New Probability Distribution for Lifetime Data Analysis

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Abstract: In various fields, the analysis of lifetime data is a significant focus. This analysis relies on selecting an appropriate probability distribution that accurately reflects the characteristics of the phenomenon under study and accurately determines the behavior of the data. This study proposes a new probability distribution for modeling and representing a various set of lifetime data. Some The mathematical properties of the distribution are derived. The maximum likelihood approach is used to estimate the distribution parameters($\underline{\omega} = \alpha, \beta, \theta, \gamma$). The behavior for the pdf $f(x; \underline{\omega})$ of the proposed new distribution demonstrates that it has the flexibility to represent data through its ability to take on different forms. The flexibility of the distribution is examined by applying it to lifetime data and comparing it with other distributions. Based on several statistical comparison criteria (AIC the Akaike information criterion, BIC the Bayesian information criterion, and AICc the adjusted Akaike information criterion), the results indicate that the proposed distribution provides a better fit to the data.

1 INTRODUCTION

The field of statistical distributions has witnessed tremendous progress in recent decades, primarily due to its pivotal role in enabling decision-making in a wide range of fields, such as medicine, engineering, and economics. This field is a fundamental component of statistical modeling, providing a precise mathematical framework for determining the appropriate distribution for analyzing data and gaining a deeper understanding of its nature. However, choosing the wrong distribution can lead to increased estimation error and erroneous results, which can have serious consequences for decision-making processes. While the normal distribution is widely used in the current literature, many real-world phenomena lack consistency with its underlying assumptions, prompting researchers to construct new distributions or propose more flexible extensions of standard distributions.

Statistical distributions are an essential tool in data analysis, used to describe random events and accurately determine the nature of a wide range of variables. In this context, lifetime distributions have proven to be a critical application area of statistical distributions, given their prominent role in determining the lifetime of components and predicting the time to failure or the time of use to failure. They are, therefore, a crucial tool in reliability studies. Moreover, in medicine, they are widely used to estimate the survival time of patients after the onset of certain diseases or following a specific treatment. This application provides a scientific basis for making intelligent medical decisions and formulating effective treatment plans, according to Taketomi et al [1]. Therefore, accurate modeling of lifetime data is crucial for gaining insight into the nature of the phenomenon and for formulating more realistic, practical, and scientifically informed decisions based on sound statistical principles. Although the exponential and the Weibull distributions are commonly used in the field of lifetime data analysis, they have some limitations that restrict their ability to accurately represent all types of data, as pointed out by Tomy et al [2]. The exponential distribution is suitable for modeling failure or death times, such as those resulting from accidental events that are not related to operational or biological age. It is characterized by a constant hazard rate and a memoryless property (i.e., the probability of failure/death at the next moment is independent of the previous operational or biological age). Therefore, it is employed medically to represent sudden deaths that are not caused by gradual deterioration (such as severe bleeding or sudden infection), and in engineering to represent random failures, such as a sudden mechanical failure that causes a fracture in a healthy component. However, its suitability declines when the risk changes over time or takes on non-monotonic patterns, necessitating choosing distributions that are flexible enough to suit the data.

Within this framework, the exponential distribution was developed to produce the weibull distribution. which is characterized by a flexibility that exceeds that of the exponential distribution. because it represents hazard rates that increase or decrease. Therefore, it is suitable for modeling monotonous hazard rates. However, it does not provide a good fit when the hazard rates are non-monotonic [3], in which the lifespan of the device is divided into three main stages [4]:

1) Infant Mortality / Early Failures stage. The hazard rate is high and then decreases over time due to manufacturing defects/postoperative complications. At this stage, the β parameter of the Weibull distribution is takes a value less than one ($\beta < 1$), indicating a decreasing risk.
2) Stability stage (Useful Life / Constant–Low Hazard). The hazard rate is low and close to constant, where the hazard rate is relatively constant and the probability of failure/death is low (momentary failure not related to operational life/sudden deaths not caused by gradual deterioration). At this stage, the β parameter of the Weibull distribution equals one ($\beta = 1$), indicating a constant risk, which reduces it to an exponential distribution.
3) Wear-Out / Aging stage. The hazard rate begins to increase exponentially due to the end of the life of components, the aging of materials, and the worsening of chronic diseases. The probability of failure/death increases. At this stage, the β parameter of the Weibull distribution is takes a value greater than one ($\beta > 1$), indicating an increasing risk.

Therefore, the weibull and the exponential distributions are incapable of representing more complex patterns of hazard functions, such as the bathtub-shaped hazard function, which combines three phases. Therefore, models with only monotonic failure rates are not appropriate or sufficient for modeling populations that generate such data. Relying solely on the exponential or Weibull distribution in such cases may lead to inaccurate or biased conclusions, especially when the causes of accidental failure/mortality overlap with those of wear and aging. These limitations reveal the limitations of modeling with the exponential and Weibull distributions when faced with complex patterns of lifetime data; thus, a research need has emerged for more flexible distributions.

Therefore, the need arose for more flexible families of lifetime distributions that could accommodate data characteristics such as positive skewness and heavy tails, and provide hazard rate functions that accommodate various failure/death patterns (decreasing, constant, increasing, single-peaked, and bathtub-shaped). In response to these challenges, researchers have developed several methods for generating more flexible distributions, such as generalizations, mixtures, competing risks, and others, with the aim of achieving a better fit to the data. From a methodological perspective, developing a distribution that fits the data characteristics or choosing a more expressive model is often more feasible than modifying the data itself in transformative ways that might distort its properties.

To address the limitations of traditional distributions, the competing risks methodology was adopted as a constructive framework for generating more flexible lifetime distributions that support multiple data representation patterns and expand the scope of applications. The competing risks framework assumes that an event (failure/death) occurs when the first of several independent causes occurs. The event time is formulated as: $T = \min(T_1, T_2, \ldots, T_K)$, where Ti represents the event time associated with cause (i). These causes compete to cause the event. Therefore, this case is called competing risks. Within reliability, the system lifetime is the smallest of its independent components, and the cause is defined as the component that failed first. In medical terms, within survival analysis, the same idea emerges when there are multiple mechanisms (e.g., cancer-specific death versus death from other causes) [5].

