

Partial Metric Spaces and Fixed Points for Computational Applications

Ghaidaa Sadon Ashiaa and Salwa Salman Abed

Department of Mathematics, College of Education for Pure Science (Ibn Al-Haitham), University of Baghdad, 10053 Baghdad, Iraq

ghaidaa.hasan2203@ihcoedu.uobaghdad.edu.iq1, salwaalbundi@yahoo.com, salwa.s.a@ihcoedu.uobaghdad.edu.iq

Keywords: Applications, Coincidence Points, Fixed Point, Multivalued Mappings, Partial Hausdorff Distance, Partial Metric Space.

Abstract: A Partial metric space (G, ρ) gives a generalized mathematical basis that coordinates well with the approximate theory and invariant elements. It serves as an essential tool in theoretical computer science due to its applications in domain theory, recursion theory, denotation semantics, and distributed computing. This advantage prompted the current study to support the theoretical aspect of the subject. Here, there are two cases of work. The first one is finding a common fixed point in an ordered complete partial metric space (G, ρ, \preceq) for a (M, ψ, φ) -contractive map $\Gamma M: G \rightarrow G$ w.r.t. a map $M: G \rightarrow G$ and a sufficient contractive condition if Γ, M are two weakly increasing maps where ψ, φ are altering distance functions. The second is finding a coincidence point of a pair (Γ, f) and a coincidence point of a pair (M, m) , where Γ, M are multi-valued maps, f, m are single-valued maps. All these maps are involved in setting up a contractionary integral condition adopted in this part of the paper. The pair (Γ, f) is the strongest straight relying on (M, m) . For this, the partial Hausdorff distance, as proposed by H. Aydi et al., was recalled. Numerical examples were also presented to illustrate these results and support our work.

1 INTRODUCTION

The partial metric spaces (shortly, PMS) have valuable employment in computer science, especially in domain theory, fixed-point computations, and semantics of programming languages. This is due to its ability to handle self-referential structures and incomplete information, making it well-suited for computational models. The PMS provides a flexible and powerful environment for studying spaces with self-referencing distances, making them valuable in theoretical computer science [1].

In fact, Iterations, or repeating a process multiple times, are a fundamental principle in computer science. When it comes to finding roots of equations, iterative techniques play a crucial role. Instead of solving complex equations algebraically, iterations involve using algorithms that repeatedly refine an estimate of the root until it reaches a desired level of accuracy [2].

Over the past decades, many authors have generalized the concept of the metric space, such as the quasi-metric space by Wilson and Nguyen [3], [4]. b-metric space by Bakhtin in 1989,

see [5]-[6], G-metric space by Mustafa and Sims in 2006, see [7], [8] for this definition and other results. PMS by Matthews, see [9]-[12] for various results and applications. While Wangwe and Kumar [12] reported varied results on the fixed points of F-Hardy-Roger multi-valued maps in a PMS with ordering. For spaces with special assumptions, see [13], [14], where the proximity properties in fuzzy normed spaces and modular spaces are discussed, respectively. By using the weak compatibility through F -contraction, Vijayabaskerreddy and Srinivas [15] proved a common fixed-point theorem for two pairs of self-maps. The authors in [16] proposed the notion of double-controlled partial metric type spaces and studied the existence of fixed points for the Kannan-type contractions.

The work in this search depends on the PMS to present some outputs about the existence of common fixed points and coincidence points. Our work is, firstly, an extension of the work of Ampadu [17], [18] and later researchers who adopted his work. Tiwari, see [19], [20], who started the concept of almost contractions and studied many valuable results on fixed points regarding a Ćirić strong almost

contraction in metric spaces. Babu [21] and Ćirić [22] proved results for some kinds of this contractive condition in an ordered metric space.

2 PRELIMINARIES

In the following, some basic notions are recalled in PMS.

Definition 1: [1] Assume that $\rho: G \times G \rightarrow [0, \infty)$ is a function where G is a nonempty set. The pair (G, ρ) is a PMS if for all $g, s, h \in G$,

1. $\rho(g, g) \leq \rho(g, s)$.
2. if $0 \leq \rho(g, g) = \rho(g, s) = \rho(s, s)$ then, $g = s$.
3. $\rho(g, s) = \rho(s, g)$.
4. $\rho(g, h) + \rho(s, s) \leq \rho(g, s) + \rho(s, h)$.

Remark 1: [1]:

- 1) If $\rho(g, s) = 0$, then from 1 and 2 in Definition 4, $g = s$ holds.
- 2) Any metric space is a partial metric space, but the converse is not always true, as the following: if $G = \mathbb{R}^+$ and $\rho(g, s) = \max\{g, s\}$. Then (G, ρ) is a partial metric space and $\rho(g, g) \neq 0$. For all $g \in G \setminus \{0\}$, but it is not a metric space.
- 3) A partial metric ρ on G generates a T_0 topology \mathfrak{T}_ρ , with a base of the family of open ρ -balls $B_\rho(g, \epsilon) = \{s \in G: \rho(g, s) < \rho(g, g) + \epsilon\}$ for all $g \in G$ and $\epsilon > 0$.

Definition 2: [23] Let (G, ρ) be a PMS and $\{g_n\}$ be a sequence in G . Then:

- 1) $\{g_n\}$ converges to $g \in G$ if $\rho(g, g) = \lim_{n \rightarrow \infty} \rho(g, g_n)$;
- 2) $\{g_n\}$ is called a Cauchy sequence if $\lim_{n, m \rightarrow \infty} \rho(g_m, g_n)$ exists (and is finite);
- 3) The space G is called complete (under \mathfrak{T}_ρ) if every Cauchy sequence $\{g_n\} \subset G$ converges to $g \in G$ such that $\rho(g, g) = \lim_{n, m \rightarrow \infty} \rho(g_m, g_n)$.

To illustrate the difference between PMS and the usual metric, see the following example:

Example 1: Recall a PMS $G = [0, 1]$ with $\rho(g, s) = \max\{g, s\}$. Let $\{\frac{1}{n}\}_{n \in \mathbb{N}}$ be a sequence in G converging to both 0 and 1 to verify Definition 5, observe that:

$$\rho(0, 0) = \lim_{n \rightarrow \infty} \rho\left(0, \frac{1}{n}\right) = \lim_{n \rightarrow \infty} \max\left\{0, \frac{1}{n}\right\} = 0$$

$$\text{and } \rho(1, 1) = \lim_{n \rightarrow \infty} \rho\left(1, \frac{1}{n}\right) = \lim_{n \rightarrow \infty} \max\left\{1, \frac{1}{n}\right\} = 1$$

This illustrates the difference from the usual metric convergence. Moreover, because $\rho(g, g) = \lim_{n \rightarrow \infty} \rho(g, g_n)$ holds for any $g \in G = [0, 1]$, $\{\frac{1}{n}\}_{n \in \mathbb{N}}$ converges to all points in this PMS $G = [0, 1]$.

