Fractional Diffusion and Lévy Processes for Financial Derivative Pricing

Muhannad F. Al- Saadony and Nasir A. Naser
Department of Statistics, Administration and Economics College, University of Al-Qadisiyah, 58001 Ad Diwaniyah, Iraq
Muhannad.alsaadony@qu.edu.iq,naser.ibrahim24@qu.edu.iq

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Abstract: This paper develops a joint framework for the fractional diffusion equation driven by Lévy processes within a stochastic volatility setting, serving as an extension of the classical Heston model. We present the mathematical foundations of the model, emphasizing the dynamics of asset prices influenced simultaneously by fractional Brownian motion, which introduces long memory, and Lévy processes, which capture discontinuous jumps. The integration of these components allows the model to account for both fat-tailed and skewed return distributions. A Fourier transform representation is derived for option pricing under this generalized framework, and a Monte Carlo simulation scheme is proposed to numerically evaluate the associated stochastic integrals. Furthermore, maximum likelihood estimation (MLE) is employed for parameter estimation, and a systematic procedure for applying the proposed model to empirical financial data is outlined. The results demonstrate that the model provides a more realistic representation of market dynamics than classical approaches, particularly in explaining volatility clustering, extreme events, and the heavy-tailed nature of financial returns. Overall, the fractional diffusion–Lévy process framework offers a robust alternative for both theoretical research and practical applications in financial modeling.

1 INTRODUCTION

In recent decades, the modeling of financial markets has moved far beyond classical approaches such as the Black–Scholes framework. While these traditional models provided a foundation for option pricing and risk management, they often fail to capture crucial empirical features of asset returns, including volatility clustering, heavy tails, and sudden jumps. Among the more advanced models, the Heston stochastic volatility model has become widely used because it allows volatility itself to evolve as a random process rather than a fixed parameter.

Despite its popularity, the Heston model is built on standard Brownian motion. This assumption limits its ability to reflect the complex behavior observed in financial time series. In particular, empirical evidence shows that financial returns exhibit long-range dependence and persistent deviations from normality, features that cannot be adequately explained by Gaussian dynamics alone. These limitations have encouraged researchers to seek more flexible processes to drive asset prices [1].

Two promising directions have emerged. The first is fractional Brownian motion, which introduces a memory effect through the Hurst parameter, allowing the model to capture persistent or anti-persistent structures in returns. The second is the use of Lévy processes, which incorporate discontinuous jumps and are therefore able to reproduce extreme market events. Each approach adds realism, but combining them within a unified framework can provide a more comprehensive description of asset dynamics [1], [2].

The purpose of this paper is to extend the Heston model by embedding it within a fractional diffusion framework enriched by Lévy jumps. In doing so, we aim to build a model capable of reflecting both continuous price variations and sudden shocks, while maintaining mathematical tractability for applications in derivative pricing and risk management. We derive the theoretical formulation of the model, establish its option pricing representation through Fourier techniques, and implement Monte Carlo simulations to explore its numerical properties. Furthermore, we investigate parameter estimation using maximum likelihood methods and demonstrate applications to real financial data, including the S\&P 500 index and the iShares Europe ETF [2].

This integration of fractional and jump dynamics represents a step toward more realistic financial modeling. By addressing both long memory and discontinuities, the proposed model has the potential to improve the accuracy of option valuation and the measurement of market risk.

Classical models in financial economics, such as the Black–Scholes framework, assume normally distributed returns and constant volatility. While elegant, these assumptions prevent the model from explaining well-documented empirical phenomena such as volatility clustering, heavy tails, and sudden jumps in asset prices. To overcome these shortcomings, Heston (1993) [2] introduced a stochastic volatility model in which the variance follows a mean-reverting process. This provided a more realistic description of option pricing and helped explain the volatility smile observed in practice.

Later extensions incorporated jump components to capture abrupt market shocks. Notable examples include the Merton jump-diffusion model and the Variance Gamma process, both of which improved the ability to model heavy-tailed and asymmetric return distributions. However, these models still lack the capacity to represent long-memory effects, which are frequently observed in high-frequency financial data.

To address persistence and long-range dependence, researchers turned to fractional Brownian motion (fBm), which incorporates memory through the Hurst exponent. At the same time, Lévy processes have gained popularity as they can jointly capture continuous price fluctuations and discontinuous jumps [3], [4].