With this concept, the framework enables the generation of highly flexible, generative distributions that account for multiple sources of failure/death, as well as diverse patterns of hazard functions. The hazard rate function is not restricted to a single form. It can be (decreasing, constant, increasing, bathtub-like, or more complex) depending on the chosen distributions and parameters of each cause, reflecting the practical reality of multiple sources of failure/death in both engineering and medicine.

When no cause information is available, it is sufficient to model the total event time using an additive hazard model (we consider the hazards to work together to form the total rate) Focusing on the total effect of the causes rather than identifying a specific cause. Specifically, this approach is related to the competing risks framework in terms of its computational structure. However, it does not require recording the cause in the data, and it is practically suitable when the cause is unknown (for example, a complex system fails without a precise identification of the source of the failure).

In recent decades, the Weibull distribution has been further developed due to its pivotal importance in lifetime data analysis, reliability, and survival applications. Xie and Lai introduced the (AW) distribution by combining two weibull distributions [6]. In a later search, Lai, Xie, and Murthy proposed a (mw) distribution, which is characterized by different patterns to the hazard function [7]. Bebbington, Zitikis, and Lai also developed the flexible (w) extension, which provided a basic rule for generating more flexible posterior distributions [8]. In addition, Almalki and Yuan proposed a new model, the New Modified Weibull (NMW) distribution, within the framework of competing risks [3], while Cordeiro, Ortega, and Lemonte introduced the Exponential-Weibull distribution [9]. In the same context, He, Cui, and Du formulated the (AMW) distribution [10], and Oluyede et al, the (Log-Logistic (W)) distribution [11]. Mdlongwa et al, introduced the (Burr XII-W) distribution [12]. Later, Tarvirdizade and Ahmadpour introduced the w–c distribution [13], while Shakhatreh, Lemonte, and Moreno-Arenas discussed the ((Log-normal) - (MW)) distribution [14]. The (L–W) distribution was introduced by Osagie and Osemwenkhae [15], and the Flexible W Extension was introduced by Kamal and Ismail [16]. More recently, Thach and Bris have introduced the A (C–W) distribution [17], and Khalil et al, have introduced another flexible distribution, the Flexible AW, as the mixture between three W distributions [18].

Using framework the competing risks, sequential a sequential system with two independent components working in series to present the Weibull-WW distribution. The following is relevant:

- The Weibull distribution is used to determine the lifespan of the first component, X_1, and the WW distribution is used to determine the lifespan of the second component, X_2 [19].
- The minimum of the two components, X= min (X_1, X_2), represents the system's total lifespan and it is distributed weibull-ww.

The Reliability (or Survival) function for the weibull distribution takes the following form:

$$S_1(x;\theta,\beta) = e^{-\theta\, x^{\beta}}, \quad x > 0;\ \theta,\beta > 0 \qquad (1)$$

And the hazard function of the weibull distribution:

$$h_1(x;\theta,\beta) = \theta\beta x^{\beta-1},\ x > 0;\ \theta,\beta > 0. \qquad (2)$$

Where: θ scale parameter, β shape parameter.

The following are the expressions for the WW distribution's hazard and reliability (or survival) functions, respectively:

$$S_2(x;\alpha,\gamma) = e^{-\frac{\alpha}{\gamma}(\frac{x}{\gamma})^{2\alpha-1}}, \quad x > 0;\ \alpha,\gamma > 0, \qquad (3)$$

$$h_2(x;\alpha,\gamma) = \frac{\alpha(2\alpha-1)}{\gamma^{2\alpha}}\,(x)^{2(\alpha-1)}, x > 0;\ \alpha,\gamma > 0. \qquad (4)$$

Where γ scale parameter, α shape parameter.

2 THE NEW DISTRIBUTION

This section explains how the new distribution is constructed, and graphical illustration of the probability density function (pdf) are presented.

2.1 Constructing the New Distribution (weibull-ww)

Refers to the distribution of weibull-ww as a generalization of the weibull and ww distributions, in order to add more flexibility for modeling lifetime data, Let X be a non-negative random variable.

- The reliability (or survival) function to the new distribution can be derived by multiplying the reliability (or survival) functions of the weibull distribution and ww distribution as follows:

$$S(x;\underline{\omega}) = \prod_{i=1}^{2} S_i(x) = e^{-\theta\, x^{\beta}-\frac{\alpha}{\gamma}(\frac{x}{\gamma})^{2\alpha-1}}, x > 0;\ \underline{\omega} > 0 \qquad (5)$$

Where $\underline{\omega} = (\theta,\beta,\alpha,\gamma)$ denotes the vector of parameters

- The cumulative distribution function for the new distribution as follows:

$$F(x;\underline{\omega}) = 1 - S(x;\underline{\omega}) = 1 - e^{-\theta\, x^{\beta}-\frac{\alpha}{\gamma}(\frac{x}{\gamma})^{2\alpha-1}},\ x > 0;\ \underline{\omega} > 0 \qquad (6)$$

- The hazard function of the new distribution. Using (2) and (4), we obtain:

$$h(x;\underline{\omega}) = h_1(x;\theta,\beta) + h_2(x;\alpha,\gamma) = \theta\beta x^{\beta-1} + \frac{\alpha(2\alpha-1)}{\gamma^{2\alpha}}\,(x)^{2(\alpha-1)} \qquad (7)$$

- The probability density function of the new distribution as follows:

$$f(x;\underline{\omega}) = h(x;\underline{\omega})\, S(x;\underline{\omega}) = \left(\theta\beta x^{\beta-1} + \frac{\alpha(2\alpha-1)}{\gamma^{2\alpha}}\,(x)^{2(\alpha-1)}\right) e^{-\theta\, x^{\beta}-\frac{\alpha}{\gamma}\left(\frac{x}{\gamma}\right)^{2\alpha-1}}, x>0, \underline{\omega}>0. \quad (8)$$