Lemma 1: [24]

- 1) Let $\{g_n\}$ and $\{s_n\}$ be two sequences in a PMS G such that

$$\rho(g, g) = \lim_{n \rightarrow \infty} \rho(g_n, g) = \lim_{n \rightarrow \infty} \rho(g_n, g_n) \quad \text{and} \\ \rho(s, s) = \lim_{n \rightarrow \infty} \rho(s_n, s) = \lim_{n \rightarrow \infty} \rho(s_n, s_n), \quad \text{Then} \\ \rho(g, s) = \lim_{n \rightarrow \infty} \rho(g_n, s_n).$$

In particular, $\rho(g, h) = \lim_{n \rightarrow \infty} \rho(g_n, h)$, for every $h \in G$.

- 2) For g, s in G

- If $\rho(g, s) = 0$, then $g = s$,
- If $g \neq s$, then $\rho(g, s) > 0$.

Ćirić et al. [22] introduced a type of monotonic map as follows:

Definition 3: The pair of mappings $\Gamma, M: G \rightarrow G$ is called:

- 1) Weakly increasing if $\Gamma g \leq M\Gamma g$ and $Mg \leq \Gamma Mg$ for all $g \in G$.
- 2) Strictly weakly increasing if $\Gamma g < M\Gamma g$ and $Mg < \Gamma Mg$ for all $g \in G$.

Clearly, a strictly weakly increasing map implies a weakly increasing mapping, for example, see [22].

In PMS, Aydi et al. [25, 26] introduced a version of Hausdorff distance as follows.

Definition 4: Let G be PMS and $CB(G) = \{A \subseteq G: \emptyset \neq A \text{ is closed and bounded, then}$

$$H_\rho(A, B) = \max\{\sup \rho(a, B), \sup \rho(b, A): a \in A, b \in B\}$$

for every $A, B \in CB(G)$, where $\rho(a, B) = \inf\{\rho(a, b), b \in B\}$.

Proposition 1: [25] In PMS, for every $A, B \in CB(G)$, the following are true

- 1) $H_\rho(A, A) \leq H_\rho(A, B)$,
- 2) $H_\rho(A, B) = H_\rho(B, A)$,
- 3) $H_\rho(A, B) + \inf_{c \in C} \rho(c, c) \leq H_\rho(A, C) + (C, B)$.

Lemma 2: [25] Let (G, ρ) be a PMS, such that $\rho(a, b) \leq \mu H_\rho(A, B)$. For $\mu > 1$, $a \in A, \exists b = b(a) \in B$ and $A, B \in CB(G)$.

Aspects of this field require the following concept:

Definition 5: [27] A function $\varphi: [0, \infty) \rightarrow [0, \infty)$ is called an altering distance if it is non-decreasing, continuous, and $\varphi(t) = 0$ if and only if $t = 0$

Shatanawi and Al-Rawashdeh. [28] treat with the following kind in an ordered metric space $(G, \rho \preceq)$:

Definition 6: [28] Let (G, ρ) be a metric space. A map $\Gamma: (G, \rho) \rightarrow (G, \rho)$ is called almost generalized (ψ, φ) -contractive if $\exists \kappa \geq 0$ and two altering distance functions ψ, φ such that:

$$\psi(\rho(\Gamma(g), \Gamma(s))) \leq \psi(A(g, s)) - \varphi(A(g, s)) + \kappa\psi(B(g, s)),$$

$$A(g, s) = \max \left[\rho(g, s), \rho(s, \Gamma s), \rho(g, \Gamma g), \frac{\rho(g, \Gamma s) + \rho(s, \Gamma g)}{2} \right].$$

$$B(g, s) = \min [\rho(g, \Gamma g), \rho(s, \Gamma g)] \quad \text{for all comparable } g, s \in G.$$

And then Shatanawi and Al-Rawashdeh proved the following:

Theorem 1: [28] In a complete partially ordered metric space (G, ρ, \preceq) , and Γ is defined as in Definition 6. Then Γ has a fixed point if there exists $g_0 \in G$ such that $g_0 \preceq \Gamma g_0$.

Chauhan et al. [29] introduced the concept of strongly tangential property and generalized forms of a multitude of common fixed points in a metric space.

Definition 7: [29] Let (G, ρ) be a metric space. A map $f: S \rightarrow G$ be a single valued, $\Gamma: S \rightarrow CB(G)$ be multivalued map, such that $S \subseteq G$ thus:

- 1) The pair (f, Γ) is called quasi coincident if $fg \in \Gamma g$ for $g \in G$ and $fg, \Gamma g \in S$.
- 2) The map f is called coincidentally idempotent w.r.t. Γ if $fg \in \Gamma g$ for $g \in G$ and $fg \in S$ imply $ffg = fg$.

3 MAIN RESULTS

3.1 Common Fixed Points

To generalize Theorem 1, the following new type of map will be suggested.

Definition 8: Let $(G, \rho \preceq)$ be a PMS with ordering. A map $\Gamma: (G, \rho) \rightarrow (G, \rho)$ is called (M, ψ, φ) -contractive if there is a map $M: (G, \rho) \rightarrow (G, \rho)$ such that:

$$\psi(\rho(\Gamma(g), M(s))) \leq \psi(A(g, s)) - \varphi(A(g, s)) + \kappa\psi(B(g, s)), \tag{1}$$

$$A(g, s) = \max \left[\frac{\rho(g, s), \rho(s, Ms), \rho(g, \Gamma g), \rho(\Gamma g, Ms) + \rho(g, \Gamma g) - \rho(\Gamma g, \Gamma g)}{2}, \left[\frac{\rho(g, Ms) + \rho(s, \Gamma g)}{2} \right] \right]$$

$$B(g, s) = \min [\rho(g, s), \rho(g, \Gamma g)],$$

for some $\kappa \geq 0$ all comparable $g, s \in G$, where ψ, φ are altering distance functions.

Proposition 2: Let $(G, \rho \preceq)$ be a complete PMS with ordering, $\Gamma: G \rightarrow G$ is (M, ψ, φ) -contractive w.r.t. a map $M: G \rightarrow G$. If Γ, M are two weakly increasing maps satisfying (1), then u is a fixed point of Γ if and only if u is a fixed point of M .

Proof: If u is a fixed point of Γ , then $\Gamma u = u$, and $u \preceq u$, putting in (1), thus getting:

$$\psi(\rho(\Gamma u, Mu)) = \psi(\rho(u, Mu)) \leq \psi \left(\max \left[\frac{\rho(u, u), \rho(u, Mu), \rho(u, \Gamma u), \frac{\rho(\Gamma u, Mu) + \rho(u, \Gamma u) - \rho(\Gamma u, \Gamma u)}{2}}{\frac{\rho(u, Mu) + \rho(u, \Gamma u)}{2}}, \left[\frac{\rho(u, Mu) + \rho(u, \Gamma u)}{2} \right] \right) \right)^- \\ \varphi \left(\max \left[\frac{\rho(u, u), \rho(u, Mu), \rho(u, \Gamma u), \frac{\rho(\Gamma u, Mu) + \rho(u, \Gamma u) - \rho(\Gamma u, \Gamma u)}{2}}{\frac{\rho(u, Mu) + \rho(u, \Gamma u)}{2}}, \left[\frac{\rho(u, Mu) + \rho(u, \Gamma u)}{2} \right] \right) \right)^+ \\ \kappa\psi(\min [\rho(u, u), \rho(u, \Gamma u)]) \text{ such that } \kappa \geq 0,$$

$$\psi(\rho(u, Mu)) = \psi(\rho(u, Mu)) - \varphi(\rho(u, Mu)) + \kappa\rho(u, u).$$

Then $\varphi(\rho(u, Mu)) = 0$ thus $\rho(u, Mu) = 0$ therefore $u = Mu$

The same way can be used to prove the converse, meaning if $u = M(u)$, implies $u = \Gamma(u)$.

