Hybrid Wavelet-Adomian Decomposition Methods for Solving Nonlinear Differential Equations

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Keywords: CAS-Haar Wavelets, Adomian Decomposition, Ordinary Differential Equations, Operational Matrix.

Abstract: In this research paper, we had the study two of different types wavelets for solving some types of non-homogeneous with variable-coefficient nonlinear ordinary differential equations. Where the two wavelets were chosen from among the large number of wavelets known today in the field of function representation. The aim of studying this hybrid method, is to combine the wavelet method with the famous Adomian decomposition method, to find a more accurate and suitable approximation than using the wavelet method alone. In this research, referred to the two wavelets, after hybridizing with the Adomian decomposition method (ADM), are referred to as CAS-AM, CAS-Haar-AM. Additionally, the convergence theory of the proposed method was studied in L^2 space for CAS-Haar-AM series. The numerical results obtained when solving three examples using the proposed two-wavelet method were compared with the exact solution. Numerical stability was obtained even when the step size was large, and the results of the CAS-Haar-AM were better than using the CAS-AM. The proposed method CAS-Haar-AM is better, more accurate, and closer to the exact solution.

1 INTRODUCTION

The subject of analysis is one of the subject that has occupied a large space in the history of mathematics, as it is a tool used to solve ordinary and partial differential equations, whether linear or non-linear, in addition to solving differential equations, numerically integral equations are also used in a very important field, image processing, signal analysis also numerical solutions for linear and nonlinear partial differential equations plays a role important in the study many physical phenomena appear in various scientific and engineering fields, such as plasma physics, biology, fluid mechanics, optical fibers, chemical physics, solid state physics [1]-[3]. The study of wavelet analysis is beneficial many scientific applications such as signal analysis, data compression, time – frequency analysis, as well as in artificial intelligence and machine learning [4], [5].

Currently, wavelet analysis contributes to improving and accelerating the accuracy of solution to different differential and integral equations that uses in field such as medicine, engineering, physics, chemistry, biology, and other sciences. Therefore, these methods have attracted the attention of many researchers from various disciplines, prompting them to write and compose many different research papers and books in the field of wavelet analysis [6]-[8].

The method of wavelet analysis has got wide-ranging publically in academia due to its ease of use and the accuracy of its results in solving many mathematical, physical, and engineering problems, making it a focus of attention for researchers. Using these methods, the Haar integral matrix was derived by the two scientists Siddu C. S. and Lata [9], [10]. Siddu C. S. as well as R. A M. describe an accurate solution to Fredholm integral equation by using the wavelet method [11]. While the Abel equation was solved using CAS wavelets, error analysis of these wavelets was also studied [12]. A coupled system of fractional Fredholm integral equations was also solved using Haar wavelets [13]. Al-Rawi E. S. and Qasim A. M. are solved the ordinary differential equations by using NEW wavelet method and solved partial differential equation [14]. The researchers Yassen A. A. and Al-Rawi E. S. were able to solve an equation of Kortewg-de Vries numerically using nodding points and the NEW wavelet method. Dalal A. Maturi, Honaida M. Malaikah are solved the heterogeneous heat equation using the Adominant analysis method. with using Maple, where a high accuracy was obtained in the results, which are very close the exact solution [15].

In this research, we will apply the Adomian decomposition method with the CAS-Haar wavelet, to find solution many different nonlinear ordinary differential equations and nonhomogeneous with the assistant of Matlab and Maple software. Where the algorithms and programming were written using Matlab.

2 MAIN CONCEPTS

We introduce the required definitions and mathematical principles of wavelets. We have the following set of continuous waves when the expansion and translation parameters a and b are constantly fluctuating:

$$\Psi_{a,b}(x) = |a|^{-\frac{1}{2}}\,\Psi\left(\frac{x-b}{a}\right).\ a,b \in R, a \neq 0\ . \quad (1)$$

Lets determine parameter a, b to represent not continuous values which are:

$$a = a_0^{-k}, b = nb_0a_0^{-k}, a_0 > 1, b_0 > 0.$$

Where n, k are integer positive value numbers, then we have a group of discrete wavelets:

$$\Psi_{k,n}(x) = |a_0|^{\frac{k}{2}}\Psi(a_0^k\,x - n\,b_0). \quad (2)$$

When $\Psi_{k,n}(x)$ is a base of wavelet for $L^2(R)$. In especially, where $a_0 = 2, b_0 = 1$, then formula $\Psi_{k,n}(x)$ is an perpendicular bases [14], [16].

2.1 CAS Wavelets and Integrals

The function approximation of CAS wavelets are introduced $\Psi_{w,m}^{CAS}(x) = \Psi_{w,m}^{CAS}(k, w, m, x)$ have four parameters, $w = 0,1,2,\dots,2^k - 1$, CAS waves perpendicular to the interval [0, 1) are defined by:

$$\Psi_{w,m}^{CAS}(x) = \begin{cases} 2^k 2\, CAS_m\,(2^k x - w\,), & for\ \frac{w}{2^k} \le x < \frac{w+1}{2^k}, \\ 0, & othewise \end{cases} \quad (3)$$

Where $CAS_m(x) = cos(2m\pi x) + sin(2m\pi x)$, and $m \in \{-M, -M+1 \dots, M\}$. The CAS wavelets are perpendicular with regard for weighting function. $w(x) = 1$, [11]. Defined The integrals of CAS wavelet in (3) analytically [14].

