\alpha cl(L,D)]$. Hence $(L,D) \in sp\beta\alpha o(Q,\tilde{\tilde{\tau}},D)$.

Definition 3.6: A S/T/S$(Q,\tilde{\tilde{\tau}},D)$ is named $s\beta\alpha$ – Hyperconnected if every non – empty $s\beta o -$ set is α – dense.

Proposition 3.7: For $s\beta\alpha$-hyperconnected space $(Q,\tilde{\tilde{\tau}},D)$,the only $sp\beta\alpha o$ – sets are $\widetilde{\emptyset}$ and $\tilde{Q}$ α-dense.

Proof: Let (L, D) be soft subset of S/T/S$(Q, \tilde{\tilde{\tau}}, D)$ if $(L, D) = \widetilde{\emptyset}$,then by proposition 3.2,$(L, D) \in sp\beta\alpha o(Q, \tilde{\tilde{\tau}}, D)$. Otherwise, $(L, D) \neq \widetilde{\emptyset}$ and since $(L, D) \in sp\beta\alpha o(Q, \tilde{\tilde{\tau}}, D)$,then $\beta int[\,\alpha cl(L, D)] \neq \emptyset$. Set $(H, D) = \beta int[\,\alpha cl(L, D)]\; s\beta o$ −set in $s\beta\alpha -$ Hyperconnected space $(Q, \tilde{\tilde{\tau}}, D)$, then $\alpha cl(H, D) = \tilde{Q}$.Since $(H, D) \subseteq \alpha cl(L, D)$,then $\tilde{Q} = \alpha cl(H, D) \subseteq \alpha cl(L, D)$. Hence (L, D) is α- dense.

Proposition 3.8: In $s\beta\alpha$-Hyperconnected space $(Q, \tilde{\tilde{\tau}}, D)$, $s\alpha$- interior of every proper $s\beta$-closed is null soft.

Proof: Assume $(L, D) \in s\beta o(Q, \tilde{\tilde{\tau}}, D)$, then by complement $(L, D)^c = (L^c, D) = Q - (L, D) \in s\beta c(Q, \tilde{\tilde{\tau}}, D)$ is $s\beta c$ −set and $\alpha int\left(Q - (L, D)\right) = \tilde{Q} - \alpha cl(L, D) = \tilde{Q} - \tilde{Q} = \widetilde{\emptyset}$. [by definition 3.6].

Proposition 3.9: In soft extremely disconnected $(Q, \tilde{\tilde{\tau}}, D)$, the following are equivalent

1) $(L, D) \in spo(Q, \tilde{\tilde{\tau}}, D)$;
2) (L, D) is $sp\beta\alpha o -$ set.

Proof: (1)→(2) it is clear by corollary 3.3; (2)→ (1) since $(L, D) \in sp\beta\alpha o(Q, \tilde{\tilde{\tau}}, D)$, then there is $(H, D) \in s\beta o(Q, \tilde{\tilde{\tau}}, D)$ for which $(L, D) \subseteq (H, D) \subseteq \alpha cl(L, D)$. Since $(Q, \tilde{\tilde{\tau}}, D)$ is soft extremely disconnected, then by proposition 2.29, $(H, D) \in \tilde{\tilde{\tau}}$ and since $(L, D) \subseteq (H, D) \subseteq \alpha cl(L, D) \subseteq cl(L, D)$ where $(H, D) \in \tilde{\tilde{\tau}}$.

Hence $(L, D) \in spo(Q, \tilde{\tilde{\tau}}, D)$.

Proposition 3.10: Arbitrary union of $sp\beta\alpha o -$ sets is also $sp\beta\alpha o$ −set.

Proof: Let $(H_i, D) \in sp\beta\alpha o(Q, \tilde{\tilde{\tau}}, D)$ for $i \in I = \{1,2,3, \ldots\}$ then for $i \in I, (L_i, D) \subseteq \beta int[\alpha cl(L_i, D)] \rightarrow \bigcup_{i\in I}(L_i\,, D) \subseteq \bigcup_{i\in I} \beta int\,[\alpha cl(L_i\,, D)] = \beta int\,(\bigcup_{i\in I} \alpha cl(L_i\,, D)) \subseteq \beta int\,(\alpha cl\, \bigcup_{i\in I}(L_i\,, D))$.

Hence $\bigcup_{i\in I}(L_i\,, D) \in sp\beta\alpha o(Q, \tilde{\tilde{\tau}}, D)$.

Proposition 3.11: Let $(Q, \tilde{\tilde{\tau}}, D)$ be S/T/S, if $(L, D) \in \tilde{\tilde{\tau}}$ and $(H, D) \in sp\beta\alpha o(Q, \tilde{\tilde{\tau}}, D)$, then $(L, D) \cap (H, D) \in sp\beta\alpha o(Q, \tilde{\tilde{\tau}}, D)$.