Theorem 2: Let $(G, \rho \preceq)$ be a complete PMS with ordering, Γ, M as in Definition 8 that satisfy (1) if Γ or M is continuous. Then Γ and M have a common fixed point.

Proof: Construct an iterative sequence, suppose that $g_0 \in G$, and (g_n) in G such that $g_{2n+1} = \Gamma g_{2n}$, and $g_{2n+2} = M g_{2n+1}$ for all $n \in \mathbb{N}$. Also, Γ, M are two weakly increasing maps with respect to \preceq , so getting:

$$g_1 = \Gamma g_0 \preceq M \Gamma g_0 = g_2 = M g_1 \\ \preceq \Gamma M g_1 = g_3 \dots \Gamma g_{2n} \preceq M \Gamma g_{2n} \dots$$

If $g_{2n} = g_{2n+1}$ for all $n \in \mathbb{N}$, then $g_{2n} = \Gamma g_{2n}$ and hence g_{2n} is a fixed point of Γ . In the same way, getting g_{2n} is a fixed point of M . And assuming that if $g_n \neq g_{n+1}$ for all $n \in \mathbb{N}$ to show that $\lim_{n \rightarrow +\infty} \rho(g_n, g_{n+1}) = 0$.

Suppose:

$$g_{2n+1} = \Gamma g_{2n}, \text{ and } g_{2n+2} = M g_{2n+1}, \text{ putting in (1). Then:} \\ \psi(\rho(\Gamma g_{2n}, M g_{2n+1})) \leq \psi(A(g_{2n}, g_{2n+1})) - \varphi(A(g_{2n}, g_{2n+1})) \tag{2} \\ + \kappa\psi(B(g_{2n}, g_{2n+1})).$$

$$\begin{aligned}
 & \frac{A(g_{2n}, g_{2n+1})}{\rho(g_{2n}, g_{2n+1})}, \\
 & \rho(g_{2n+1}, Mg_{2n+1}), \\
 & \rho(g_{2n}, \Gamma g_{2n}), \\
 = \max & \left[\frac{\rho(\Gamma g_{2n}, Mg_{2n+1}) + \rho(g_{2n}, \Gamma g_{2n}) - \rho(\Gamma g_{2n}, \Gamma g_{2n})}{2}, \right. \\
 & \left. \frac{\rho(g_{2n}, Mg_{2n+1}) + \rho(g_{2n+1}, \Gamma g_{2n})}{2} \right] \\
 & B(g_{2n}, g_{2n+1}) \\
 = \min & n[\rho(g_{2n}, g_{2n+1}), \rho(g_{2n}, \Gamma g_{2n})].
 \end{aligned}$$

Then:

$$\begin{aligned}
 & A(g_{2n}, g_{2n+1}) \\
 & \rho(g_{2n}, g_{2n+1}), \rho(g_{2n+1}, g_{2n+2}), \rho(g_{2n}, g_{2n+1}), \\
 = \max & \left[\frac{\rho(g_{2n+1}, g_{2n+2}) + \rho(g_{2n}, g_{2n+1}) - \rho(g_{2n+1}, g_{2n+1})}{2}, \right. \\
 & \left. \frac{\rho(g_{2n}, g_{2n+2}) + \rho(g_{2n+1}, g_{2n+1})}{2} \right] \\
 \leq \max & \left[\frac{\rho(g_{2n}, g_{2n+1}), \rho(g_{2n+1}, g_{2n+2})}{\rho(g_{2n}, g_{2n+2}), \frac{\rho(g_{2n}, g_{2n+2}) + \rho(g_{2n+1}, g_{2n+1})}{2}} \right] \\
 = \max & [\rho(g_{2n}, g_{2n+1}), \rho(g_{2n+1}, g_{2n+2})]. \quad (3)
 \end{aligned}$$

And

$$\begin{aligned}
 & B(g_{2n}, g_{2n+1}) \\
 = \min & [\rho(g_{2n}, g_{2n+1}), \rho(g_{2n}, \Gamma g_{2n})] \quad (4) \\
 = \min & [\rho(g_{2n}, g_{2n+1}), \rho(g_{2n}, g_{2n+1})] \\
 = & \rho(g_{2n}, g_{2n+1})
 \end{aligned}$$

Putting (3) and (4) in (2), getting:

$$\begin{aligned}
 & \psi(\rho(\Gamma g_{2n}, Mg_{2n+1})) \\
 \leq & \psi(A(g_{2n}, g_{2n+1})) - \varphi(A(g_{2n}, g_{2n+1})) \\
 & + \kappa\psi(B(g_{2n}, g_{2n+1})) \quad (5) \\
 \leq & \psi(\max[\rho(g_{2n}, g_{2n+1}), \rho(g_{2n+1}, g_{2n+2})]) \\
 - & \varphi(\max[\rho(g_{2n}, g_{2n+1}), \rho(g_{2n+1}, g_{2n+2})]) \\
 & + \kappa\psi(\rho(g_{2n}, g_{2n+1}))
 \end{aligned}$$

Suppose that, if

$$\begin{aligned}
 & \max[\rho(g_{2n}, g_{2n+1}), \rho(g_{2n+1}, g_{2n+2})] = \\
 & \rho(g_{2n}, g_{2n+1}), \text{ putting in (5). Then} \\
 & \psi(\rho(g_{2n+1}, g_{2n+2})) \leq \psi(\rho(g_{2n}, g_{2n+1})) \\
 - & \varphi(\rho(g_{2n}, g_{2n+1})) + \kappa\psi(\rho(g_{2n}, g_{2n+1})) \quad (6) \\
 \leq & \psi(\rho(g_{2n}, g_{2n+1}))
 \end{aligned}$$

Now,

$$\begin{aligned}
 & \text{If } \max[\rho(g_{2n}, g_{2n+1}), \rho(g_{2n+1}, g_{2n+2})] = \\
 & \rho(g_{2n+1}, g_{2n+2}), \text{ putting in (5). Then} \\
 & \psi(\rho(g_{2n+1}, g_{2n+2})) \leq \psi(\rho(g_{2n+1}, g_{2n+2})) - \\
 - & \varphi(\rho(g_{2n+1}, g_{2n+2})) \leq \psi(\rho(g_{2n+1}, g_{2n+2})). \quad (7)
 \end{aligned}$$

That is contraction (7)

By (6) and (7) and monotonically ψ , getting $\{\rho(g_n, g_{n+1}): n \in \mathbb{N}\}$ is a non-increasing real sequence, then $\exists t \geq 0$ such that $\lim_{n \rightarrow +\infty} \rho(g_n, g_{n+1}) = t$, as $n \rightarrow +\infty$ in (6), therefore $\psi(t) \leq \psi(t) - \varphi(t) \leq \psi(t) - 0$ where $\psi(t) = 0$, then $t = 0$, and by