$$P_{i,v}^{CAS}(x) = \begin{cases} 0 & 0 \le x < \frac{n}{2^k} \\ 2^{\frac{k}{2}}\frac{(-1)^{a_s}}{(2^{k+1}\pi m)^s}\cos\left(2\pi m(2^k x - w)\right) + 2^{\frac{k}{2}}\frac{(-1)^{b_s}}{(2^{k+1}\pi m)^s}\sin\left(2\pi m(2^k x - w)\right) - \sum_{jj=0}^{s-1} 2^{\frac{k}{2}}\frac{1}{jj\,!}\frac{(-1)^{a_s}}{(2^{k+1}\pi m)^{s-jj}}\left(x - \frac{w}{2^k}\right)^{jj}, & \frac{w}{2^k} \le x < \frac{w+1}{2^k} \\ \sum_{jj=0}^{s-1}\frac{1}{j\,!}\left(x - \frac{w+1}{2^k}\right)^{j}, . \left(\left(2^{\frac{k}{2}}\frac{(-1)^{a_s}}{(2^{k+1}\pi m)^s}\cos(2\pi m)\ 2^{\frac{k}{2}}\frac{(-1)^{b_s}}{(2^{k+1}\pi m)^s}\sin(2\pi m) - \sum_{jj=0}^{s-1} 2^{\frac{k}{2}}\frac{1}{jj\,!}\frac{(-1)^{a_s}}{(2^{k+1}\pi m)^{s-jj}}\left(\frac{1}{2^k}\right)^{jj}\right). & \frac{w+1}{2^k} \le x < 1 \end{cases} \quad (4)$$

Where:

$$a_s = \begin{cases} 0 & if\ \ s = 3,4,7,8,11,\dots. \\ 1 & if\ \ s = 1,2,5,6,9,10,\dots.' \end{cases}$$

and

$$b_s = \begin{cases} 0 & if\ \ s = 1,4,5,8,9,12,\dots\dots \\ 1 & if\ \ s = 2,3,6,7,10,11,\dots\dots. \end{cases}$$

The case $s = 0$ corresponds to CAS function $\Psi_{w,m}^{CAS}(x)$ in (3).

2.2 NEW Wavelets and Integrals

The NEW wavelet is denoted by CAS-Haar wavelets which is derived from CAS and Haar wavelets, and are consist of terms sine and cosine trigonometric functions with cyclist function and definite integration [14].

$$\Psi_{w,m}^{CAS-Haar}(x) = \begin{cases} \frac{2^{\frac{-k}{2}-1}}{\pi m}[\cos\left(2\pi m(2^k x - w)\right)) - sin\left(2\pi m(2^k x - w)\right) + 2sin(\pi m) - 2cos(\pi m) + 1], & \frac{w}{2^k} \le x < \frac{w+1}{2^k} \\ 0 & ,\ otherwise \end{cases} \quad (5)$$

Such that $m \in \{-M, -M+1, \dots, M\}$. The formula integrals of the NEW wavelets (CAS-Haar) specified in (5) are the generic formula:

$$P_{i,s}^{CAS-Haar}(x) = \begin{cases} 0, & 0 \le x < \frac{w}{2^k} \\ \frac{1}{(2.2^k)^s(\pi m)^{s+1}}\Big[\left(\frac{(-1)^s(2)^{s+1}}{s!}\right)sin(\pi m)(\pi m)^s w^s + \left(\frac{(-1)^{s+1}(2)^{s+1}}{s!}\right)cos(\pi m)(\pi m)^s w^s + (-1)^{b_s}\sin\left(2\pi m(2^k x - w)\right) + (-1)^{a_s}(\pi m)^2 cos(2\pi m(2^k x - w)) + (-1)^{c_s} + \sum_{j=1}^{r}\left(\frac{(-1)^{s+j+1}(2)^{s+1}}{(s-j)!}\right)\left(cos(\pi m)\right)(2^k x)^j (\pi m)^s w^{s-j} + \left(\frac{(-1)^{s+j}(2)^{s+1}}{(s-j)!.j!}\right)(sin(\pi m)(2^k x)^j(\pi m)^s w^{s-j}).\frac{(2\pi m(2^k x - w))^j}{j!}\Big], & \frac{w}{2^k} \le x < \frac{w+1}{2^k} \\ \sum_{j=1}^{s}[(-1)^{a_s}(2\pi m)^{s-j}cos(2m\pi) + (-1)^{b_s}(2m\pi)^{s-j}sin(2m\pi) - \frac{2^{s+1}}{j!}(\pi m)^s cos(\pi m) + \frac{2^{s+1}}{j!}(m\pi)^s\sin(\pi m) + \sum_{u=0}^{s-1}\frac{2^{s-u}}{(j-u)!}(\pi m)^{s-u}] + (-1)^{a_s}. & \frac{w+1}{2^k} \le x < 1 \end{cases} \quad (6)$$

Where:

$$a_s = \begin{cases} 0, & if\ s = 1,4,5,8,\dots \\ 1, & if\ s = 2,3,6,7,\dots \end{cases}$$

$$b_s = \begin{cases} 0, & if\ s = 1,2,5,6,\dots \\ 1, & if\ s = 3,4,7,8\dots \end{cases}$$

And,

$$c_s = \begin{cases} 0, & if\ s = 2,3,6,7,\dots \\ 1, & if\ s = 1,4,5,8,\dots \end{cases}$$

If $u > j\ then\ (j-u)! = 0, s$ is order of the integration. Any function $y(x) \in L^2[0,1)$ might be extended involving $\Psi_{w,m}^{\text{CAS-Haar}}(x)$ wavelets as:

$$y(x) = \sum_{w=1}^{\infty}\sum_{m\in z} C_{w,m}\ \Psi_{w,m}^{CAS-Haar}(x). \quad (7)$$

Where $C_{w,m} = < y(x), \Psi_{w,m}^{\text{CAS-Haar}} >$.