In recent years, efforts have been made to integrate these two perspectives. For example, Aljethi (2022) developed a fractional Lévy framework for option pricing, supported by numerical sensitivity analysis. More recently, new studies derived fractional differential equations based on Lévy processes to better align theoretical models with empirical data (see Science Direct, 2023). A comprehensive survey published in 2024 highlights the advances of fractional Black–Scholes equations, emphasizing their ability to capture extreme events and nonlinear volatility patterns, along with analytical and numerical solution techniques (MDPI, 2024).

Nevertheless, despite this progress, a fully unified framework that systematically integrates fractional diffusion dynamics with Lévy-driven jumps for practical applications in option pricing and risk management is still underdeveloped.

The contribution of this paper lies in bridging this gap. We propose a generalized extension of the Heston model that integrates fractional Brownian motion with Lévy jumps into a single coherent diffusion framework. The approach provides a Fourier-based option pricing representation, a Monte Carlo simulation scheme, and a maximum likelihood estimation strategy for parameter calibration. Taken together, these innovations offer both theoretical novelty and empirical applicability.

2 MATHEMATICAL FORMULATION

The Heston model is described by the following system of stochastic differential equations (SDEs). The stock price S_t follows the SDE [2], [5]:

$$dS_t = \mu s_t dt + \sqrt{V_t} S_t dW_t^H. \quad (1)$$

Where μ: The drift rate of the stock price, V_t: The instantaneous variance (volatility squared) and W_t^H: A fractional Brownian motion for the stock.

And the variance V_t follows the SDE [5]:

$$dV_t = \theta(\mu_v - V_t)dt + \sigma_v \sqrt{V_t} dW_t^V. \quad (2)$$

Where θ is the rate of mean reversion, μ_v is the long-term mean level of volatility and σ_v is the volatility of volatility. The two Wiener processes $W_t^H$and W_t^V are correlated with a correlation coefficient ρ.

$$dW_t^H \, dW_t^V = \rho dt.$$

The variance process (2) was originally used by Cox, Ingersoll, and Ross (1985) for modeling the short term interest rate. It is defined by three parameters: θ, σ^2, and γ. In the context of stochastic volatility models they can be interpreted as the long term variance, the rate of mean reversion to the long term variance, and the volatility of variance, respectively, if we assume that the Wiener process in the Heston model is replaced by a Lévy process, the model can be generalized to incorporate jumps in addition to the continuous price movements. This leads to a more complex framework known as a Lévy-driven stochastic volatility model. We have the model:

When we introduce a Lévy process L_t to account for jumps, the stock price dynamics become [6]:

$$dS_t = \mu s_t dt + \sqrt{V_t} S_t dW_t^H + S_t dL_t. \quad (3)$$

Where dL_t: Represents the jump component of the stock price, modeled by a Lévy process, and the variance dynamics remain the same as in the original Heston model (2), where W_t^V can still be correlated with the Lévy process.

The empirical evidence suggests that the tail of the risk-neutral return distribution exhibits "fat" characteristics as maturity increases, traditionally interpreted as slow convergence to normality. This phenomenon has been attributed to persistent stochastic volatility processes. However, this modeling approach proposes an alternative perspective, positing that the observed data may indicate a violation of the conditions necessary for the central limit theorem, leading to a risk-neutral distribution that does not converge to normality [7], [8].

To explain the fat tails in the return distribution, the model introduces a parameter αα, which controls tail thickness relative to the central mass. In the Black-Scholes model, Brownian motion represents a special case of the αα-stable class with $\alpha = 2$. To generate fat tails, α must be less than 2, but this results in infinite variance for log returns, raising concerns about the existence of a martingale measure and the finiteness of option values [7], [9].

To address this issue, the model incorporates a second parameter, β, which ranges from -1 to 1, governing the skewness of the distribution. The key innovation is focusing on the case where $\beta = -1$, inducing maximum negative skewness in the Lévy α-stable motion. Under this condition, it is proven that all conditional moments for the index level exist, despite the log return having infinite variance. This specification effectively captures the highly skewed nature of the implied density for log returns, a characteristic that neither Brownian motion nor symmetric Lévy αα-stable motion can adequately represent [10].

The standardized Lévy αα-stable motion, denoted as $L(\alpha, \beta)$, is martingale and serves as a potential driver for the risk-neutral process of discounted index levels. As a Lévy process, it possesses independent and stationary increments, with the increment $dL(\alpha, \beta)$ following an αα-stable distribution characterized by zero drift, a dispersion of $dt(1/\alpha)$, and a skew parameter β. The fractional Brownian motion unlike standard Brownian motion, its characterized by Hurst parameter H, which influences the correlation structure of the increments.