Lemma 1 and Definition 5, getting $\psi(\lim_{n \rightarrow +\infty} \rho(g_n, g_{n+1})) = \psi(t)$, then:

$$\lim_{n \rightarrow +\infty} \rho(g_n, g_{n+1}) = 0. \quad (8)$$

The next aim is to prove (g_n) is a Cauchy sequence. Suppose not, then $\exists \epsilon > 0$, there are two subsequences $(g_{m(i)})$ and $(g_{n(i)})$ of (g_n) such that $n(i)$ is the smallest index for which $n(i) > m(i) > i$,

$$\rho(g_{2m(i)}, g_{2n(i)}) \geq \epsilon, \quad (9)$$

$$\rho(g_{2m(i)}, g_{2n(i)-2}) < \epsilon, \quad (10)$$

From (9), (10), and Definition 1, getting:

$$\begin{aligned}
 \epsilon & \leq \rho(g_{2m(i)}, g_{2n(i)}) \leq \rho(g_{2m(i)}, g_{2m(i)-1}) + \\
 & \rho(g_{2m(i)-1}, g_{2n(i)}) - \rho(g_{2m(i)-1}, g_{2m(i)-1}) \quad \text{by} \\
 & \text{Definition 4, thus}
 \end{aligned}$$

$$\begin{aligned}
 & \leq \rho(g_{2m(i)}, g_{2m(i)-1}) + \rho(g_{2m(i)-1}, g_{2n(i)-1}) \\
 & + \rho(g_{2n(i)-1}, g_{2n(i)}) - \rho(g_{2m(i)-1}, g_{2m(i)-1}) \\
 & \leq 2[\rho(g_{2m(i)}, g_{2m(i)-1})] + \epsilon + \rho(g_{2n(i)-1}, g_{2n(i)}) \\
 & \quad - \rho(g_{2m(i)-1}, g_{2m(i)-1})
 \end{aligned}$$

For all $i \rightarrow +\infty$ and using (8), getting:

$$\begin{aligned}
 \lim_{i \rightarrow +\infty} \rho(g_{2m(i)}, g_{2n(i)}) & = \lim_{i \rightarrow +\infty} \rho(g_{2m(i)-1}, g_{2n(i)}) \\
 & = \rho(g_{2m(i)}, g_{2n(i)-1}) \\
 & = \rho(g_{2m(i)-1}, g_{2n(i)-1}) = \epsilon
 \end{aligned}$$

Then, from (1), the following was obtained:

$$\begin{aligned}
 & \psi(\rho(g_{2n(i)+1}, g_{2m(i)+2})) \\
 & = \psi(\rho(\Gamma(g_{2n(i)}), M(g_{2n(i)+1}))) \\
 \leq & \psi(A(g_{2n(i)}, g_{2m(i)+1})) - \varphi(A(g_{2n(i)}, g_{2m(i)+1})) + \\
 & \kappa\psi(B(g_{2n(i)}, g_{2m(i)+1})) \quad (11)
 \end{aligned}$$

Thus

$$\begin{aligned}
 & A(g_{2n(i)}, g_{2m(i)+1}) \\
 & \rho(g_{2n(i)}, g_{2m(i)+1}), \\
 & \rho(g_{2m(i)+1}, Mg_{2m(i)+1}), \rho(g_{2n(i)}, \Gamma g_{2n(i)}) \\
 & \rho(\Gamma g_{2n(i)}, Mg_{2m(i)+1}) + \\
 = \max & \left[\frac{\rho(g_{2n(i)}, \Gamma g_{2n(i)}) - \rho(\Gamma g_{2n(i)}, \Gamma g_{2n(i)})}{2}, \right. \\
 & \left. \frac{\rho(g_{2n(i)}, Mg_{2m(i)+1}) + \rho(g_{2m(i)+1}, \Gamma g_{2n(i)})}{2} \right] \\
 & \rho(g_{2n(i)}, g_{2m(i)+1}), \rho(g_{2m(i)+1}, g_{2m(i)+2}), \\
 & \rho(g_{2n(i)}, g_{2n(i)+1}) \\
 \leq \max & \left[\frac{\rho(g_{2n(i)}, g_{2m(i)+2})}{2}, \right. \quad (12) \\
 & \left. \frac{\rho(g_{2n(i)}, g_{2m(i)+2}) + \rho(g_{2m(i)+1}, g_{2n(i)+1})}{2} \right]
 \end{aligned}$$

$$\begin{aligned}
 & \text{And } B(g_{2n(i)}, g_{2m(i)+1}) = \\
 & \min [\rho(g_{2n(i)}, \Gamma g_{2n(i)}), \rho(g_{2m(i)+1}, \Gamma g_{2n(i)}), \rho(g_{2n(i)}, Mg_{2m(i)+1})] \\
 = \min & [\rho(g_{2n(i)}, g_{2n(i)+1}), \rho(g_{2m(i)+1}, g_{2n(i)+1}), \quad (13) \\
 & \rho(g_{2n(i)}, g_{2m(i)+2})]
 \end{aligned}$$

For all $i \rightarrow +\infty$ putting in (12), and (13), getting

$$\lim_{i \rightarrow +\infty} A(g_{2n(i)}, g_{2m(i)+1}) = \varepsilon, \quad (14)$$

$$\lim_{i \rightarrow +\infty} B(g_{2n(i)}, g_{2m(i)+1}) = 0. \quad (15)$$

For all $i \rightarrow +\infty$ putting (14) and (15) in (12), (13)

By Definition 5 the contradiction is given, since $\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon) \leq \psi(\varepsilon)$, where $\varphi(\varepsilon) = 0$. Then (g_n) is Cauchy, and the completeness of G guarantees convergence of (g_n) to v in G .

Finally, since v is a fixed point of M , then v is a fixed point of Γ by Proposition 2.

Remark 2: The continuity condition in Theorem 2 can be replaced by another condition

Theorem 3: By the same assumptions in Theorem 2, without the continuity condition of maps, and assume that $g_n \in G$ whenever (g_n) is a non-decreasing sequence such that $g_n \rightarrow g$ imply $g_n \preceq g, \forall n \in \mathbb{N}$. Then Γ and M have a fixed point $g \in G$.

Proof: Steps of proof as in Theorem 2, and there exists a non-decreasing sequence (g_n) in G such that $g_n \rightarrow u$ for some $u \in G$. According to the hypothesis $g_n \preceq u$. Then to show $\Gamma u = Mu = u$. Using (1) and (2) therefore:

$$\psi(\rho(\Gamma(g_{2n}), M(u))) = \psi(\rho(g_{2n+1}, M(u))) \quad (16)$$

$$\leq \psi(A(g_{2n}, u)) - \varphi(A(g_{2n}, u))$$

$$+ \kappa\psi(B(g_{2n}, u))$$

Getting:

$$\psi(\rho(u, Mu)) = \psi(\rho(u, Mu)) - \varphi(\rho(u, Mu)) + \kappa\psi(\rho(u, Mu))$$