The sequence was unlimited in (7) is rounded, then (7) can be expressed as:

$$y(x) = \sum_{w=0}^{2^k-1}\sum_{m=-M}^{M} C_{w,m}\,\Psi_{w,m}^{CAS-Haar}(x) = C^T\Psi_{w,m}^{CAS-Haar}(x). \quad (8)$$

Where C and $\Psi_{w,m}^{\text{CAS-Haar}}$ are $2^k(2M+1)\times 1$ matrices given by:

$$C = \left[c_{0,(-M)}, c_{0,(-M+1)}, \dots, c_{0,M}, c_{1,(-M)}, \dots, c_{1,(M)}, c_{2^k-1,(-M)}, \dots, c_{2^k-1,(M)}\right]^T$$

$$\Psi_{w,m}^{CAS-Haar}(x) = \left[\Psi_{0,(-M)}(x), \Psi_{0,(-M+1)}(x), \dots, \Psi_{0,M}(x), \Psi_{1,(-M)}(x), \dots, \Psi_{2^k-1,(-M)}(x), \dots, \Psi_{2^k-1,M}(x)\right]^T$$

Now, we rewrite (8) as follows:

$$y(x) = \sum_{j=1}^{2^k(2M+1)} C_j\ \Psi_{2^k(2M+1),i}^{\text{CAS-Haar}}(x). \quad (9)$$

Where k is any positive integer number, x is variable, $w = 0,1,2,\dots,2^k-1$, M is any positive integer number.

2.3 Adomian Decomposition Method

The first to propose a new method for solving ordinary and partial differential equations was the Armenian scientist, which later became known as the Adomian decomposition method, named after him. This method essentially involves transforming any function y(x) into an infinite sum of terms, and by observing this sum and its closer to a previously known function. To solve any differential equation by this Method, we decompose the unknown function y(x) of any equation into the sum of an infinite number of components, using a decomposition series called the Adomian Decomposition.

We can write the function $y(x)$ by Adomian Decomposition Method as:

$$y(x) = \sum_{i=0}^{\infty} y_i(x). \quad (10)$$

Where the components $y_i(x), i \ge 0$, and for representing any ordinary differential equation as:

$$L[y(x)] + N[y(x)] + R[y(x)] = g(x) \quad (11)$$

When L is an operator of linear differential, N is a nonlinear terms, R is denotes the rest of terms, and $g(x)$ is a well-known function.

The nonlinear term $N[y(x)]$ is separated into a sequence of Adomian polynomials A_i:

$$N[y(x)] = \sum_{i=0}^{\infty} A_i. \quad (12)$$

Where the Adomian terms A_i are constructed from the components of $y_i(x)$.

$$A_i = \frac{1}{i!}\left[\frac{d^i}{d\lambda^i}N\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}. \quad (13)$$

Now by taking L^{-1} for both side of (11) we get:

$$y(x) - \vartheta = L^{-1}g(x) - L^{-1}N[y(x)] - L^{-1}R[y(x)]. \quad (14)$$

Where ϑ is represented the initial conditions, and from (10), (12) we get:

$$\sum_{i=0}^{\infty} y_i(x) = \vartheta + L^{-1}g(x) - L^{-1}\sum_{i=0}^{\infty} A_i(x) - L^{-1}R[\sum_{i=0}^{\infty} y_i(x)] \quad (15)$$

Now from (15) we have:

$$y_0(x) = \vartheta + L^{-1}g(x), y_{i+1}(x) = -L^{-1}(A_i(x) + R[y_i(x)]), i = 0,1,2, \dots . \quad (16)$$

Form the first term $y_0(x)$, that can be conclude the other terms respectively. But if any term of $y_0(x)$ equal to zero then the following terms are zeros [17]-[19].

3 METHODOLOGY

In this section, is done hybridize two wavelets, the first of wavelet is CAS, with method of Adomian, and the second wavelet is CAS-Haar, with method of Adomian. The introduced method, which has processed a nonlinear term in the nonlinear equation, with variable coefficients and nonhomogeneous ordinary differential equations as, will derive its formula, and then makes an algorithm for it