For $H \in (0,1)$ [1], [3]:

- if $H = 0.5$, it reduces to standard Brownian motion;
- if $H > 0.5$, the process exhibits persistent behavior (long-rang dependence);
- if $H < 0.5$, it exhibits anti-persistent behavior.

3 LÉVY PROCESS

A Lévy process has independent and stationary increments, allowing for both continuous changes (like Brownian motion) and discrete jumps. The characteristics of the Lévy process can be defined by its characteristic function or Lévy-Khintchine representation [11], [12]:

$$\phi(u,t) = E\left[e^{iuL_t}\right] = \exp\left(i\delta u + \int e^{iux} - 1 - iux.1_{|x|<1}\upsilon dx\right), \tag{4}$$

where δ: Drift term of the Lévy process and υdx: Lévy measure, which describes the jump intensity and distribution. In a risk-neutral framework, replace μ with the risk-free rate r. However, if we consider instead of fractional Brownian motion, we need to adjust our understanding of the stochastic calculus involved. To derive the solution for S_t, we can use the Ito formula adapted for fractional Brownian motion, the stochastic differential equation (SDE) (3) [13].

$$dS_t = \mu s_t dt + \sqrt{V_t} S_t dW_t^H + S_t dL_t.$$

To solve this equation, we can use Itos formula for a function $f(t, S_t)$, for some function f, then [13]-[15]:

$$df(t,S_t) = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu S_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 S_t^2 + jump\ terms\right)dt + \frac{\partial f}{\partial x}\sigma S_t dW_t + \frac{\partial f}{\partial x} S_t dL_t. \tag{5}$$

In our case. Let's define $f(t, S_t) = \log S_t$. Thus, we can express S_t in terms of $f(t, S_t)$.

We have, $\frac{\partial f}{\partial t} = 0$ (since f does not explicitly depend on t), $\frac{\partial f}{\partial x} = \frac{1}{S_t}$, and $\frac{\partial^2 f}{\partial x^2} = \frac{1}{S_t^2}$.

Substituting these derivatives into the Fractional Ito formula gives

$$df(t,S_t) = \left(\mu - \frac{1}{2}V_t\right)dt + \sqrt{V_t}dW_t^H + dL_t. \tag{6}$$

Integrating both sides from 0 to t:

$$\log S_t - \log S_0 = \int_0^t \left(\mu - \frac{1}{2}V_s\right)ds + \int_0^t \sqrt{V_s}dW_s^H + \int_0^t dL_s,$$

$$\log S_t = \log S_0 + \int_0^t \left(\mu - \frac{1}{2}V_t\right) ds + \int_0^t \sqrt{V_s} dW_s^H + \int_0^t dL_s.$$

Exponentiating both sides yields

$$S_t = S_0 \exp\left(\int_0^t \left(\mu - \tfrac{1}{2}V_s\right) ds + \int_0^t \sqrt{V_s} dW_s^H + \int_0^t dL_s\right),$$

$$S_t = S_0 \exp\left(\mu t - \frac{1}{2}\int_0^t V_s\, ds + \int_0^t \sqrt{V_s} dW_s^H + L_t\right), \quad (7)$$

To solve the integral $\int_0^t \sqrt{V_s} dW_s^H$, using numerical methods such as Monte Carlo simulations.

3.1 Define the Simulation Parameters

To perform the numerical simulation of the proposed fractional diffusion–Lévy model, the following key parameters are defined:

- Time Interval: $[0, t]$;
- Number of Simulations: N $(e.g., 10{,}000)$;
- Time Steps: M $(e.g., 100)$;
- Hurst Parameter: H $(e.g., 0.7)$;
- Final Time: t $(e.g., 1.0)$;
- Numerical Simulation.

3.1.1 Discretization of Time

The time interval is discretized into Msteps:

$$\Delta t = \frac{t}{M}.$$

Let $t_k = k \cdot \Delta t \;\; for\; k = 0,1, \dots, M$.

3.1.2 Simulating Fractional Brownian Motion

Fractional Brownian motion is generated using Cholesky decomposition of the covariance matrix, ensuring the correct correlation structure of the increments.

3.1.3 Simulating V_s

The variance V_t may be assumed deterministic ($V_t = \sigma^2$) or simulated as a stochastic process. For example, it can follow a geometric Brownian motion or the Heston stochastic variance dynamics.