Thus:

$$\begin{aligned} & (g_{2n}, u) \\ & \rho(g_{2n}, u), \rho(g_{2n}, Mu), \rho(g_{2n}, \Gamma g_{2n}), \\ & = \max \frac{\rho(\Gamma g_{2n}, Mu) + \rho(u, \Gamma g_{2n}) - \rho(\Gamma g_{2n}, \Gamma g_{2n})}{2}, \\ & \left[\frac{\rho(u, Mu) + \rho(u, \Gamma g_{2n})}{2} \right] \\ & \leq \max \left[\frac{\rho(g_{2n}, u), \rho(g_{2n}, Mu), \rho(g_{2n}, g_{2n+1})}{2}, \frac{\rho(u, Mu) + \rho(u, g_{2n+1})}{2} \right] \quad (17) \end{aligned}$$

$$B(g_{2n}, u) = \min [\rho(g_{2n}, u), \rho(g_{2n}, \Gamma g_{2n})]$$

Therefore:

$$B(g_{2n}, u) = \min [\rho(g_{2n}, u), \rho(g_{2n}, g_{2n+1})] \quad (18)$$

For all $n \rightarrow +\infty$ putting in (17) and (18), getting

$$\lim_{n \rightarrow +\infty} A(g_{2n}, u) = \rho(u, Mu) \quad (19)$$

$$\lim_{n \rightarrow +\infty} B(g_{2n}, u) = 0 \quad (20)$$

For all $n \rightarrow +\infty$ putting in (19) and (20), hence:

$$\begin{aligned} \psi(\rho(u, M(u))) & \leq \psi(\rho(u, Mu)) - \varphi(\rho(u, Mu)) \\ & + \kappa\psi(\rho(u, Mu)) \end{aligned}$$

By Proposition 2, getting $u = \Gamma u = Mu$.

Corollary 1: Let $(G, \rho \preceq)$ be a complete PMS with ordering, Γ, M as in Definition 6 that satisfy (1), if there exists $g_0 \in G$ such that $g_0 \in \Gamma g_0$ for some $0 \leq k < 1$, that satisfy

$$\rho(\Gamma(g), \Gamma(s)) \leq \kappa(A(g, s)) + \kappa(B(g, s)) \quad (21)$$

Where:

$$\begin{aligned} A(g, s) & = \max \frac{\rho(g, s), \rho(s, Ms), \rho(g, \Gamma g), \rho(\Gamma g, Ms) + \rho(g, \Gamma g) - \rho(\Gamma g, \Gamma g)}{2}, \\ B(g, s) & = \min \left[\frac{\rho(g, Ms) + \rho(s, \Gamma g)}{2}, \rho(g, s), \rho(g, \Gamma g) \right] \quad \text{for all comparable } g, s \in G. \end{aligned}$$

If Γ or M is continuous. Then Γ and M have a common fixed point.

Proof: Steps of proof as in Theorem 3, and taking $\psi(t) = t$ and $(t) = t - t\kappa, \forall 0 \leq t < +\infty$. Hence, Γ and M have a fixed point $g \in G$.

To illustrate Theorem 3:

Example 2: Consider (G, ρ) a PMS $G = [0, \infty)$ and $\rho: G \times G \rightarrow \mathbb{R}^+$

Define the maps $\Gamma, M: (G, \rho) \rightarrow (G, \rho)$, such that:

$$\begin{aligned} \rho(g, s) & = \begin{cases} 0 & \text{where } g = s \\ \max \{g, s\} & \text{where } g \neq s \end{cases} \text{ and} \\ \Gamma g & = \begin{cases} 0 & \text{where } 0 \leq g < 3 \\ g - 3 & \text{where } g \geq 3 \end{cases} \\ Mg & = \begin{cases} 0 & \text{where } 0 \leq g < 4 \\ g - 4 & \text{where } g \geq 4 \end{cases} \end{aligned}$$

And Γ, M have fixed point where $\psi: [0, \infty) \rightarrow [0, \infty)$ $\psi(t) = t^2, \varphi(t) = t$ as altering distance function, then:

- 1) The maps Γ and M are weakly increasing with respect to \preceq on a PMS G .
- 2) Γ is (M, ψ, φ) -contractive if it is a map $M, 0 < \kappa \leq 1$ such that:

$$\psi(\rho(\Gamma(g), M(s))) \leq \psi(A(g, s)) - \varphi(A(g, s)) + \kappa\psi(B(g, s))$$

Where

$$\begin{aligned} A(g, s) & = \max \frac{\rho(g, s), \rho(s, Ms), \rho(g, \Gamma g), \rho(\Gamma g, Ms) + \rho(g, \Gamma g) - \rho(\Gamma g, \Gamma g)}{2}, \\ B(g, s) & = \min \left[\frac{\rho(g, Ms) + \rho(s, \Gamma g)}{2}, \rho(g, s), \rho(g, \Gamma g) \right] \quad \text{for all comparable } g, s \in G. \end{aligned}$$

Remark 3: In Corollary 1, it is not necessary to have a continuity condition of the map, as shown in the following

Corollary 2: The same assumption in Corollary 1, but without the continuity condition of the map, and assume that $g_n \in G$ whenever (g_n) is a non-

decreasing sequence such that $g_n \rightarrow g$ imply $g_n \preceq g, \forall n \in \mathbb{N}$. Then Γ and M have a fixed point $g \in G$.

Proof: Steps of proof as in Theorem 3, and taking $\psi(t) = t$ and $(t) = t - t\kappa$, for all $0 \leq t < +\infty$. So Γ and M have a fixed point $g \in G$.

Corollary 3: Let $(G, \rho \preceq)$ be a complete PMS with ordering, $\Gamma = I$: (I is identity map) as Definition 3 that satisfy (1), if there exists $g_0 \in G$ such that $g_0 \preceq \Gamma g_0$ for some $0 \leq k < 1$, that satisfy

$$\psi(\rho(\Gamma(g), \Gamma(s))) \leq \psi(A(g, s)) - \varphi(A(g, s)) + \kappa\psi(B(g, s)),$$

where:
 $A(g, s)$

$$= \max \left[\frac{\rho(g, s), \rho(s, \Gamma s), \rho(g, \Gamma g), \rho(\Gamma g, s) + \rho(g, \Gamma g) - \rho(\Gamma g, \Gamma g)}{2}, \frac{\rho(g, \Gamma s) + \rho(s, \Gamma g)}{2} \right]. \quad (22)$$

$B(g, s) = \min [\rho(g, s), \rho(g, \Gamma g)]$ for all comparable $g, s \in G$. If Γ is continuous. Then Γ has a common fixed point.

Corollary 4: Let $(G, \rho \preceq)$ be a complete PMS with ordering. The map Γ is a non-decreasing continuous function. Assume that $\exists k \in (0, 1]$, such that

$$\rho(\Gamma(g), \Gamma(s)) \int_0^{\rho(\Gamma(g), \Gamma(s))} \lambda(t) dt \leq k \int_0^{\max \left[\frac{\rho(g, s), \rho(s, \Gamma s), \rho(g, \Gamma g), \rho(\Gamma g, s) + \rho(g, \Gamma g) - \rho(\Gamma g, \Gamma g)}{2}, \frac{\rho(g, \Gamma s) + \rho(s, \Gamma g)}{2} \right]} \lambda(t) dt + \int_0^{\min [\rho(g, \Gamma g), \rho(s, \Gamma s), \rho(s, \Gamma g)]} \lambda(t) dt,$$

for all comparable $g, s \in G$, if $\exists g_0 \in G$ such that $g_0 \preceq \Gamma g_0$. Then Γ has a fixed point $g \in G$.