- The inverse hazard function and cumulative hazard function of the new distribution can be expressed as follows:

$$r(x;\underline{\omega}) = \frac{f(x;\underline{\omega})}{F(x;\underline{\omega})} = \frac{\left(\theta\beta x^{\beta-1} + \frac{\alpha(2\alpha-1)}{\gamma^{2\alpha}}\,(x)^{2(\alpha-1)}\right) e^{-\theta\, x^{\beta}-\frac{\alpha}{\gamma}\left(\frac{x}{\gamma}\right)^{2\alpha-1}}}{1-e^{-\theta\, x^{\beta}-\frac{\alpha}{\gamma}\left(\frac{x}{\gamma}\right)^{2\alpha-1}}}, x > 0, \underline{\omega} > 0 \quad (9)$$

And

$$H(x;\underline{\omega}) = -\ln R(x;\underline{\omega}) = -\ln\left(e^{-\theta\, x^{\beta}-\frac{\alpha}{\gamma}\left(\frac{x}{\gamma}\right)^{2\alpha-1}}\right) = \theta\, x^{\beta} - \frac{\alpha}{\gamma}\left(\frac{x}{\gamma}\right)^{2\alpha-1}, x>0, \underline{\omega}>0 \quad (10)$$

The new distribution introduces a methodology for modeling the lifetime of a component or individual subject to two independent failure mechanisms simultaneously, such that the occurrence of either of them leads to the overall failure of the system, where representing a series system composed of two independent components.

2.2 Graphical Illustration

In this section, the probability density function plots of the new distribution are presented to illustrate the flexibility of the distribution.

As shown in Figure 1a, the influence of the parameter α in determining the shape of the Weibull–WW distribution probability density function is clearly evident. The graph shows that the curve gradually decreases when α is small, begins to form a distinct peak, and transforms into a unimodal shape as α increases. We also notice that the peak's location shifts toward larger values of x and its height increases significantly, indicating that failure times become more concentrated at specific values as the dispersion decreases.

This behavior highlights the crucial role of the parameter α in controlling the distribution's behavior. It emphasizes the model's flexibility in representing different patterns of failure rates across the system's life cycle, from early failures to terminal wear.

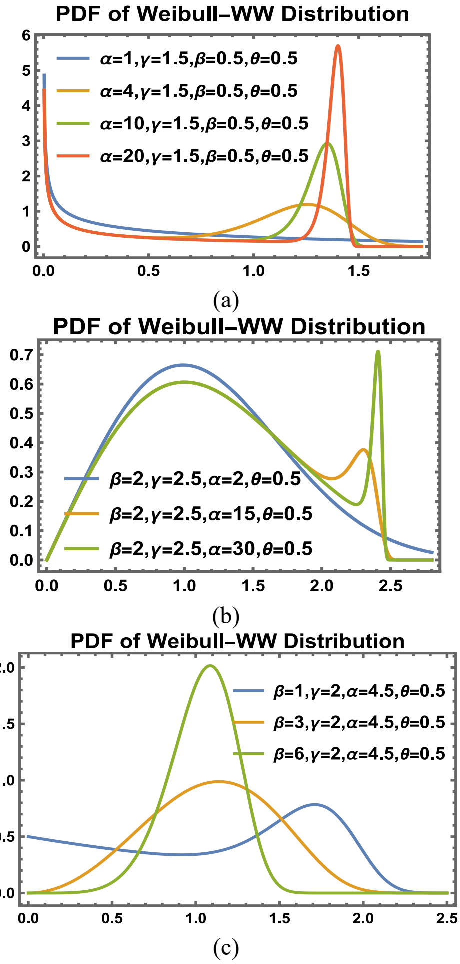

Figure 1: Plot of pdf of the new distribution.

From Figure 1b, it is clear that:

Increasing α while holding other parameters constant means:

- Disappearance of early failure (decrease of f(x) near zero)
- Centrality of failure times around a value close to γ, a higher and narrower peak
- A much shorter right tail (more closely spaced lifetimes).

Practically: α is an indicator of the system's transition from dispersed behavior to a convergent operational life, expressing clear late wear.

As Figure 1c shows, increasing β while holding other parameters constant results in the disappearance of the peak at zero, and the appearance of a higher and more compressed peak that shifts slightly toward lower values of x, with an apparent shortening of the right tail; Reflecting a decrease in early failure and a greater concentration of failure times around a specific operational value.

In practical terms, this behavior indicates that the Weibull–WW distribution has a high degree of flexibility in representing various patterns of lifetime data. At small values of α, the model can represent systems that experience high failure rates at the start of operation due to manufacturing defects or harsh starting conditions (the early failure phase). At large values of α, the distribution becomes suitable for systems that operate stably for an extended period and then gradually increase their failure rates due to wear and tear (late failure phase). Therefore, α can be used as a primary indicator for determining a system's reliability pattern, giving the model broad interpretive and representative capabilities in applications such as reliability analysis, maintenance engineering, and modeling the lifetimes of industrial and medical devices and components.

3 THE PROPERTIES OF THE NEW DISTRIBUTION

The derivation some of properties to the new distribution.

3.1 The Quantile Function

The generation of RVs from continuous probability distributions depends on the quantile function, denoted Q(u). The uth quantile, denoted by $x_u = Q(u)$, for the Weibull-WW distribution, the value of this quantity is extracted by solving the following:

$$F(x;\underline{\omega}) = u,\ 0 < u < 1, \quad (11)$$

$$1 - e^{-\theta\, {x_u}^{\beta} - \frac{\alpha}{\gamma}(\frac{x_u}{\gamma})^{2\alpha-1}} = u, \quad (12)$$

$$e^{-\theta\, {x_u}^{\beta} - \frac{\alpha}{\gamma}(\frac{x_u}{\gamma})^{2\alpha-1}} = 1 - u\,, \quad (13)$$

$$-\theta\, {x_u}^{\beta} - \frac{\alpha}{\gamma}\,(\frac{x_u}{\gamma})^{2\alpha-1} = ln(1-u)\,, \quad (14)$$

$$-\,\theta\, {x_u}^{\beta} - \frac{\alpha}{\gamma}\left(\frac{x_u}{\gamma}\right)^{2\alpha-1} - ln(1-u) = 0, \quad (15)$$

$$\theta\, {x_u}^{\beta} + \frac{\alpha}{\gamma}\left(\frac{x_u}{\gamma}\right)^{2\alpha-1} + ln(1-u) = 0. \quad (16)$$

Using u = 0.5 in (16) yields the median of the proposed distribution. Similarly, u = 0.25 and u = 0.75 yield the first and third quartiles of the distribution, respectively.