Proof: Since $(H, D) \in sp\beta\alpha o(Q, \tilde{\tilde{\tau}}, D)$, then by proposition 3.2, there is $(M, \breve{A}) \in s\beta o(Q, \tilde{\tilde{\tau}}, D)$ for which $(H, D) \subseteq (M, D) \subseteq \alpha cl(H, D)$.

It follow that $(H, D) \cap (L, D) \subseteq (M, D) \cap (L, D) \subseteq \alpha cl(H, D) \cap (L, D)$

By proposition 2.31,$(M, D) \cap (L, D) \in s\beta o(Q, \tilde{\tilde{\tau}}, D)$. Since $(L, D) \in s\alpha o(Q, \tilde{\tilde{\tau}}, D)$Then by proposition 2.30, $\alpha cl(H, D) \cap (L, D) \subseteq \alpha cl((H, D) \cap (L, D))$. Consequently,$(H, D) \cap (L, D) \subseteq (M, D) \cap (L, D) \subseteq \alpha cl((H, D) \cap (L, D))$ where $(M, D) \cap (L, D) \in s\beta o(Q, \tilde{\tilde{\tau}}, D)$.

Hence $(H, D) \cap (L, D) \in sp\beta\alpha o(Q, \tilde{\tilde{\tau}}, D)$.

Proposition 3.12: Let $(S, \tilde{\tilde{\tau}}_S, D)$ be a α-clopen subspace of $(Q, \tilde{\tilde{\tau}}, D)$and let $(L, D) \cong (S, \tilde{\tilde{\tau}}_S, D) \cong (Q, \tilde{\tilde{\tau}}, D)$ if $(L, D) \in sp\beta\alpha o(Q, \tilde{\tilde{\tau}}, D)$, then $(L, D) \in sp\beta\alpha o(S, \tilde{\tilde{\tau}}_S, D)$.

Proof: Since $(L, D) \in sp\beta\alpha o(Q, \tilde{\tilde{\tau}}, D)$, then by proposition 3.2, $(H, D) \in s\beta o(Q, \tilde{\tilde{\tau}}, D)$ for which $(L, D) \subseteq (H, D) \subseteq \alpha cl(L, D)$.It follow that $(L, D) \tilde{\cap} \tilde{S} \subseteq (H, D) \cap \tilde{S} \subseteq \alpha cl(L, D) \cap \tilde{S}$. By proposition 2.25,$(L, D) \subseteq (K, D)_S \subseteq \alpha cl_S(L, D)$ where $(K, D)_S \in s\beta o(S, \tilde{\tilde{\tau}}_S, D)$. Hence $(L, D) \in sp\beta\alpha o(S, \tilde{\tilde{\tau}}_S, D)$.

Proposition 3.13: Let (L, D) be soft subset of soft subspace $(S, \tilde{\tilde{\tau}}_S, D)$ of S/T/S$(Q, \tilde{\tilde{\tau}}, D)$.If $(L, D) \in sp\beta\alpha o(S, \tilde{\tilde{\tau}}_S, D)$, then $(L, D) \in sp\beta\alpha o(Q, \tilde{\tilde{\tau}}, D)$.

Proof: Since $(L, D) \in sp\beta\alpha o(S, \tilde{\tilde{\tau}}_S, D)$, there is $(H, D) \in s\beta o(S, \tilde{\tilde{\tau}}_S, D)$ for which

$(L, D) \subseteq (H, D) \subseteq \alpha cl_S(L, D)$. Since $\alpha cl_S(L, D) \subseteq \alpha cl_Q(L, D) \cap \tilde{S}$,then $(L, D) \cap \tilde{S} \subseteq (K, D) \cap \tilde{S} \subseteq \alpha cl_Q(L, D) \cap \tilde{S}$ where $(H, D) = (K, D) \cap \tilde{S}, (K, D) \in s\beta o(Q, \tilde{\tilde{\tau}}, D)$.

Consequently,$(L, D) \subseteq (K, D) \subseteq \alpha cl_Q(L, D)$. Hence $(L, D) \in sp\beta\alpha o(Q, \tilde{\tilde{\tau}}, D)$.

Definition 3.14: Let $(Q, \tilde{\tilde{\tau}}, D)$ be S/T/S, (L, D) be soft in $\tilde{Q}.(L, D)$ is labeled as $sp\beta\alpha c -$ set if it is complement is $sp\beta\alpha o -$ set.