Corollary 5: Let $(G, \rho \preceq)$ be a complete PMS with ordering. If the maps Γ, M are two weakly increasing. Assume that $\exists k \in (0, 1]$, and $\mathcal{L} \geq 0$ such that

$$\rho(\Gamma(g), M(s)) \int_0^{\rho(\Gamma(g), M(s))} \lambda(t) dt \leq k \int_0^{\max \left[\frac{\rho(g, s), \rho(s, Ms), \rho(g, \Gamma g), \rho(\Gamma g, Ms) + \rho(g, \Gamma g) - \rho(\Gamma g, \Gamma g)}{2}, \frac{\rho(g, Ms) + \rho(s, \Gamma g)}{2} \right]} \lambda(t) dt + \mathcal{L} \int_0^{\min [\rho(g, \Gamma g), \rho(s, \Gamma s), \rho(s, Ms)]} \lambda(t) dt$$

for all comparable $g, s \in G$, if there exists $g_0 \in G$ such that $g_0 \preceq \Gamma g_0$. Then Γ and M have a common fixed point.

3.2 Coincidence Points

Definition 9: Let (G, ρ) be a PMS. The maps $f: G \rightarrow G$, and $\Gamma: G \rightarrow CB(G)$. A point $g \in G$ is called a coincidence point of f and Γ if $fg \in \Gamma g$.

Below, there are new sorts of generalized results in PMS about the existence of a coincidence point.

Definition 10: Let (G, ρ) be a PMS. The maps $f, m: S \rightarrow G$ be single-valued, and $\Gamma, M: S \rightarrow CB(G)$ be multivalued maps such that $S \subseteq (G, \rho)$ when a sequence $\{g_n\}$ and $\{s_n\}$ in S . If

$$\lim_{n \rightarrow \infty} \Gamma g_n = \lim_{n \rightarrow \infty} M g_n = D \in CB(G),$$

and

$$\lim_{n \rightarrow \infty} f g_n = \lim_{n \rightarrow \infty} m g_n = q \in D, q \in f(S) \cap m(S)$$

Then the pair (f, m) is called the strongest straight w.r.t. the pair (Γ, M) .

The following example satisfied Definition 10.

Example 3: Let $G = S = \mathbb{R}^+$ and $\rho: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be $\rho(g, s) = \max\{g, s\}$, if $g \neq s$ which is a PMS [1] such that $f, m: S \subseteq \mathbb{R}^+ \rightarrow \mathbb{R}^+$, and $\Gamma, M: S \subseteq \mathbb{R}^+ \rightarrow CB(G)$ are defined by

$$f(g) = \begin{cases} \frac{g}{4}, & \text{if } g \in [0, 1) \\ 1, & \text{if } g \in [1, \infty) \end{cases},$$

$$m(g) = \begin{cases} 1, & \text{if } g \in [0, 1] \\ \frac{1}{4g}, & \text{if } g \in [1, \infty) \end{cases}$$

$$\Gamma(g) = \begin{cases} [4, g + 4], & \text{if } g \in [0, 1) \\ [1, 2], & \text{if } g \in [1, \infty) \end{cases},$$

$$M(g) = \begin{cases} [1, 4 - g], & \text{if } g \in [0, 1] \\ [\frac{1}{2}, 1], & \text{if } g \in [1, \infty) \end{cases}$$

$\lim_{n \rightarrow \infty} \Gamma g_n = \lim_{n \rightarrow \infty} M g_n = [1, 2] \in CB(G)$, when $\{g_n\}$ and $\{s_n\}$ in S therefore

$\lim_{n \rightarrow \infty} f g_n = \lim_{n \rightarrow \infty} m g_n = 1 \in [1, 2]$ and $1 \in f(S) \cap m(S)$. Thus (f, m) is the strongest straight w.r.t. the pair (Γ, M) .

Now, a result that combines the tools of single-valued, multi-valued maps and the strongest straight is presented, and thus the fixed point can be proven.

Note, in the next results, a function $\lambda: [0, \infty) \rightarrow [0, \infty)$ is a summable nonnegative Lebesgue integrable and $\int_0^\varepsilon \lambda(t) dt > 0$, for $\varepsilon > 0$.

Theorem 4: Let (G, ρ) be a PMS, $f, m: S \rightarrow G$ and $\Gamma, M: S \subseteq G \rightarrow CB(G)$. If $H_\rho(\Gamma g, Ms)$

$$\int_0^\sigma \lambda(t) dt \leq \sigma \int_0^{\max} \left[\frac{\rho(fg, ms), \rho(fg, \Gamma g), \rho(ms, Ms)}{\rho(fg, Ms) + \rho(ms, \Gamma g)} \right] \lambda(t) dt, \quad (23)$$

for all $g, s \in S$, $0 \leq \sigma < 1$. If the property of the strongest straight for the pair (f, m) regarding the pair (Γ, M) holds. Then

- 3) There is u in S such that $\Gamma u = fu$ (coincidence point).
- 4) There is v in S such that $Mv = mv$ (coincidence point).

Proof: Since (f, m) is the strongest straight with respect to (Γ, M) , then:

$\lim_{n \rightarrow \infty} \Gamma g_n = \lim_{n \rightarrow \infty} M g_n = D \in CB(G)$, whenever a sequences $\{g_n\}$ and $\{s_n\}$ in S thus, $\lim_{n \rightarrow \infty} f g_n = \lim_{n \rightarrow \infty} m g_n = q \in D$ and $q \in f(S) \cap m(S)$

If $q \in f(S)$, then $\exists u \in S, f(u) = q$, also, since $q \in m(S)$ then $\exists v \in S, m(v) = q$,

Hence $f(u) = m(v) = \lim_{n \rightarrow \infty} f g_n = \lim_{n \rightarrow \infty} m g_n = q \in D = \lim_{n \rightarrow \infty} \Gamma g_n = \lim_{n \rightarrow \infty} M g_n$,

q must belong to $F(u)$, if not, then $g = u$, and $s = s_n$, putting in (23), getting

$$\int_0^\sigma \lambda(t) dt \leq \sigma$$

For all $n \rightarrow \infty$ putting in (24), with $M s_n = D$, $m s_n = q$ then

$$\int_0^{H_\rho(\Gamma u, D)} \lambda(t) dt \leq \sigma \int_0^{\max} \left[\frac{\rho(fu, q), \rho(fu, \Gamma u), \rho(q, D)}{\rho(fu, D) + \rho(q, \Gamma u)} \right] \lambda(t) dt \quad (25)$$

$$\int_0^\sigma \lambda(t) dt \leq \sigma \int_0^{\max} \left[\frac{\rho(q, q), \rho(q, \Gamma u), \rho(q, D)}{\rho(q, D) + \rho(q, \Gamma u)} \right] \lambda(t) dt \quad (26)$$

$\leq \int_0^{\rho(q, \Gamma u)} \lambda(t) dt$ by Lemma 2, thus,
 $\leq \sigma \int_0^{H_\rho(\Gamma u, D)} \lambda(t) dt$. That is a contradiction.