3.2 Moments

Moments are a fundamental tool in statistical analysis, representing an effective means of extracting the properties of various distributions, especially in practical applications. They enable the study of essential measures such as central tendency, dispersion, skewness, and kurtosis. If the variable X follows the Weibull-WW distribution with parameters $(\theta,\beta,\alpha,\gamma)$, the moment of order r can be derived as follows:

$$E[x^r] = \int_0^{\infty} x^r\, f(x)\, d\,, \quad (17)$$

$$let\ u = x^r\ \rightarrow du = r\, x^{r-1} dx\,,$$
$$dv = f(x)dx\ \ \rightarrow\ \ v = F(x)\,,$$
$$\int_0^{\infty} u\, dv = uv - \int_0^{\infty} v\, du\,,$$
$$\int_0^{\infty} x^r\, f(x)\, dx = [x^r F(x)]_0^{\infty} - \int_0^{\infty} F(x)\, r\, x^{r-1} dx, \quad (5)$$
$$\int_0^{\infty} x^r\, f(x)\, dx = [x^r(1 - S(x))]_0^{\infty} - \int_0^{\infty}\big(1 - S(x)\big) r\, x^{r-1} dx, \quad (19)$$

$$\int_0^{\infty} x^r\, f(x)\, dx = r\int_0^{\infty} S(x)\, x^{r-1} dx\,, \quad (20)$$

$$E[x^r] = \int_0^{\infty} x^r\, f(x)\, dx = r\int_0^{\infty} S(x)\, x^{r-1} dx\,, \quad (21)$$

$$E[x^r] = r\int_0^{\infty} x^{r-1}\, e^{-\theta\, x^{\beta} - \frac{\alpha}{\gamma}(\frac{x}{\gamma})^{2\alpha-1}}\, dx\,, \quad (22)$$

$$e^{-\frac{\alpha}{\gamma}(\frac{x}{\gamma})^{2\alpha-1}} = e^{-\frac{\alpha}{\gamma^{2\alpha}}\, x^{2\alpha-1}} = \sum_{j=1}^{\infty}\frac{(-1)^j}{j!}\,(\frac{\alpha}{\gamma^{2\alpha}})^j\, x^{j(2\alpha-1)}\,, \quad (23)$$

$$E[x^r] = r\sum_{j=1}^{\infty}\frac{(-1)^j}{j!}\,(\frac{\alpha}{\gamma^{2\alpha}})^j \int_0^{\infty} x^{r-1+j(2\alpha-1)}\, e^{-\theta\, x^{\beta}}\, dx\,, \quad (24)$$

$$let{:}\ u = \theta\, x^{\beta}\ \rightarrow x^{\beta} = \frac{u}{\theta}\ \rightarrow x = \left(\frac{u}{\theta}\right)^{\frac{1}{\beta}}, dx = \frac{1}{\beta}\left(\frac{1}{\theta}\right)^{\frac{1}{\beta}} (u)^{\frac{1}{\beta}-1} du \qquad, m = r + j(2\alpha-1),$$

$$E[x^r] = r\sum_{j=1}^{\infty}\frac{(-1)^j}{j!}\left(\frac{\alpha}{\gamma^{2\alpha}}\right)^j \int_0^{\infty}\left(\left(\frac{u}{\theta}\right)^{\frac{1}{\beta}}\right)^{m-1} e^{-u}\, \frac{1}{\beta}\left(\frac{1}{\theta}\right)^{\frac{1}{\beta}} (u)^{\frac{1}{\beta}-1} du, \quad (25)$$

$$E[x^r] = r\sum_{j=1}^{\infty}\frac{(-1)^j}{j!}\,(\frac{\alpha}{\gamma^{2\alpha}})^j\, \frac{1}{\beta}\int_0^{\infty}(\frac{1}{\theta})^{\frac{m-1}{\beta}+\frac{1}{\beta}}\,(u)^{\frac{m-1}{\beta}+\frac{1}{\beta}-1}\, e^{-u}\, du, \quad (26)$$

$$E[x^r] = r\sum_{j=1}^{\infty}\frac{(-1)^j}{j!}\,(\frac{\alpha}{\gamma^{2\alpha}})^j\, \frac{1}{\beta}\,(\frac{1}{\theta})^{\frac{m}{\beta}} \int_0^{\infty}\ (u)^{\frac{m}{\beta}-1}\, e^{-u}\, du,$$
$$\int_0^{\infty}\ (u)^{\frac{m}{\beta}-1}\, e^{-u}\ du\ = \Gamma\left(\frac{m}{\beta}\right),$$
$$E[x^r] = r\sum_{j=1}^{\infty}\frac{(-1)^j}{j!}\,(\frac{\alpha}{\gamma^{2\alpha}})^j\, \frac{1}{\beta}\,(\frac{1}{\theta})^{\frac{m}{\beta}}\Gamma\left(\frac{m}{\beta}\right), \quad (27)$$

$$E[x^r] = \frac{r}{\beta}\sum_{j=1}^{\infty}\frac{(-1)^j}{j!}\,(\frac{\alpha}{\gamma^{2\alpha}})^j\,(\frac{1}{\theta})^{\frac{r+j(2\alpha-1)}{\beta}}\,\Gamma\left(\frac{r+j(2\alpha-1)}{\beta}\right), r > 0 \quad (28)$$

Set:

- r=1 for the mean
- r=2 for the second moment (29)
- the variance as V(x)= E[X^2] – (E[X])^2

4 MAXIMUM LIKELIHOOD ESTIMATION

Suppose $(x1, \ldots, xn)$ be a observations values of size n from a random sample that follows the new distribution with parameters ($\underline{\omega}$ = α, β, θ, γ).