3.2 Monte Carlo Simulation Algorithm

The stochastic integral $\int_0^t \sqrt{V_s}\, dW_s^H$ is approximated numerically using the trapezoidal rule. Monte Carlo simulations are conducted with $N = 10{,}000$ repetitions, each discretized into $M = 100$ time steps. The mean and variance of the simulated integrals are then computed to obtain statistical properties of the process.

4 ESTIMATE THE PARAMETERS OF THE HESTON MODEL

To estimate the parameters of the Heston model with fractional Brownian motion using Quasi-Maximum Likelihood Estimation (QMLE), the Table 1 shows the estimated parameters of the model, the QMLE, also known as pseudo-maximum likelihood estimation, is a statistical technique employed to estimate model parameters when the true likelihood function is either unknown or computationally intractable. Rather than maximizing the true likelihood function, QMLE optimizes a "quasi-likelihood" function, which is constructed under the assumption of a specific distribution that may differ from the actual data-generating process. Despite the potential divergence between the assumed and true distributions, QMLE frequently produces parameter estimates that are consistent and asymptotically efficient under certain regularity conditions. The method relies on an assumed likelihood function based on a given distribution (e.g., normal or Gaussian), even when the true underlying distribution of the data does not conform to this assumption. Consistency of the parameter estimates is maintained provided the quasi-likelihood accurately captures the mean and variance structure of the data. Furthermore, QMLE achieves asymptotic efficiency if the quasi-likelihood coincides with the true likelihood; otherwise, efficiency may be diminished, though consistency remains intact under mild assumptions. QMLE is extensively applied in econometrics, time series analysis, and generalized linear models, particularly in scenarios where the true error distribution is unknown or complex. Its ability to yield reliable estimates despite distributional misspecification makes it a valuable tool in statistical modeling and inference, we will follow a systematic approach [16], [17]. This includes defining the model, deriving the likelihood function, and then estimating the parameters. To apply QMLE, we need to discretize the model. Assume we observe S_t at discrete time $t_0, t_1, \dots, t_n$. We define the time increments as $\Delta t = t_{i+1} - t_i$, the discretized stock price can be approximated as

$$S_{t_{i+1}} = S_{t_i}\left(1 + \mu\Delta t + \sqrt{V_t}\, \Delta W_{t_i}^{H} + \Delta L_{t_i}\right). \quad (8)$$

Where $\Delta W_{t_i}^{H}$ is the increment of fractional Brownian motion And variance.

$$V_{t_{i+1}} = V_{t_i} + \theta\left(\mu_V - V_{t_i}\right)\Delta t + \sigma_V \sqrt{V_{t_i}}\Delta W_i. \quad (9)$$

The MLE is constructed based on the joint distribution of observed data, we can express the log-likelihood as follows. [27] The log returns can be computed as

$$R_t = \log S_t - \log S_{t-1} = \log \frac{S_{t_{i+1}}}{S_{t_i}} = \left(\mu - \frac{1}{2} V_t\right) dt + \sqrt{V}_t\, W_i^H + dL_t. \quad (10)$$

Assuming the increments of R_t are approximately normality distributed, the log-likelihood function L can be expressed as:

$$L\left(\mu, \theta, \mu_V, \sigma_V | R\right) = \sum_{i=0}^{n-1} f\left(R_t | \mu, V_{t_i}\right), \quad (11)$$

where $f(R_i|\mu, V_{t_i})$ is the probability density function of the log-normal distribution, we will take the log-likelihood function becomes [17], [18]:

$$\ell\left(\mu, \theta, \mu_V, \sigma_V | R\right) = \sum_{i=0}^{n-1}\left[-\frac{1}{2}\log\left(2\pi \hat{V}_t\right) - \frac{(R_t - \mu)^2}{2\hat{V}_t}\right], \quad (12)$$

where $\hat{V}_t$ is the estimated volatility at time t. To estimate $\hat{V}_t$, we can use realized volatility which is computed from the observed return.

$$\hat{V}_t = \frac{1}{\Delta t}\sum_{i=1}^{k}(R_{t-i} - \bar{R})^2. \quad (13)$$

Where, Δt is time interval and $\bar{R}$ its mean return over the interval. For fractional Brownian motion, the volatility can be modeled as:

$$\hat{V}_t = V_0 e^{\theta t + \sigma W_t^H}. \quad (14)$$

To find the parameters estimates, maximize log-likelihood function with respect to the parameters $\mu, \theta, \mu_v, \sigma_v$:

$$\left(\hat{\mu}, \hat{\theta}, \hat{\mu}_v, \hat{\sigma}_v\right) = \arg max_{\mu,\theta,\mu_v,\sigma_v} L\left(\mu, \theta, \mu_v, \sigma_v\right). (15)$$