Thus, $q \in D$. So, $q \in F(u)$. Then $q = fu \in F(u)$. That is proof (i). q must belong to $M(v)$, if not, then $g = g_n$, and $s = v$, putting in (23), getting $H_\rho(\Gamma g_n, Mv)$

$$\int_0^\sigma \lambda(t) dt \leq \sigma \int_0^{\max} \left[\frac{\rho(fg_n, mv), \rho(fg_n, \Gamma g_n), \rho(mv, Mv)}{\rho(fg_n, Mv) + \rho(mv, \Gamma g_n)} \right] \lambda(t) dt \quad (27)$$

For all $n \rightarrow \infty$ putting in (27), with $\Gamma g_n = D$, and $f g_n = q$ then

$$\int_0^{H_\rho(D, Mv)} \lambda(t) dt \leq \sigma \int_0^{\max} \left[\frac{\rho(q, mv), \rho(q, D), \rho(mv, Mv)}{\rho(q, Mv) + \rho(mv, D)} \right] \lambda(t) dt \quad (28)$$

getting

$$\int_0^{H_\rho(D, Mv)} \lambda(t) dt \leq \sigma \int_0^{\max} \left[\frac{\rho(q, q), \rho(q, D), \rho(q, Mv)}{\rho(q, Mv) + \rho(q, D)} \right] \lambda(t) dt \quad (29)$$

$\leq \sigma \int_0^{\rho(q, Mv)} \lambda(t) dt$ by Lemma 2, thus,
 $\leq \sigma \int_0^{H_\rho(D, Mv)} \lambda(t) dt$. That is a contradiction.

Thus, $q \in Mv$. So, $q \in Mv$. Then $q = mv \in Mv$. That is proof (ii).

We need to recall Definition (11).

Definition 11: Let (G, ρ) be a PMS. The maps $f: S \rightarrow G$ be single-valued, and $\Gamma: S \rightarrow CB(G)$ be multivalued map such that $S \subseteq G$. Then

- 5) The pair (f, Γ) is called to be quasi-coincidentally commuting if $fg \in \Gamma g$ for $g \in G$ with $fg, \Gamma g \in S$ such that $f\Gamma g \subset \Gamma fg$.
- 6) The map f is called coincidentally idempotent w.r.t Γ if $fg \in \Gamma g$ for $g \in G$ with $fg \in S$ such that $ffg = fg$.

In Theorem 4, using $f = m$ and $\Gamma = M$ yields the following

Corollary 6: Let (G, ρ) be a PMS, $f: S \rightarrow G$ and $\Gamma: S \subseteq G \rightarrow CB(G)$. If:

$$H_\rho(\Gamma g, \Gamma s) \leq \sigma \int_0^{H_\rho(\Gamma g, \Gamma s)} \lambda(t) dt$$

$$\leq \sigma \int_0^{\max\{\rho(fg, fs), \rho(fg, \Gamma g), \rho(fs, \Gamma s), \frac{\rho(fg, \Gamma s) + \rho(fs, \Gamma g)}{2}\}} \lambda(t) dt,$$

for all $g, s \in S$, $0 \leq \sigma < 1$ such that (1) holds. There is u in S such that $\Gamma u = fu$ (coincidence point). In addition, when the pair (f, Γ) is quasi-coincidentally commuting and coincidentally idempotent, then $\exists u \in S \ni \Gamma u - fu = u$.

Next, a special case of Theorem 4 by putting $\lambda(t) = 1$:

Corollary 7: Let (G, ρ) be a PMS, $f, m: S \rightarrow G$ and $\Gamma, M: S \subseteq G \rightarrow CB(G)$. If the property of the strongest straight for the pair (f, m) regarding the pair (Γ, M) holds and

$$H_\rho(\Gamma g, Ms) \leq \sigma \max \left[\begin{array}{l} \rho(fg, ms), \\ \rho(fg, \Gamma g), \rho(ms, Ms), \\ \frac{\rho(fg, Ms) + \rho(ms, \Gamma g)}{2} \end{array} \right]$$

for all $g, s \in S$, $0 \leq \sigma < 1$. Then

- 7) There is u in S such that $\Gamma u = fu$ (coincidence point).
- 8) There is v in S such that $Mv = mv$ (coincidence point).

In addition, when the pair (f, Γ) is quasi-coincidentally commuting and coincidentally idempotent, then Γ, M, f , and m have a common fixed point.

A case similar to Theorem 4 is presented in the next version, and the proof will also be similar to its proof.

Theorem 5: Let (G, ρ) be a PMS, $f, m: S \rightarrow G$ and $\Gamma, M: S \subseteq G \rightarrow CB(G)$. If

$$\int_0^{H_\rho(\Gamma g, Ms)} \lambda(t) dt \leq \psi \int_0^{\max \left[\begin{array}{l} \rho(fg, ms), \\ \rho(fg, \Gamma g), \rho(ms, Ms), \\ \frac{\rho(fg, Ms) + \rho(ms, \Gamma g)}{2} \end{array} \right]} \lambda(t) dt \tag{30}$$

for all $g, s \in S$, and $\psi: [0, \infty) \rightarrow [0, \infty)$ is non-decreasing and $\psi(0) = 0, \psi(r) > r$, for $r > 0$. Then

- 9) There is u in S such that $\Gamma u = fu$ (coincidence point).
- 10) There is v in S such that $Mv = mv$ (coincidence point).

Also, when the pair (f, Γ) is quasi-coincidentally commuting and coincidentally idempotent, then Γ, M, f , and m have a common fixed point.

In Theorem 5, if $\lambda(t) = 1, \forall t$, then the following will be satisfied.

Corollary 8: Let (G, ρ) be a PMS, $f, m: S \rightarrow G$ and $\Gamma, M: S \subseteq G \rightarrow CB(G)$. If the pair (f, m) is the strongest straight with respect to the pair (Γ, M) and

$$H_\rho(\Gamma g, Ms) \leq \psi \max \left[\begin{array}{l} \rho(fg, ms), \\ \rho(fg, \Gamma g), \rho(ms, Ms), \\ \frac{\rho(fg, Ms) + \rho(ms, \Gamma g)}{2} \end{array} \right]$$

for all $g, s \in S$, and for non-decreasing $\psi: [0, \infty) \rightarrow [0, \infty)$ where $\psi(0) = 0, \psi(r) > r$, for $r > 0$. Then

- 11) There is u in S such that $\Gamma u = fu$ (coincidence point).
- 12) There is v in S such that $Mv = mv$ (coincidence point).

Moreover, the maps Γ, M, f , and m have a common fixed point provided the pair (f, Γ) and (M, m) are quasi-coincidentally commuting and coincidentally idempotent.

4 APPLICATIONS

Finally, the authors encourage reading the results in [30] and formulating a suitable study in the PMS space. Chauhan et al. [30] have successfully extended the Suzuki-type contraction framework to a more generalized form, facilitating the identification of common fixed points for three mappings in metric spaces. Upon Suzuki-type contraction, the results obtained could be applied to solve specific classes of functional equations in dynamic programming. In the future, the authors look forward to applying current results to dynamic programming problems. This application may be inspired by the work of Chauhan et al. [30], also in fractals, as in [31]. It is also possible to delve into the role of the iterative method in data mining, where iterative theory provides the

mathematical foundation that ensures algorithms such as PageRank, k-means clustering, and gradient descent converge to meaningful solutions (often fixed points of a data-defined transformation).