The likelihood function ($L(x;\underline{\omega})$) for the sample can be formulated as follows:

$$f(x;\underline{\omega}) = \left(\theta\beta x^{\beta-1} + \frac{\alpha(2\alpha-1)}{\gamma^{2\alpha}}(x)^{2(\alpha-1)}\right)e^{-\theta\, x^{\beta} - \frac{\alpha}{\gamma}(\frac{x}{\gamma})^{2\alpha-1}}, \quad (30)$$

$$L(x;\underline{\omega}) = \prod_{i=1}^{n} f(x_i;\underline{\omega})$$

$$L(x;\underline{\omega}) = \prod_{i=1}^{n}\left(\theta\beta x_i^{\beta-1} + \frac{\alpha(2\alpha-1)}{\gamma^{2\alpha}}(x_i)^{2(\alpha-1)}\right)e^{-\theta\sum_{i=0}^{n} x_i^{\beta} - \frac{\alpha}{\gamma^{2\alpha}}\sum_{i=0}^{n} x_i^{2\alpha-1}}, \quad (31)$$

Let $D_i = \theta\beta x^{\beta-1} + \frac{\alpha(2\alpha-1)}{\gamma^{2\alpha}}(x)^{2(\alpha-1)} = \theta\beta x_i^{\beta-1} + \alpha(2\alpha-1)\,\gamma^{-2\alpha}(x_i)^{2(\alpha-1).}$

The natural logarithmic form of the likelihood function is:

$$\ell = \ln L(x;\underline{\omega}) = \sum_{i=0}^{n} \ln D_i - \theta\sum_{i=0}^{n} x_i^{\beta} - \frac{\alpha}{\gamma^{2\alpha}}\sum_{i=0}^{n} x_i^{2\alpha-1}. \quad (32)$$

When finding the partial derivatives of (32) with respect to the parameters $\underline{\omega}$ = α, β, θ, γ, and then setting them equal to zero, we have:

$$\frac{\partial\ell}{\partial\gamma} = \sum_{i=0}^{n}\frac{1}{D_i}\frac{\partial D_i}{\partial\gamma} - \frac{\partial\ell}{\partial\gamma}\theta\sum_{i=0}^{n} x_i^{\beta} - \frac{\partial\ell}{\partial\gamma}\frac{\alpha}{\gamma^{2\alpha}}\sum_{i=0}^{n} x_i^{2\alpha-1},$$

$$\frac{\partial D_i}{\partial\gamma} = \frac{\partial}{\partial\gamma}\left[\theta\beta x_i^{\beta-1} + \alpha(2\alpha-1)\,\gamma^{-2\alpha}x_i^{2(\alpha-1)}\right] = -2\alpha^2(2\alpha-1)\,\gamma^{-2\alpha-1}x_i^{2(\alpha-1)},$$

$$-\frac{\partial\ell}{\partial\gamma}\frac{\alpha}{\gamma^{2\alpha}}\sum_{i=0}^{n} x_i^{2\alpha-1} = \frac{\partial\ell}{\partial\gamma}\left[-\alpha\,\gamma^{-2\alpha}\sum_{i=0}^{n} x_i^{2\alpha-1}\right] = 2\alpha^2\,\gamma^{-2\alpha-1}\sum_{i=0}^{n} x_i^{2\alpha-1},$$

$$\frac{\partial\ell}{\partial\gamma} = \sum_{i=0}^{n}\frac{-2\alpha^2(2\alpha-1)\,\gamma^{-2\alpha-1}x_i^{2(\alpha-1)}}{\theta\beta x_i^{\beta-1}+\alpha(2\alpha-1)\,\gamma^{-2\alpha}(x_i)^{2(\alpha-1)}} + 2\alpha^2\,\gamma^{-2\alpha-1}\sum_{i=0}^{n} x_i^{2\alpha-1},$$

$$\frac{\partial\ell}{\partial\gamma} = 2\alpha^2\,\gamma^{-2\alpha-1}\sum_{i=0}^{n} x_i^{2\alpha-1} - \sum_{i=0}^{n}\frac{x_i^{2}\left[2\alpha^2(2\alpha-1)\,\gamma^{-2\alpha-1}x_i^{2\alpha}\right]}{x_i^{2}\left[\theta\beta x_i^{\beta+1}+\alpha(2\alpha-1)\,\gamma^{-2\alpha}(x_i)^{2\alpha}\right]},$$

$$\frac{\partial\ell}{\partial\gamma} = 2\alpha^2\,\gamma^{-2\alpha-1}\sum_{i=0}^{n} x_i^{2\alpha-1} - \sum_{i=0}^{n}\frac{2\alpha^2(2\alpha-1)\,\gamma^{-2\alpha-1}x_i^{2\alpha}}{\theta\beta x_i^{\beta+1}+\alpha(2\alpha-1)\,\gamma^{-2\alpha}(x_i)^{2\alpha}} = 0, \quad (33)$$

$$\frac{\partial\ell}{\partial\alpha} = \sum_{i=0}^{n}\frac{1}{D_i}\frac{\partial D_i}{\partial\alpha} - \frac{\partial\ell}{\partial\alpha}\theta\sum_{i=0}^{n} x_i^{\beta} - \frac{\partial\ell}{\partial\alpha}\alpha\,\gamma^{-2\alpha}\sum_{i=0}^{n} x_i^{2\alpha-1},$$

$$\frac{\partial D_i}{\partial\alpha} = \frac{\partial}{\partial\alpha}\left[\theta\beta x_i^{\beta-1} + \alpha(2\alpha-1)\,\gamma^{-2\alpha}x_i^{2(\alpha-1)}\right] = \frac{\partial}{\partial\alpha}\alpha(2\alpha-1)\,\gamma^{-2\alpha}x_i^{2\alpha-2},$$

Let $U(\alpha) = \alpha(2\alpha-1)$, $V(\alpha) = \gamma^{-2\alpha}$, $W(\alpha) = x_i^{2\alpha-2}$.