This maximization can be performed using numerical optimization techniques. We will use the Newton-Raphson method. To apply the Newton-Raphson method we will the gradient of the log-likelihood function with respect to each parameter

$$\frac{\partial L}{\partial \mu} = \sum_{t=1}^{n} \frac{(R_t - \mu)}{\hat{V}_t}. \quad (16)$$

And derivative with respect to θ we need requires differentiating with respect to the estimated volatility, which itself is a function of θ. Assuming $\hat{V}_t$ depends on θ

$$\frac{\partial L}{\partial \theta} = \sum_{t=1}^{n}\left[-\frac{1}{2}\frac{\partial \hat{V}_t}{\partial \theta} - \frac{(R_t - \mu)}{2\hat{V}_t^2}\frac{\partial \hat{V}_t}{\partial \theta}\right], \quad (17)$$

μ_V and σ_v. These derivatives will also depend on the specific formulation of $\hat{V}_t$. Now, we calculate the Hessian matrix consists of the second derivatives of the log-likelihood function [19], [18]:

$$H = \begin{bmatrix} \frac{\partial^2 L}{\partial \mu^2} & \frac{\partial^2 L}{\partial \mu \partial \theta} & \frac{\partial^2 L}{\partial \mu \partial \mu_v} & \frac{\partial^2 L}{\partial \mu \partial \sigma_v} \\ \frac{\partial^2 L}{\partial \theta \partial \mu} & \frac{\partial^2 L}{\partial \theta^2} & \frac{\partial^2 L}{\partial \theta \partial \mu_v} & \frac{\partial^2 L}{\partial \theta \partial \sigma_v} \\ \frac{\partial^2 L}{\partial \mu_v \partial \mu} & \frac{\partial^2 L}{\partial \mu_v \partial \theta} & \frac{\partial^2 L}{\partial \mu_v^2} & \frac{\partial^2 L}{\partial \mu_v \partial \sigma_v} \\ \frac{\partial^2 L}{\partial \sigma_v \partial \mu} & \frac{\partial^2 L}{\partial \sigma_v \partial \theta} & \frac{\partial^2 L}{\partial \sigma_v \partial \mu_v} & \frac{\partial^2 L}{\partial \sigma_v^2} \end{bmatrix}.$$

Thin we update parameters using Newton-Raphson method (N-R), the N-R update rule is given by

$$\theta^{(k+1)} = \theta^{(k)} - H^{-1}\nabla L\left(\theta^{(k)}\right). \quad (18)$$

Where, $\theta^{(k)}$ is the parameter vector at iteration k, ∇L is the gradient vector and H^{-1} is the inverse of the Hessian matrix, we can write the iterative algorithm.

1) Initialize: set initial guesses for $\mu, \theta, \mu_v, \sigma_v$.
2) Compute log-likelihood: calculate $L\left(\mu, \theta, \mu_v, \sigma_v\right)$
3) Calculate Gradient and Hessian: compute the gradient and Hessian matrix.
4) Update parameters: use the newton-Raphson update rule.
5) Check Convergence: if the change in parameters is below a certain threshold, stop; otherwise, return to step 2.

Given the parameters, we can analyze the conditional moments of the returns. For the Lévy process with $\alpha < 2$ and $\beta = -1$, the moments can be expressed as follows [20], [21], [17]:

$$E[R_t|F_{t-1}] = \mu + \theta(\mu_v - V_{t-1}).$$

and

$$Var(R_t|F_{t-1}) = V_{t-1}.$$

The parameters were estimated using S&P 500 index data using the maximum likelihood method for the period from 1/1/2007 to 12/31/2013 and the results are as shown below:

Table 1: Represents the values of the parameters estimating using the QMLE method.