Proposition 3.15: Let $(Q, \tilde{\tilde{\tau}}, D)$ be S/T/S and let (L, D) be soft set in $\tilde{Q}$,then (L, D) is $sp\beta\alpha c -$ set if and only if there is $(H, D) \in s\beta c(Q, \tilde{\tilde{\tau}}, D)$ for which $s\alpha int\,(L, D) \subseteq (H, D) \subseteq (L, D)$.

Proof: Suppose that $(L, D) \in sp\beta\alpha c(Q, \tilde{\tilde{\tau}}, D)$,then $\tilde{Q} - (L, D) = (L^c, D) \in sp\beta\alpha o(Q, \tilde{\tilde{\tau}}, D)$, by proposition 3.2,there is $(H, D) \in s\beta o(Q, \tilde{\tilde{\tau}}, D)$ for which $(L^c, D) \subseteq (H, D) \subseteq s\alpha cl(L^c, D)$.since $(H, D) \subseteq s\alpha cl(L^c, D)$,then $\tilde{Q} - s\alpha cl(L^c, D) \subseteq \tilde{Q} - (H, D) \rightarrow s\alpha int\,(L, D) \subseteq (H^c, D)$ and since $(L^c, D) \subseteq (H, D)$, then $(H^c, D) \subseteq (L, D)$. Consequently, $s\alpha int\,(L, D) \subseteq (H^c, D) \subseteq (L, D)$, where $(H^c, D) \in s\beta o(Q, \tilde{\tilde{\tau}}, D)$. Conversely, Assume there is $(K, D) \in s\beta c(Q, \tilde{\tilde{\tau}}, D)$ for which $s\alpha\, int\,(L, D) \subseteq (K, D) \subseteq (L, D)$. since $s\alpha int\,(L, D) \subseteq (K, D)$, then $s\beta cl(s\alpha int\,(L, D) \subseteq s\beta cl(K, D) = (K, D) \subseteq (L, D)$. It follow that $s\beta cl(s\alpha int\,(L, D) \subseteq (L, D)$. Hence $(L, D) \in sp\beta\alpha c(Q, \tilde{\tilde{\tau}}, D)$.

Proposition 3.16: Assume (L, D) be soft set over S/T/S$(Q, \tilde{\tilde{\tau}}, D)$. If $(L, D) \in s\beta c(Q, \tilde{\tilde{\tau}}, D)$,then $(L, D) \in sp\beta\alpha c(Q, \tilde{\tilde{\tau}}, D)$.

Corollary 3.17: Every soft closed, soft pre-closed is $sp\beta\alpha c -$ set.

Proposition 3.18: Let (H_i, D) be soft sets over S/T/S$(Q, \tilde{\tilde{\tau}}, D)$, then the arbitrary intersection of $sp\beta\alpha c -$ sets is $sp\beta\alpha c$ −set.

Proof: Given $\{(H_i, D): i \in \wedge\}$ be the aggregate of $sp\beta\alpha c - sets$, then by proposition 3.15, there is

$(K_i, D) \in s\beta c(Q, \tilde{\tilde{\tau}}, D)$ for which $s\alpha int\ (H_i, D) \subseteq (K_i, D) \subseteq (H_i, D)$. It follow that $\bigcap_{i\in\wedge} s\alpha int\ (H_i\ , D) \subseteq \bigcap_{i\in\wedge}(K_i\ , D) \subseteq \bigcap_{i\in\wedge}(H_i\ , D)\ \rightarrow\ s\alpha\ int\ \bigcap_{i\in\wedge}(H_i\ , D) \subseteq \bigcap_{i\in\wedge}(K_i\ , D) \subseteq \bigcap_{i\in\wedge}(H_i\ , D)$ where $\bigcap_{i\in\wedge}(K_i\ , D) \in s\beta c(Q, \tilde{\tilde{\tau}}, D)$. Consequently $\bigcap_{i\in\wedge}(H_i\ , D) \in sp\beta\alpha c(Q, \tilde{\tilde{\tau}}, D)$.

4 CONCLUSIONS

In this paper, the concepts of soft pβα-open sets and soft pβα-closed sets in soft topological spaces were introduced and investigated. Several fundamental properties and characterizations of these classes were established. Relationships between the proposed notions and other existing types of soft open and soft closed sets were also studied.

In addition, the behavior of soft pβα-open sets under standard set-theoretic operations such as union, intersection, and soft subspaces was examined. Various propositions and counterexamples were presented to clarify the structure and properties of the newly introduced classes.

The obtained results contribute to the development of generalized soft topological structures and provide a broader framework for studying generalized openness and closedness in soft topology. It is expected that these concepts may support further theoretical investigations and future applications of soft topology in mathematical analysis, uncertainty modeling, and decision-making problems.

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