Then $\acute{U}(\alpha) = 4\alpha - 1$, $\acute{V}(\alpha) = -2\ln(\gamma)\,\gamma^{-2\alpha}$, $\acute{W}(\alpha) = 2\ln(x_i)\,x_i^{2\alpha-2}$,

$$\frac{\partial D_i}{\partial\alpha} = \frac{\partial}{\partial\alpha}\alpha(2\alpha-1)\,\gamma^{-2\alpha}x_i^{2\alpha-2} = (4\alpha-1)\,\gamma^{-2\alpha}x_i^{2\alpha-2} + \alpha(2\alpha-1)\,(-2\ln(\gamma)\,\gamma^{-2\alpha})x_i^{2\alpha-2} + \alpha(2\alpha-1)\,\gamma^{-2\alpha}(2\ln(x_i)\,x_i^{2\alpha-2}),$$

$$\frac{\partial D_i}{\partial\alpha} = (4\alpha-1)\,\gamma^{-2\alpha}x_i^{2\alpha-2} - 2\alpha(2\alpha-1)\,\ln(\gamma)\,\gamma^{-2\alpha}\,x_i^{2\alpha-2} + \alpha(2\alpha-1)\,\gamma^{-2\alpha}(2\ln(x_i)\,x_i^{2\alpha-2}),$$

$$\frac{\partial D_i}{\partial\alpha} = \alpha(2\alpha-1)\gamma^{-2\alpha}x_i^{2\alpha-2}\left[\frac{(4\alpha-1)}{\alpha(2\alpha-1)} - 2\ln(\gamma) + 2\ln(x_i)\right],$$

$$-\frac{\partial\ell}{\partial\alpha}\alpha\,\gamma^{-2\alpha}\sum_{i=0}^{n} x_i^{2\alpha-1} = -[1 * \gamma^{-2\alpha}\sum_{i=0}^{n} x_i^{2\alpha-1} + \alpha(-2\ln(\gamma)\,\gamma^{-2\alpha})\sum_{i=0}^{n} x_i^{2\alpha-1} + \alpha\,\gamma^{-2\alpha}(2\ln(x_i)\,x_i^{2\alpha-2})],$$

$$-\frac{\partial\ell}{\partial\alpha}\alpha\,\gamma^{-2\alpha}\sum_{i=0}^{n} x_i^{2\alpha-1} = -\gamma^{-2\alpha}\sum_{i=0}^{n} x_i^{2\alpha-1}[1 - 2\alpha\,\ln(\gamma) + 2\alpha\ln(x_i)],$$

$$\frac{\partial\ell}{\partial\alpha} = \sum_{i=0}^{n}\frac{\alpha(2\alpha-1)\gamma^{-2\alpha}x_i^{2\alpha-2}\left[\frac{(4\alpha-1)}{\alpha(2\alpha-1)} - 2\ln(\gamma) + 2\ln(x_i)\right]}{\theta\beta x_i^{\beta-1}+\alpha(2\alpha-1)\,\gamma^{-2\alpha}(x_i)^{2(\alpha-1)}} - \gamma^{-2\alpha}\sum_{i=0}^{n} x_i^{2\alpha-1}[1 - 2\alpha\,\ln(\gamma) + 2\alpha\ln(x_i)],$$

$$\frac{\partial\ell}{\partial\alpha} = \sum_{i=0}^{n}\frac{\alpha(2\alpha-1)\gamma^{-2\alpha}x_i^{2\alpha}\left[\frac{(4\alpha-1)}{\alpha(2\alpha-1)} - 2\ln(\gamma) + 2\ln(x_i)\right]}{\theta\beta x_i^{\beta+1}+\alpha(2\alpha-1)\,\gamma^{-2\alpha}(x_i)^{2\alpha}} - \gamma^{-2\alpha}\sum_{i=0}^{n} x_i^{2\alpha-1}[1 - 2\alpha\,\ln(\gamma) + 2\alpha\ln(x_i)] = 0 \quad (34)$$

$$\frac{\partial\ell}{\partial\beta} = \sum_{i=0}^{n}\frac{1}{D_i}\frac{\partial D_i}{\partial\beta} - \frac{\partial\ell}{\partial\beta}\theta\sum_{i=0}^{n} x_i^{\beta} - \frac{\partial\ell}{\partial\beta}\alpha\,\gamma^{-2\alpha}\sum_{i=0}^{n} x_i^{2\alpha-1},$$

$$\frac{\partial D_i}{\partial\beta} = \frac{\partial}{\partial\beta}\left[\theta\beta x_i^{\beta-1} + \alpha(2\alpha-1)\,\gamma^{-2\alpha}x_i^{2(\alpha-1)}\right] = \frac{\partial}{\partial\beta}\theta\beta x_i^{\beta-1},$$

$$\frac{\partial D_i}{\partial\beta} = \frac{\partial}{\partial\beta}\theta\beta x_i^{\beta-1} = \theta x_i^{\beta-1} + \theta\beta\ln(x_i)\,x_i^{\beta-1} = \theta x_i^{\beta-1}[1 + \beta\ln(x_i)],$$

$$-\frac{\partial\ell}{\partial\beta}\theta\sum_{i=0}^{n} x_i^{\beta} = -\theta\sum_{i=0}^{n} x_i^{\beta}\ln(x_i),$$

$$\frac{\partial\ell}{\partial\beta} = \sum_{i=0}^{n}\frac{1}{D_i}\frac{\partial D_i}{\partial\beta} - \frac{\partial\ell}{\partial\beta}\theta\sum_{i=0}^{n} x_i^{\beta} = \sum_{i=0}^{n}\frac{\theta x_i^{\beta-1}[1+\beta\ln(x_i)]}{\theta\beta x_i^{\beta-1}+\alpha(2\alpha-1)\,\gamma^{-2\alpha}x_i^{2(\alpha-1)}} - \theta\sum_{i=0}^{n} x_i^{\beta}\ln(x_i) = 0 \quad (35)$$