3.1 Mathematical Formulation of the Hybrid Method CAS-Haar-AM

The method of Adomian decomposition often employ to convert a nonlinear term into the linear term. In each wavelet-centered method, the mathematical formulas specific to the employed wavelet are used, when applying the CAS wavelet method, the CAS wavelet formulas are used, when applying the CAS-Haar wavelet method, the CAS-Haar wavelet formulas are used, and similarly, when applying the Haar wavelet method, the formulas corresponding to the Haar wavelet are exclusively employed. Consider the ordinary differential equation:

$$D^{(n)}y(x) + h(x)W(y(x)) + f(x)N(y(x)) = g(x) \quad (17)$$

Where $D^{(n)}y(x) = \frac{d^n y}{dx^n}$, $W(y(x))$ is a linear term, $N(y(x))$ is the nonlinear term, and $h(x), f(x), g(x)$ is known functions, with the initial conditions,

$$y(0) = \alpha_0, Dy(0) = \alpha_1, D^2y(0) = \alpha_2, \dots, D^{n-1}y(0) = \alpha_{n-1}. \quad (18)$$

Now applying the method of Adomian decomposition, can be written a solution of (17) as follows:

$$y(x) = \sum_{i=0}^{\infty} y_i(x). \quad (19)$$

Rounded Adomian sequence to make the differential equation close the solution as follows:

$$y(x) \approx \sum_{i=0}^{r} y_i(x), r\epsilon N \quad (20)$$

Substituting (20) in (17) we get:

$$D^{(n)}(\sum_{i=0}^{r} y_i(x)) + h(x)W(\sum_{i=0}^{r} y_i(x)) + f(x)(\sum_{i=0}^{r-1} A_i) = g(x).$$

Where $N(y(x)) = \sum_{i=0}^{r-1} A_i$, and A_i is define in (13). We will calculate $r+1$ sub-problems using the Adomian method of analysis and linear superposition as follows:

$$D^{(n)}(y_0(x)) + h(x)W(y_0(x)) = g(x). \quad (22)$$

With initial conditions in (18), and,

$$D^{(n)}(y_i(x)) + h(x)W(y_i(x)) + f(x)A_{i-1} = 0, \; i > 0. \quad (23)$$

With initial conditions:

$$y_i(0) = Dy_i(0) = D^2y_i(0) = \dots = D^{(n-1)}y_i(0) = 0. \quad (24)$$

Now by using the wavelets technique on (22) to find the approximate solution $y_0(x)$:

$$D^{(n)}y_0(x) = \sum_{i=1}^{m} c_j^0 \Psi_j(x) = C^T\Psi_j(x). \quad (25)$$

The symbol $\Psi_j(x)$ refers to the use of the wavelets (CAS and CAS-Haar). When the wavelet CAS is used, the corresponding symbol is $\Psi_{w,m}^{CAS}(x)$, as defined in (3), and the resulting solution $y(x)$ is obtained using the first proposed method CAS-AM. On the other hand, the symbol $\Psi_{w,m}^{CAS-Haar}(x)$, defined in (5), refers to the use of the second proposed method CAS-Haar-AM.

$$\Psi_j(x) = \begin{cases} \Psi_{w,m}^{CAS}(x), & by\ using\ CAS\ wvelet. \\ \Psi_{w,m}^{CAS-Haar}(x), & by\ using\ CAS-Haar\ wvelet. \end{cases} \quad (26)$$

Now by integrating (25) n times for x, from 0 to x, and use initial conditions in (18), we obtain:

$$y_0(x) = \sum_{j=1}^{m} c_j^0 P_{j,n}(x) + \phi(x). \quad (27)$$

Where $P_{j,n}(x)$ are the operational matrices of both wavelets:

$$P_{j,n}(x) = \begin{cases} P_{j,n}^{CAS}(x), & by\ using\ CAS\ wvelet. \\ P_{j,n}^{CAS-Haar}(x), & by\ using\ CAS-Haar\ wvelet. \end{cases} \quad (28)$$

And $\phi(x)$ is:

$$\phi(x) = \begin{cases} y(0), & if\ n=1 \\ y(0) + xDy(0), & if\ n=2 \\ y(0) + xDy(0) + \frac{1}{2!}x^2D^2y(0), & if\ n=3 \\ \vdots \end{cases} \quad (29)$$

Now, putting (25) and (27) in (22), with according to the value of $n = 1,2,3, \dots$.

$$\sum_{j=1}^{m} c_j^0\, \Psi_j(x) + h(x)W\left(\sum_{j=1}^{m} c_j^0\, P_{j,n}(x) + \phi(x)\right) = g(x). \quad (30)$$

Now we applying the collocation points x_j, where $x_j = \frac{j-0.5}{m}$, in (31), we have:

$$\sum_{j=1}^{m} c_j^0\, \Psi_j(x_j) + h(x)W\left(\sum_{j=1}^{m} c_j^0\, P_{j,n}(x_j) + \phi(x_j)\right) = g(x) \quad (31)$$

By extracting, c_i^0 is a common factor, and calculating the value of c_i^0 from (30) with using Matlab and Maple, then putting value of c_i^0 it into (27), we get the solution $y_0(x)$ this is the first term of a series Adomian decomposition.