5 CONCLUSIONS

The PMS offers a versatile and powerful environment for investigating spaces with self-referencing distances, making them useful in theoretical computer science. This versatility allows researchers to explore complex phenomena such as recursion, continuity, and convergence in various computational contexts. The current results are divided into two folds, the first Theorem 2 of which includes the derivation of common fixed points for Γ , M , as in Definition 8, that satisfy (1), if Γ , M are weakly increasing maps and Γ or M is continuous. Theorem 2 gives a similar result but with an alternative condition for continuity. Also, three corollaries and an illustrative example are presented. The second axis is presented in Theorem 2, which includes results about coincidence points, jointly, for a pair of single-valued maps and a pair of multi-valued maps. This is done by employing the concept of the strongest straight in Definition 9.

ACKNOWLEDGMENTS

This work was supported by the authors themselves. Thanks for the help provided to complete this work.

REFERENCES

- [1] S. Matthews, Partial metric topology, Research Report 212, Department of Computer Science, University of Warwick, 1992.
- [2] H. Jabar, M. Abed, and S. Behadili, "Unveiling patterns of nomophobia using data mining techniques," *Iraqi Journal of Science*, vol. 65, no. 8, pp. 4623–4632, 2024.
- [3] W. A. Wilson, "On quasi-metric spaces," *American Journal of Mathematics*, vol. 53, no. 3, pp. 675–684, 1931.
- [4] T. Nguyen and T. Michel, "Fixed points of regular set-valued mappings in quasi-metric spaces," *Journal of Mathematical Programming and Operations Research*, 2024. doi: 10.1080/02331934.2024.2399231.
- [5] M. Mamud and K. Tola, "Fixed point results for generalized (α, ψ) -contraction mapping in rectangular b-metric spaces," *Abstract and Applied Analysis*, pp. 1–12, 2022.
- [6] S. Albundi, "Iterated function system in 9-metric spaces," *Bulletin of Paraná's Mathematical Society*, vol. 4, no. 3s, pp. 1–10, 2022.
- [7] R. Thirumalai and S. Thalpathiraj, "A novel approach for some fixed point results in complete G-metric space by using asymptotically regular mapping in digital image," *International Journal of Applied Mathematics*, vol. 37, no. 3, pp. 311–332, 2024.
- [8] S. Latif and S. Abed, "Types of fixed points of set-valued contraction mappings for comparable elements," *Iraqi Journal of Science*, pp. 190–195, 2020.
- [9] S. Khatun, A. Banerjee, and R. Mondal, "Rough ideal convergence in a partial metric space," *General Topology*, 2025, pp. 1–10.
- [10] D. Gopal and S. Jain, *Fixed point theory in partial metric spaces*, 1st ed., New York: Chapman and Hall/CRC, 2021, pp. 1–16.
- [11] E. Güner and H. Aygün, "A new approach to fuzzy partial metric spaces," *Hacettepe Journal of Mathematics and Statistics*, vol. 1, no. 12, pp. 1563–1576, 2022.
- [12] L. Wangwe and S. Kumar, "Fixed point theorems for multi-valued α -F-contractions in partial metric spaces with an application," *Results in Nonlinear Analysis*, vol. 4, no. 3, pp. 130–148, 2021.
- [13] S. Seyed and M. Morteza, "Diameter approximate best proximity pair in fuzzy normed spaces," *Sahand Communications in Mathematical Analysis*, vol. 16, no. 1, pp. 17–34, 2019.
- [14] N. Mohammed and S. Abed, "On invariant approximations in modular spaces," *Iraqi Journal of Science*, vol. 62, no. 9, pp. 3097–3101, 2021.
- [15] B. Vijayabaskerreddy and V. Srinivas, "Matthews partial metric space using F-contraction," *Communications on Applied Nonlinear Analysis*, vol. 31, no. 5s, pp. 449–459, 2024.
- [16] H. Ahmad and M. Younis, "Mehmet Emir Köksal double controlled partial metric type spaces and convergence results," *Journal of Mathematics*, vol. 11, no. 1, pp. 1–11, 2021.
- [17] C. Ampadu, "Interpolative Berinde weak mapping theorem on partial metric spaces," *Earthline Journal of Mathematical Sciences*, vol. 15, no. 4, pp. 489–494, 2025.
- [18] C. Ampadu, "Interpolative Berinde Meir-Keeler weak contraction mapping theorem," *Earthline Journal of Mathematical Sciences*, vol. 15, no. 4, pp. 455–459, 2025.
- [19] H. Tiwari, Padmavati, and S. Sharma, "Some fixed point results on τ - ϕ -Berinde-contraction mappings in partial metric spaces," *Engineering Mathematics*, 2025.
- [20] C. Ampadu, "Interpolative Berinde weak cyclic contraction mapping principle," *Journal of Mathematical Sciences*, vol. 15, no. 4, pp. 235–238, 2025.
- [21] D. Babu, "Some best proximity theorems for generalized proximal Z-contraction maps in b-metric spaces with applications," *Communications in Mathematical Analysis*, vol. 22, no. 2, pp. 201–222, 2025.

- [22] Lj. B. Ćirić, M. Abbas, R. Saadati, and N. Hussain, "Common fixed points of almost generalized contractive mappings in ordered metric spaces," *Applied Mathematics and Computation*, vol. 217, pp. 5784–5789, 2012.
- [23] D. Błaszczuk, P. Matusiak, and R. Wang, "On compactness and fixed point theorems in partial metric spaces," *Fixed Point Theory*, vol. 23, no. 1, pp. 163–178, 2022. doi: 10.48550/arXiv.2005.06327.
- [24] M. Jleli, B. Samet, and C. Vetro, "Fixed point theory in partial metric spaces via ϕ -fixed point's concept in metric spaces," *Journal of Inequalities and Applications*, vol. 2014, no. 426, pp. 1–9, 2014.
- [25] H. Aydi, M. Abbas, and C. Vetro, "Partial Hausdorff metric and Nadler's fixed point theorem on partial metric space," *Topology and Its Applications*, vol. 159, pp. 3234–3242, 2012.
- [26] M. Priya and R. Uthayakumar, "Fractal set of generalized countable partial iterated function system with generalized contractions via partial Hausdorff metric," *Topology and Its Applications*, vol. 308, 2022.
- [27] V. Gupta, R. Saini, and M. Verma, "Fixed point theorem by altering distance technique in complete fuzzy metric spaces," *International Journal of Computer Aided Engineering and Technology*, vol. 13, no. 4, pp. 437–447, 2020.
- [28] W. Shatanawi and A. Al-Rawashdeh, "Common fixed points of almost generalized (ψ, ϕ) -contractive mappings in ordered metric spaces," *Fixed Point Theory and Applications*, pp. 1–14, 2012.
- [29] S. Chauhan, M. Imdad, E. Karapınar, and B. Fisher, "An integral type fixed point theorem for multi-valued mappings employing strongly tangential property," *Journal of the Egyptian Mathematical Society*, vol. 22, pp. 258–264, 2014.
- [30] S. Chauhan, M. D., M. J., and M. S., "Common fixed points results for three mappings under generalized contraction of Suzuki-type in b-metric spaces with application," *Baghdad Science Journal*, vol. 22, no. 6, pp. 2044–2061, 2025.
- [31] S. Albundi, "New create of Julia sets and some properties," *Bulletin of the Paraná Mathematical Society*, vol. 43, no. 3s, pp. 1–10, 2025.