Parameters	Sample size (n)
	n=1000
μ	-0.09308951
θ	0.09999268
μ_v	0.20022537
σ_v	0.1000030

The parameter μ represents the drift term in the returns equation, which captures the average growth or decline rate of the asset's log returns over time. A negative value ($\mu = -0.09308951$) suggests that the returns exhibit a downward trend on average, indicating that the asset has a negative expected return over the sample period. This could reflect a bearish market trend dataset where the returns are biased downward, this result is based on real data (S&P500), it might indicate a period of market decline or other structural factors affecting in the returns. The parameter θ represents the speed at which the variance (V) reverts to its long-term mean (μ_v). A value of $\theta = 0.09999268$ indicates that the variance tends to revert to its equilibrium level $\mu_v = 0.20022537$ at a moderate rate, the mean reversion speed is relatively low ($\theta \approx 0.1$), meaning that the variance process is persistent and takes time to return to its long-term mean deviations. This is consistent with the behavior of volatility in financial markets, where high volatility tends to persist for extended periods before reverting to normal levels. The parameter μ_v represents the long-term average level of variance (V), a value of $\mu_v = 0.20022537$ suggests the variance fluctuates around this level over the long run. This value is reasonable for financial data, as it reflects a moderate level of variance, which is typical for daily returns in equity markets. In the context of the S&P500, this indicates that the typical daily variance of returns is around 0.2, which corresponds to standard deviation of approximately $\sqrt{0.2} \approx o.447$ or 44.7%. the parameter σ_v represents the volatility of the variance process, which measures how much the variance (V) fluctuates over time, a value of $\sigma_v = 0.100003$ suggests that the variance process is sufficiently stochastic fluctuations. This value indicates that the variance process is sufficiently stochastic, meaning that variance (V) is not constant but fluctuates over time due to random shocks, the volatility of variance (σ_v) is crucial for capturing realistic market dynamics, as it reflects the variability of volatility itself, which is often observed in financial markets. The results demonstrate that the stochastic volatility model provides a reasonable representation of the dynamics of returns and variance for the given sample size $(n = 1000)$. The estimated parameters $(\mu, \theta, \mu_v, \sigma_v)$are consistent with theoretical expectations and provide insights into the behavior of financial data, particularly the S&P 500 index.

5 THE FRACTIONAL DIFFUSION EQUATION WITH LÉVY PROCESS

The core fractional diffusion equation for the modified Heston model can be expressed as [6], [21], [22], [23]:

$$\frac{\partial^\beta V}{\partial t^\beta} = D\,\frac{\partial^\alpha V}{\partial x^\alpha} - rV + L_t, \quad (19)$$

Where V is option price or value of the underlying asset, D is diffusion coefficient, $\frac{\partial^\beta V}{\partial t^\beta}$ is fractional time derivative of order β, $\frac{\partial^\alpha V}{\partial x^\alpha}$ is fractional spatial derivative of order α, r is Risk-free interest rate and L_t is the jump term modeled as Lévy process [3], [30], [31], to solve this equation we will use the Fourier transform of a function $V(x,t)$ is defined as:

$$\hat{V}(k,t) = F\{V(x,,t)\} = \int_{-\infty}^{\infty} V(x,t)\, e^{-ikx} dx, \quad (20)$$

and the inverse Fourier transform is given by:

$$V(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{V}(k,t)\, e^{-kx} dk. \quad (21)$$

Applying the Fourier transform to both sides of the fractional diffusion equation gives [8], [24], [25]:

$$F\left\{\frac{\partial^\beta V(x,t)}{\partial t^\beta}\right\} = D\,F\left\{\frac{\partial^\alpha V(x,t)}{\partial x^\alpha}\right\} - r\,F\{V(x,,t)\} + F\{L_t\},$$

by using the properties of the Fourier transform, we have. Using the fractional derivative in the Fourier domain, we get

$$F\left\{\frac{\partial^\beta V(x,t)}{\partial t^\beta}\right\} = (iw)^\beta \hat{V}(k,,t),$$

where w is the frequency variable in the time domain

$$F\left\{\frac{\partial^\alpha V(x,t)}{\partial x^\alpha}\right\} = (ik)^\alpha \hat{V}(k,,t).$$

Substituting these transforms into the fractional diffusion equation yields:

$$(iw)^{\beta}\hat{V}(k,,t) = D\,(ik)^{\alpha}\hat{V}(k,,t) - r\hat{V}(k,,t) + \hat{L}_t. \quad (22)$$

Rearranging the equation gives.

$$\hat{V}(k,,t)\left(\,(iw)^{\beta} - D\,(ik)^{\alpha}\,\right) + r\hat{V}(k,,t) = \hat{L}_t.$$

$$\hat{V}(k,,t) = \frac{\hat{L}_t}{(\,(iw)^{\beta} - D\,(ik)^{\alpha}\,) + r}. \quad (23)$$

We have L_t is a Lévy process, its Fourier transform can be expressed as

$$\hat{L}_t = \emptyset L_t = \int_{-\infty}^{\infty} L_t\, e^{-iwt} dt,$$

to solve this equation, we can use the characteristic function of a Lévy process, which is given by

$$\emptyset L_t = e^{\int_0^t \Psi(w)ds},$$

where $\Psi(w)$ is the characteristic exponent of the Lévy process (Lévy-Khintchine formula) [31], [32]:

$$\Psi(w) = i\mu w - \frac{1}{2}\,\sigma^2 w^2 + \int_{-\infty}^{\infty} e^{iwx} - 1 - iwx1_{|x|<1}\,vdx, \quad (24)$$

where μ is drift parameter, $\sigma^2$is the diffusion parameter, and vdx is the Lévy measure.