Then, repeating the same previous steps on (23) when $i > 0$, the solutions are of $y_i(x)$.

$$D^{(n)} y_i(x) = \sum_{j=1}^{m} c_j^i\, \Psi_j(x) = C^T \Psi_j(x). \quad (32)$$

Where $\Psi_j(x)$ is in (26), now by integrating (32) n times for x, from 0 to x, and use initial conditions in (24), we obtain:

$$y_i\ (x) = \sum_{j=1}^{m} c_j^i\, P_{j,n}(x) + \phi(x). \quad (33)$$

Where $P_{j,n}(x)$ are defined in (28) and $\phi(x) = 0$, for all $n = 1,2,3, \dots$.

$$y_i\ (x) = \sum_{j=1}^{m} c_j^i\, P_{j,n}(x). \quad (34)$$

Now, putting (32), (34) in (23), with according to the value of $n = 1,2,3, \dots$.

$$\sum_{j=1}^{m} c_j^i\, \Psi_j(x) + h(x)W\left(\sum_{j=1}^{m} c_j^i\, P_{j,n}(x_j)\right) = -f(x)A_{i-1}(x). \quad (35)$$

Where $A_{i-1}(x_i)$ is Adomian terms as in (13), and by applying the collocation points x_j, in (35), we get:

$$\sum_{j=1}^{m} c_j^i\, \Psi_j(x) + h(x)W\left(\sum_{j=1}^{m} c_j^i\, P_{j,n}(x_j)\right) = -f(x)A_{i-1}(x_i). \quad (36)$$

By extracting, c_j^i is a common factor, and calculating the value of c_j^i from the (36) with using Matlab and Maple, then putting the value of c_j^i into (34), this yields $y_i(x), i > 0$ which is the remainder of the of an Adomian decomposition series. Finally, we collect the terms $y_0(x), y_i(x)$, to obtain the approximate solution of $y(x)$.

3.2 The Suggestion Algorithm CAS-AM

The summarize of this procedure can be written as in the following given seven steps.

Step 1: Input the formula of CAS wavelet, integrals and initial conditions.

Step 2: Put the general differential (17), with the Adomian decomposition series in (20) in (17).

Step 3: Set the expansion wavelet

$$D^n y_0(x) = \sum_{j=1}^{m} c_j^0\, \Psi_j^{CAS}(x) = C^T \Psi_j^{CAS}(x).$$

Step 4: Integrate the expansion wavelet in step (3) n times from (0) to (x), we obtain:

$$y_0\ (x) = \sum_{j=1}^{m} c_j^0\, P_{j.n}^{CAS}(x) + \phi(x).$$

where $P_{j,n}(x)$ as in (3) and (4). And $\phi(x)$ as in (29):

Step 5: Put the estimated solution $y_0\ (x)$ and differentials of $D^n y_0(x)$ in (22) with the mesh points x_j, then get a collocation of algebraic equations.

$$\sum_{j=1}^{m} c_j^0\, \Psi_j^{CAS}(x_j) + h(x)W\left(\sum_{j=1}^{m} c_j^0\, P_{j.n}^{CAS}(x_j) + \phi(x_j)\right) = g(x_j).$$

Step 6: Find the solution of the collocation of algebraic equations obtained in step (5), for the constants c_j^0 of the proposed wavelet method and computed the close solution for $y_0(x)$.

Step 7: Repeat the same steps (2-6) for calculate y_i, by setting $g(x) = 0$, with initial conditions in (24).

Step 8: Calculate $y_0\ (x) + y_i\ (x)$ to obtain $y(x)$ which represent the approximate solution of ordinary differential equation.

With substituting $\Psi_j^{CAS}(x), P_{j.n}^{CAS}(x)$ wavelet and its integrals by $P_{j.n}^{CAS--Haar}(x), \Psi_j^{CAS-Haar}(x)$ wavelet, respectively, we obtained the second presented the algorithm.

3.3 Convergence Analysis of the CAS-Haar-AM

To calculate the convergence analysis of the proposed wavelet method, it is very important to know the CAS-Haar wavelet analysis method, the method of Adomian decomposition, because the convergence analysis of the presented method depends on them. As is well known, for the approach of the CAS-Haar wavelet, we can assume $y_i(x)$, which represents the finite Adomian series. Now, let $y_i(x)$ be a differentiable function on the open interval (0,1), and suppose that the first derivative of $y_i(x)$ is bounded on the same open interval. This leads to the existence of a positive integer k (k > 0) such that the condition holds for every $x \in (0,1)$.

$|y_i'(x)| \leq k$

We know that CAS-Haar wavelet approximation for any function $y_i(x)$ is given by:

$$y_{M,i}(x) = \sum_{j=1}^{2M} c_j^i \, \Psi_j \ (x).$$

By Ahmed H. H. [20], we get this theorem:

Theorem: Suppose that $y(x)$ is a continuous function defined in $L^2[0,1]$ with a restricted partial derivative $\left|y''(x)\right| \leq M$, then the error approximation of the CAS-Haar wavelet series is defined as:

$$|y_{num}(x) - y(x)| \leq M_1 \sum_{n=2^k+1}^{\infty} \sum_{m=M+1} \left(\frac{1}{4\pi^3 2^{\frac{7}{2}k}} + \frac{3}{m\pi 2^{\frac{3}{2}k}} \right). \quad (37)$$

Where M_1 is the maximum bound of the function and its derivatives. As $M = 2^J$ and J is the maximal level of resolution. It is clear from (37) that the magnitude of the error decreases as the level of accuracy increases. Hence, (37) ensures the convergence of the CAS-Haar wavelet method $y_{num}(x)$ to the Adomian series expansion y_i (x)at a higher level of accuracy. That's mean when $M = 2^J \to \infty$, and According to the convergence of Adomian's method [2], then we get $\sum_{i=0}^{N-1} y_i(x)$ converges to $y(x)$ when N approaches to infinity. Thus, the solution obtained by the present CAS-Haar wavelet method converges to the exact solution of (17) as N and J approach infinity.