$$\frac{\partial\ell}{\partial\theta} = \sum_{i=0}^{n}\frac{1}{D_i}\frac{\partial D_i}{\partial\theta} - \frac{\partial\ell}{\partial\theta}\theta\sum_{i=0}^{n} x_i^{\beta} - \frac{\partial\ell}{\partial\theta}\alpha\,\gamma^{-2\alpha}\sum_{i=0}^{n} x_i^{2\alpha-1},$$

$$\frac{\partial \ell}{\partial \theta} = \sum_{i=0}^{n} \frac{1}{D_i}\frac{\partial D_i}{\partial \theta} - \frac{\partial \ell}{\partial \theta}\theta \sum_{i=0}^{n} x_i^{\beta} ,$$
$$\frac{\partial D_i}{\partial \theta} = \frac{\partial}{\partial \theta}\left[\theta\beta {x_i}^{\beta-1} + \alpha(2\alpha-1)\,\gamma^{-2\alpha} {x_i}^{2(\alpha-1)}\right] = \frac{\partial}{\partial \theta}\theta\beta {x_i}^{\beta-1} ,$$
$$\frac{\partial D_i}{\partial \theta} = \frac{\partial}{\partial \theta}\theta\beta {x_i}^{\beta-1} = \beta {x_i}^{\beta-1},$$
$$-\frac{\partial \ell}{\partial \theta}\theta \sum_{i=0}^{n} x_i^{\beta} = -\sum_{i=0}^{n} x_i^{\beta},$$
$$\frac{\partial \ell}{\partial \theta} = \sum_{i=0}^{n} \frac{\beta {x_i}^{\beta-1}}{\theta\beta {x_i}^{\beta-1} + \alpha(2\alpha-1)\,\gamma^{-2\alpha} {x_i}^{2\alpha-2}} - \sum_{i=0}^{n} x_i^{\beta} = 0 .$$
(36)

We can read directly from the (33), (34), (35), and (36) that the formulae have no closed-form analytical solutions.

Hence, the distribution parameters were estimated using numerical methods, and their numerical values were determined by executing the NMaximize algorithm in Wolfram Mathematica version 12.3.

5 APPLICATIONS

This section presents a real example illustrating the flexibility of the Weibull-WW distribution in modeling data. It was implemented on a data sample, presenting through the example the advantage the distribution has over the Weibull and WW distributions:

- Weibull distribution

$$f(x;\beta,\theta) = \theta\beta x^{\beta-1} e^{-\theta x^{\beta}}, x > 0, (\beta,\theta) > 0 \quad (37)$$

- WW distribution

$$f(x;\alpha,\beta) = \frac{\beta(2\beta-1)}{\alpha^{2\beta}}\,(x)^{2(\beta-1)}\, e^{-\frac{\beta}{\alpha^{2\beta}}x^{2\beta-1}},\ x > 0;\ \alpha > 0, \beta > 0 \quad (38)$$

MLE was used to estimate parameters $\underline{\omega} = \alpha, \beta, \theta, \gamma$. In addition, metrics are used to measure distribution flexibility, including:

- AIC the Akaike information criterion.
- BIC the Bayesian information criterion.
- AICc the adjusted Akaike information criterion.

$$AIC = 2r - 2L , \quad (39)$$

$$BIC = r\ln(n) - 2L , \quad (40)$$

$$CAIC = AIC + \frac{2r(r+1)}{n-r-1}. \quad (41)$$

- L the natural logarithm of the likelihood function;
- n the number of all the observations;
- r the number of the parameters.

5.1 The Data Set

The data set includes the following lifetime values for 50 devices [20]:
0.1,0.2,1,1,1,1,1,2,3,6,7,11,12,18,18,18,18,18,21,32, 36,40,45,46,47,50,55,60,63,63,67,67,67,67,72,75,79, 82,82,83,84,84,84,85,85,85,85,85,86,86.

Table 1 provides MLE of the parameters $\underline{\omega} = \alpha, \beta, \theta, \gamma$ of the weibull-ww distribution and other competing distributions, and the goodness of fit measures are provided in Table 2.

According to Table 2, the proposed model is the best among the fitted models, as it has the lowest values of AIC, AICc, and BIC. From Figure 2, it is clear that the PDF of weibull-ww fits the data well.

Table 1: Maximum likelihood estimates (MLEs) for the First Data Set.

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\gamma}$
Weibull-WW	41.667	0.70249	0.055274	84.19
Weibull		0.94904	0.027029	
WW	6.2926	0.97452		

Table 2: Summary values of the models fitted to the first set of data.

Distribution	AIC	AICc	BIC	-L
Weibull-WW	420.19	421.08	427.84	206.09
Weibull	486.004	486.259	489.828	241.02
WW	486.004	486.259	489.828	241.02

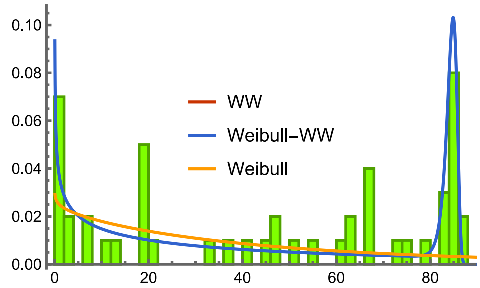

Figure 2: The PDF of the Weibull-WW, Weibull, and WW Distributions of the first data set

6 CONCLUSIONS

In this study, a new distribution is proposed based on the competing risks notion. the proposed new distribution represents an extension of the Weibull and WW distributions, incorporating the distinctive features of each. The behavior for the pdf $f(x;\underline{w})$ of the proposed new distribution in the graphical illustration paragraph indicates that it has the flexibility to represent data through its ability to take on different forms (At small values of parameter α, the distribution can represent systems that experience high failure rates at the start of operation due to manufacturing defects or harsh starting conditions (the early failure phase). At large values of α, the distribution becomes suitable for systems that operate stably for an extended period and then gradually increase their failure rates due to wear and tear (late failure phase). Therefore, parameter α can be used as a primary indicator for determining a system's reliability pattern, giving the model broad interpretive and representative capabilities in applications such as reliability analysis, maintenance engineering, and modeling the lifetimes of industrial and medical devices and components). Some of its statistical properties were derived. Its performance was evaluated through practical applications. A set of basin-shaped lifetime data was analyzed, and goodness-of-fit metrics were calculated. The results showed that the proposed new distribution(weibull--ww) provides the best fit to the data compared to baseline distributions (Weibull and ww distributions). To support the numerical results, a pdf function was plotted, demonstrating that the proposed distribution accurately represents the data, confirming its flexibility in modeling lifetime data. This distribution is expected to have broad applications to other types of lifetime data, to verify the flexibility of the distribution in diverse fields.