Substituting the characteristic exponent into the Fourier transform expression, we get [4], [26]:

$$\hat{L}_t = \emptyset L_t = e^{\int_0^t \Psi(w)ds} = e^{t\Psi(w)}. \quad (25)$$

Now, to find the final solution of $V(x,t)$, we need to take the inverse Fourier transform

$$V(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{t\Psi(w)}}{(\,(iw)^{\beta} - D\,(ik)^{\alpha}\,) + r}\, e^{ikx} dk.$$

The poles of the integrand are the solutions to the

$$\left(\,(iw)^{\beta} - D\,(ik)^{\alpha}\,\right) + r = 0,$$

let's denote the solutions as $z_1, z_2, \ldots, z_n$. These will be the poles of the integrand, the residue of the integrand at poles z_k is given by [27]:

$$Res\frac{e^{t\Psi(w)}}{(\,(iw)^{\beta} - D\,(ik)^{\alpha}\,) + r}, z_k = \lim_{w \to z_k} \frac{e^{t\Psi(w)}}{(\,(iw)^{\beta} - D\,(ik)^{\alpha}\,) + r}.$$

To compute the residue, we need to evaluate the limit and the derivative of the numerator and denominator at the poles z_k

The derivative of the numerator is

$$\frac{d}{dw} e^{t\Psi(w)} = t\, e^{t\Psi(w)}\, \Psi'(w).$$

The derivative of the denominator is

$$\frac{d}{dw}\left[\left(\,(iw)^{\beta} - D\,(ik)^{\alpha}\,\right) + r\right] = i^{\beta}\beta(iw)^{\beta-1} - iD^{\alpha}k^{\alpha}(ik)^{\alpha-1}.$$

Then, we substituting these derivatives into residue formula, we get [28], [29]:

$$Res\frac{e^{t\Psi(w)}}{(\,(iw)^{\beta} - D\,(ik)^{\alpha}\,) + r}, z_k = \frac{t\, e^{t\Psi(z_k)}\, \Psi'(z_k)}{i^{\beta}\beta z_k^{\beta-1} - iD^{\alpha}k^{\alpha}(iz_k)^{\alpha-1}}.$$

By using the residue theorem, we have

$$V(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{t\Psi(w)}}{(\,(iw)^{\beta} - D\,(ik)^{\alpha}\,) + r}\, e^{ikx} dk$$

$$= 2\pi \sum_{k=1}^{n} Res\frac{e^{t\Psi(w)}}{(\,(iw)^{\beta} - D\,(ik)^{\alpha}\,) + r}, z_k\, e^{ikx},$$

$$V(x,t) = 2\pi \sum_{k-1}^{n} \frac{t\, e^{t\Psi(z_k)}\, \Psi'(z_k)}{i^{\beta}\beta z_k^{\beta-1} - iD^{\alpha}k^{\alpha}(iz_k)^{\alpha-1}} e^{ikx}. \quad (26)$$

The final solution for $V(x,t)$ is expressed in terms of the residues and the pole locations $z_1, z_2, \ldots, z_n$, which depend on the model parameters μ, σ^2, the Lévy measure $\nu(dx)$, the diffusion coefficient D, the risk-free rate r, and the orders of the fractional derivatives α and β. To further simplify the solution, we can express the residues in terms of the model parameters. This may involve solving the equation $\left(\,(iw)^{\beta} - D\,(ik)^{\alpha}\,\right) + r = 0$ to find the pole locations z_k, and then evaluating the residue formula at these points.

6 APPLICATION OF FRACTIONAL DIFFUSION EQUATION WITH LÉVY PROCESSES TO S&P 500 AND ISHARES EUROPE ETF (IEV)

The application of the fractional diffusion equation with Lévy processes to financial markets such as the S&P 500 and iShares Europe ETF (IEV) involves extending classical financial models (like the Heston model) to better represent real-world market behaviors. Detailed insights into the concepts, methodology, and simulations discussed in the paper.