4 NUMERICAL EXAMPLES

To show the efficiency of the suggestion method, we will apply CAS-AM and CAS-Haar-AM to obtain the approximate solution of the nonlinear nonhomogeneous ordinary differential equation. All of the computations have been performed using Matlab 13. Lenovo laptop, model X380 Yoga, with 1.70GHz processor and 128 MB cache memory.

Example 1. Consider nonlinear ODE $y'' + xy' + x^2y^3 = \left(2 + 6e^{x^2}\right)e^{x^2} + x^2e^{3x^2}$, where initial conditions $y(0) = 1, y'(0) = 0$, where the exact solution $y(x) = e^{x^2}$, $N\big(y(x)\big) = y^3$,and $g(x) = \left(2 + 6e^{x^2}\right)e^{x^2} + x^2e^{3x^2}$, [21].

After using the steps of the suggestion algorithm, we got the results. In Tables 1 and 2, compare the numerical results of the hybrid wavelet method with the exact solution by using the absolute error ABSE and mean square error MSE:

$$ABSE = \left|y_{num}(x_j) - y(x_j)\right|.$$

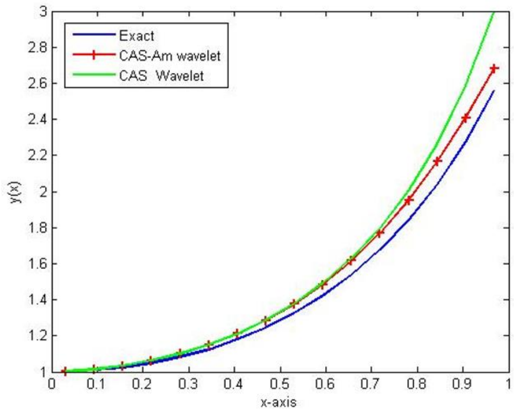
$$MSE = \frac{\sqrt{\sum_{j=0}^{m}\left(y_{num}(x_j) - y(x_j)\right)\big)^2}}{2m}.$$

Table 1: Comparison between the exact solution, CAS-AM, and CAS-Haar-AM at $m = 16$ in example 1.

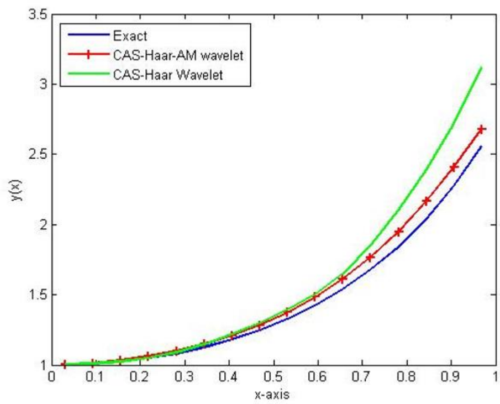
$\frac{x}{32}$	Exact	CAS	CAS -AM	CAS-Haar	CAS-Haar-AM
1	1.0009770	1.0019588	1.0019588	1.001469	1.001933
3	1.0088277	1.0137658	1.0137655	1.010197	1.013576
5	1.0247145	1.0337881	1.0337845	1.024804	1.033299
7	1.0490149	1.0625872	1.0625652	1.048550	1.061792
9	1.0823142	1.1009893	1.1009002	1.088036	1.100006
11	1.1254287	1.1501318	1.1498512	1.146945	1.149027
13	1.1794391	1.2115298	1.2107814	1.218291	1.210049
15	1.2457360	1.2871707	1.2853948	1.299213	1.284546
17	1.3260791	1.3796494	1.3757794	1.391565	1.374535
19	1.4226752	1.4923623	1.4844439	1.502487	1.482738
21	1.5382786	1.6297911	1.6143286	1.649276	1.612474
23	1.6763210	1.7979254	1.7687421	1.851965	1.767242
25	1.8410785	2.0049091	1.9511145	2.101287	1.950187
27	2.0378881	2.2620506	2.1643160	2.386409	2.163618
29	2.2734285	2.5854398	2.4089479	2.715043	2.407915
31	2.5560844	2.9986006	2.6791601	3.117695	2.677746

Table 2: The mean square error MSE of the wavelets for different values of m of example 1.

m	CAS	CAS-AM	CAS-Haar	CAS-Haar-AM
8	3.9918106e-2	2.0635195e-2	5.0794197e-2	2.0392438e-2
16	1.9890979e-2	9.1645172e-3	2.7212667e-2	9.0517873e-3
32	1.0791459e-2	4.2448017e-3	1.7674342e-2	4.2146526e-3



(a)



(b)

Figure 1: Comparison between a) CAS -AM with CAS, b) CAS-Haar-AM with CAS-Haar of example 1.

The mean square error MSE of the numerical solutions resulting from the application of the hybrid wavelets and for different values of m, $m = 8,16,32$ was calculated in Table 2, and we note that the CAS-Haar-AM gives a lower MSE, which means that increasing the value of m leads to a decrease in error. In Figure 1, we can see the behavior of the numerical solution resulting from the application of the CAS-AM and CAS-Haar-AM, which were compared to the exact solution at $m = 16$.