REFERENCES

[1] N. Taketomi, K. Yamamoto, C. Chesneau, and T. Emura, “Parametric distributions for survival and reliability analyses: A review and historical sketch,” Mathematics, vol. 10, no. 20, 2022, [Online]. Available: https://doi.org/10.3390/math10203907.

[2] L. Tomy, M. Jose, and V. G, “A review on recent generalizations of exponential distribution,” Biometrics and Biostatistics International Journal, vol. 9, no. 4, pp. 152-156, Aug. 2020, [Online]. Available: https://doi.org/10.15406/bbij.2020.09.00313.

[3] S. J. Almalki and J. Yuan, “A new modified Weibull distribution,” Reliability Engineering and System Safety, vol. 111, pp. 164-170, 2013, [Online]. Available: https://doi.org/10.1016/j.ress.2012.10.018.

[4] R. Shehla and A. A. Khan, “Reliability analysis using an exponential power model with bathtub-shaped failure rate function: A Bayes study,” SpringerPlus, vol. 5, no. 1, Dec. 2016, [Online]. Available: https://doi.org/10.1186/s40064-016-2722-3.

[5] C. D. Lai, “Constructions and applications of lifetime distributions,” Applied Stochastic Models in Business and Industry, vol. 29, no. 2, pp. 127-140, Mar. 2013, [Online]. Available: https://doi.org/10.1002/asmb.948.

[6] M. Xie and C. D. Lai, “Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function,” Reliability Engineering and System Safety, vol. 52, no. 1, pp. 87-93, 1996, [Online]. Available: https://doi.org/10.1016/0951-8320(95)00149-2.

[7] C. D. Lai, M. Xie, and D. N. P. Murthy, “A modified Weibull distribution,” IEEE Transactions on Reliability, vol. 52, no. 1, pp. 33-37, Mar. 2003, [Online]. Available: https://doi.org/10.1109/TR.2002.805788.

[8] M. Bebbington, C. D. Lai, and R. Zitikis, “A flexible Weibull extension,” Reliability Engineering and System Safety, vol. 92, no. 6, pp. 719-726, 2007, [Online]. Available: https://doi.org/10.1016/j.ress.2006.03.004.

[9] G. M. Cordeiro, E. M. M. Ortega, and A. J. Lemonte, “The exponential-Weibull lifetime distribution,” Journal of Statistical Computation and Simulation, vol. 84, no. 12, pp. 2592-2606, 2014, [Online]. Available: https://doi.org/10.1080/00949655.2013.797982.

[10] B. He, W. Cui, and X. Du, “An additive modified Weibull distribution,” Reliability Engineering and System Safety, vol. 145, pp. 28-37, Jan. 2016, [Online]. Available: https://doi.org/10.1016/j.ress.2015.08.010.

[11] B. O. Oluyede, S. Foya, G. Warahena-Liyanage, and S. Huang, “The log-logistic Weibull distribution with applications to lifetime data,” Austrian Journal of Statistics, vol. 45, no. 3, pp. 43-69, 2016, [Online]. Available: https://doi.org/10.17713/ajs.v45i3.107.

[12] P. Mdlongwa, B. O. Oluyede, A. Amey, and S. Huang, “The Burr XII modified Weibull distribution: Model, properties and applications,” Electronic Journal of Applied Statistical Analysis, vol. 10, no. 1, pp. 118-145, 2017, [Online]. Available: https://doi.org/10.1285/i20705948v10n1p118.

[13] B. Tarvirdizade and M. Ahmadpour, “A new extension of Chen distribution with applications to lifetime data,” Communications in Mathematics and Statistics, vol. 9, no. 1, pp. 23-38, Mar. 2021, [Online]. Available: https://doi.org/10.1007/s40304-019-00185-4.

[14] M. K. Shakhatreh, A. J. Lemonte, and G. Moreno-Arenas, “The log-normal modified Weibull distribution and its reliability implications,” Reliability Engineering and System Safety, vol. 188, pp. 6-22, 2019, [Online]. Available: https://doi.org/10.1016/j.ress.2019.03.014.

[15] S. A. Osagie and J. E. Osemwenkhae, “Lomax-Weibull distribution with properties and applications in lifetime analysis,” [Online]. Available: http://ijmao.unilag.edu.ng/article/view/274.

[16] R. M. Kamal and M. A. Ismail, “The flexible Weibull extension-Burr XII distribution: Model, properties and applications,” Pakistan Journal of Statistics and Operation Research, vol. 16, no. 3, pp. 447-460, 2020, [Online]. Available: https://doi.org/10.18187/pjsor.v16i3.2957.

[17] T. Thanh Thach and R. Briš, “An additive Chen-Weibull distribution and its applications in reliability modeling,” Quality and Reliability Engineering

International, vol. 37, no. 1, pp. 352-373, 2021, [Online]. Available: https://doi.org/10.1002/qre.2740.

[18] A. Khalil et al., “A novel flexible additive Weibull distribution with real-life applications,” Communications in Statistics - Theory and Methods, vol. 50, no. 7, pp. 1557-1572, 2021, [Online]. Available: https://doi.org/10.1080/03610926.2020.1732658.

[19] S. O. Ezeah, A. A. Adekola, O. O. Fabelurin, and T. O. Obilade, “On a variant Weibull-Weibull distribution: Theory and properties,” Mathematics and Statistics, vol. 12, no. 5, pp. 401-408, Sep. 2024, [Online]. Available: https://doi.org/10.13189/ms.2024.120501.

[20] M. V. Aarset, “How to identify a bathtub hazard rate,” 1987.