Example 2. Consider nonlinear ODE $y'' + yy' = xsin(2x^2) - 4x^2 sinx^2 + 2cosx^2$, with initial condition $y(0) = 0, y'(0) = 0$, where the exact solution $y(x) = sinx^2$, $N(y(x)) = yy'$, and $g(x) = xsin(2x^2) - 4x^2 sinx^2 + 2cosx^2$. [21]

After using the steps of the suggestion algorithm, we got the results. In Tables 3 and 4, compare the numerical results of the hybrid wavelet method with the exact solution by using the absolute error ABSE and mean square error MSE.

The mean square error MSE of the numerical solutions resulting from the application of the hybrid wavelets and for different values of m, $m = 8,16,32$ was calculated in Table 4, and we note that the CAS-Haar-AM gives a lower MSE, which means that increasing the value of m leads to a decrease in error. In Figure 2, we can see the behavior of the numerical solution resulting from the application of the CAS-AM and CAS-Haar-AM, which were compared to the exact solution at $m = 16$.

Example 3. Consider nonlinear ODE $y'' + xy' + 2e^y = -\frac{2(2x^4+1)}{(x^2+1)}$, with initial condition $y(0) = 0, y'(0) = 0$, where the exact solution $y(x) = -2\ln(x^2+1)$, $N(y(x)) = e^y$, and $g(x) = -\frac{2(2x^4+1)}{(x^2+1)}$.

After applying the steps of the suggestion algorithm, we got the results in Tables 5 and 6, compare the numerical results of the hybrid wavelet method with the exact solution by using the absolute error ABSE and mean square error MSE.

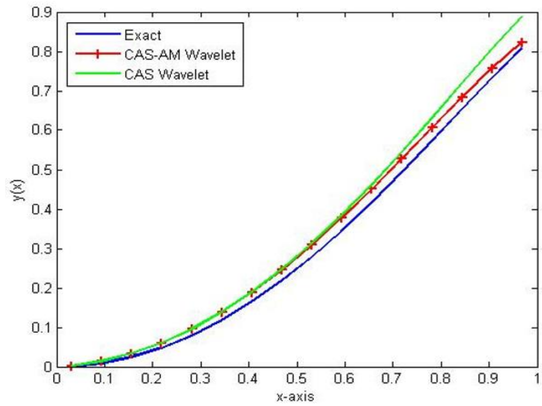
The mean square error MSE of the numerical solutions resulting from the application of the hybrid wavelets and for different values of m, $m = 8,16,32$ was calculated in Table 6, and we note that the CAS-Haar-AM gives a lower MSE, which means that increasing the value of m leads to a decrease in error. In Figure 3, we can see the behavior of the numerical solution resulting from the application of the CAS-AM and CAS-Haar-AM, which were compared to the exact solution at $m = 16$.

Table 3: Comparison between the exact solution, CAS-AM, and CAS-Haar-AM at $m = 16$ in example 2.

$\frac{x}{32}$	Exact	CAS	CAS -AM	CAS-Haar	CAS-Haar-AM
1	0.0009765	0.0019530	0.0019530	0.0014648	0.0019274
3	0.0087889	0.0136716	0.0136702	0.0101266	0.0134817
5	0.0244116	0.0332040	0.0331892	0.0243537	0.0327109
7	0.0478333	0.0605507	0.0604829	0.0466899	0.0597228
9	0.0790190	0.0957012	0.0954879	0.0819498	0.0945861
11	0.1178892	0.1386159	0.1380789	0.1317281	0.1371791
13	0.1642908	0.1892005	0.1880356	0.1893717	0.1871554
15	0.2179627	0.2472718	0.2450007	0.2519608	0.2440317
17	0.2784948	0.3125147	0.3084328	0.3193384	0.3072574
19	0.3452818	0.3844300	0.3775507	0.3927312	0.3761528
21	0.4174743	0.4622728	0.4512753	0.4747329	0.4497385
23	0.4939280	0.5449812	0.5281697	0.5661950	0.5265794
25	0.5731555	0.6310993	0.6063836	0.6616760	0.6047635
27	0.6532841	0.7186943	0.6836087	0.7576743	0.6819735
29	0.7320246	0.8052768	0.7570544	0.8525837	0.7554808
31	0.8066586	0.8877302	0.8234555	0.9436549	0.8220261

Table 4: The mean square error MSE of the wavelets for different values of m in example 2.

m	CAS	CAS-AM	CAS-Haar	CAS-Haar-AM
8	1.23132e-2	8.51177e-3	1.05844e-2	7.84623e-3
16	5.35178e-3	3.03309e-3	8.05527e-3	2.88887e-3
32	2.57629e-3	1.04331e-3	5.82305e-3	1.01009e-3



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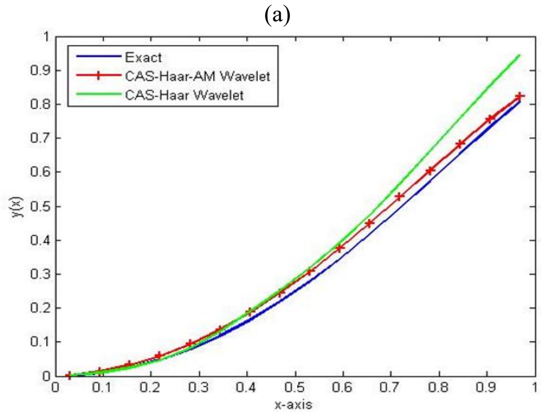


(b)

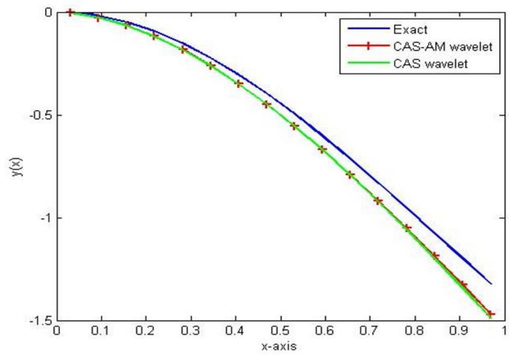
Figure 2: Comparison between a) CAS -AM with CAS, b) CAS-Haar-AM with CAS-Haar of example 2.

Table 5: Comparison between the exact solution, CAS-AM, and CAS-Haar-AM at $m = 16$.

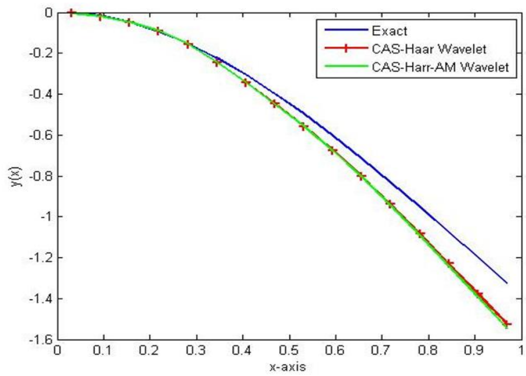
$\frac{x}{32}$	Exact	CAS	CAS -AM	CAS-Haar	CAS-Haar-AM
1	-0.0019521	-0.0038910	-0.0038910	-0.0029204	-0.0029204
3	-0.0175013	-0.0271479	-0.0271476	-0.0201381	-0.0201380
5	-0.0482416	-0.0653655	-0.0653623	-0.0480655	-0.0480642
7	-0.0934838	-0.1177964	-0.1177803	-0.0909083	-0.0908992
9	-0.1522576	-0.1835020	-0.1834463	-0.1560605	-0.1560136
11	-0.2233762	-0.2614177	-0.2612665	-0.2441983	-0.2440342
13	-0.3055092	-0.3504185	-0.3500720	-0.3432192	-0.3428328
15	-0.3972534	-0.4493782	-0.4486779	-0.4484898	-0.4477634
17	-0.4971961	-0.5572182	-0.5559323	-0.5595491	-0.5583166
19	-0.6039672	-0.6729417	-0.6707544	-0.6776730	-0.6756360
21	-0.7162774	-0.7956568	-0.7921610	-0.8058903	-0.8024452
23	-0.8329440	-0.9245869	-0.9192819	-0.9456898	-0.9398883
25	-0.9529050	-1.0590720	-1.0513666	-1.0917594	-1.0828446
27	-1.0752241	-1.1985638	-1.1877836	-1.2408807	-1.2283471
29	-1.1990890	-1.3426154	-1.3280152	-1.3926566	-1.3759938
31	-0.3238047	-1.4908695	-1.4716485	-1.5475771	-1.5261001

Table 6: The mean square error MSE of the wavelets for different values of m in example 3.

m	CAS	CAS-AM	CAS-Haar	CAS-Haar-AM
8	2.1181991e-2	1.9809759e-2	1.1236813e-2	9.5943381e-3
16	1.0183500e-2	9.3742766e-3	1.284057e-2	1.1886652e-2
32	5.4810106e-3	4.9621424e-3	1.0214036e-2	9.5827074e-3



a



b

Figure 3: Comparison between (a) CAS -AM with CAS, (b) CAS-Haar-AM with CAS-Haar of example 3.

5 CONCLUSIONS

In this research, the hybrid two wavelets CAS and CAS-Haar with the Adomian decomposition method were developed as efficient and accurate methods for solving nonlinear ordinary differential equations, on the three ordinary differential equations nonhomogeneous with variable coefficients. The suggestion methods were applied with collocation points. From a theoretical perspective, as mentioned, the study analyzed the convergence and error estimation in the L^2 space of hybrid wavelet chains. This is important because it ensures that numerical approximations converge toward the true solution as the resolution of the representation increases The numerical solutions of the three examples in the Tables 1, 3, 5 and Figures 1-3 show that, the CAS-Haar-AM produce results that are closer to the exact solution. It is clear from Tables 2, 4, 6 that if the values of m and k are small, then the CAS-Haar-AM solutions give excellent results. Also, numerical solution converge to the exact values when $2^k(2m+1) = 36, 2^k(2m+1) = 44, \ldots$. This study has contributed to providing more accurate results, with better approximations and in less time. Finally the suggestion methods is applicable for Partial differential equation, fractional ordinary differential equations and even if applied on the fractional partial differential equations.

6 ACKNOWLEDGMENTS

The completion of this research would not have been possible without the support and assistance of "College of Computer Sciences and Mathematics, University of Mosul, Mosul, Iraq". We would like to express our deepest gratitude.

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