6.1 Fractional Diffusion Equation with Lévy Processes

The classical Heston model employs stochastic volatility driven by Brownian motion, assuming continuous price movements. However, this approach is limited in its ability to capture several important features of real financial markets, including volatility clustering, fat tails, and sudden jumps.

Volatility clustering refers to alternating periods of high and low volatility, while fat tails describe the presence of extreme movements in asset returns. In addition, financial markets often exhibit jumps, which correspond to abrupt price changes caused by macroeconomic events or unexpected news.

To address these limitations, the fractional diffusion equation with Lévy processes extends the classical framework in two key ways. First, it incorporates fractional Brownian motion to model long-range dependencies and memory effects in price dynamics. Second, it introduces Lévy processes to capture discontinuities, skewness, and heavy-tailed behavior in return distributions.

6.2 Application to Financial Markets

6.2.1 S&P 500 Market Dynamics

The S&P 500, a benchmark index for U.S. equities, exhibits several well-known stylized facts of financial markets, including high liquidity, volatility clustering, fat-tailed return distributions, and sudden price jumps driven by macroeconomic events.

Volatility clustering is reflected in alternating periods of high and low volatility, while fat tails indicate the presence of extreme return values. In addition, jumps represent abrupt and significant price changes caused by unexpected news or economic shocks.

These characteristics are illustrated in Figure 1, which demonstrates fat tails in the return distribution, skewness capturing asymmetry in price movements, and volatility clustering over time.

The fractional diffusion equation with Lévy processes is particularly suitable for modeling such dynamics, as it incorporates both stochastic volatility and jump behavior. Specifically, stochastic volatility is modeled using fractional Brownian motion to capture long-range dependence, while Lévy processes are used to represent discontinuities and sudden shocks in asset prices.

Using R statistical software, the model integrates stochastic volatility driven by fractional Brownian motion and jump intensity governed by Lévy processes. In addition, risk-neutral return distributions are considered, which are essential for pricing financial derivatives such as options.

The simulation results confirm the presence of fat tails, skewness, and volatility clustering, thereby demonstrating the capability of the proposed model to capture key features of real financial market behavior.

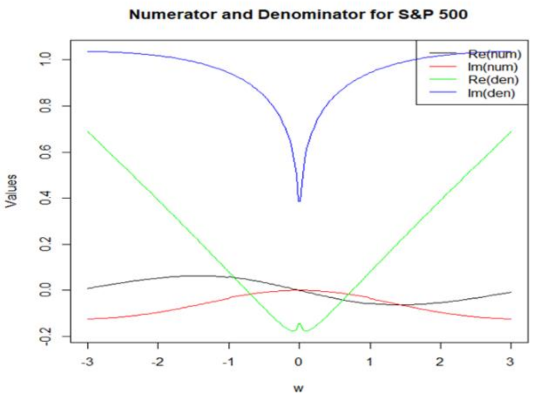

Figure 1: Reference: R statistical software by the researcher.

6.2.2 iShares Europe ETF (IEV) Market Dynamics

The IEV tracks a broad range of European equities and is influenced by various regional factors, including macroeconomic conditions such as economic growth, inflation, and interest rates, as well as political events, including elections, geopolitical tensions, and regulatory changes.

Compared to the S&P 500, the IEV exhibits distinct market characteristics. As illustrated in Figure 2, the level of volatility is generally lower and more regionally constrained than the global volatility observed in U.S. markets. Although fat-tailed behavior is still present, extreme return movements occur less frequently. In addition, price jumps are typically associated with regional shocks, such as major political events (e.g., Brexit), rather than global macroeconomic disruptions.

The fractional diffusion equation with Lévy processes is well suited to modeling these dynamics, as it accounts for both long-range dependence and discontinuities in asset prices. In this framework, stochastic volatility is modeled using fractional Brownian motion, while Lévy processes capture jump behavior, albeit with lower intensity compared to the S&P 500.

To further evaluate the model, numerical simulations were conducted using R statistical software. The simulation incorporates stochastic volatility driven by fractional Brownian motion and jump components governed by Lévy processes, while also considering risk-neutral dynamics relevant for pricing financial derivatives.

The results, presented in Figure 2, indicate moderate fat tails, reflecting less frequent extreme movements in European markets. Volatility clustering is also observed, although it is more stable and less pronounced than in the S&P 500. Furthermore, the lower intensity of jumps suggests relatively smoother market dynamics compared to U.S. equities.

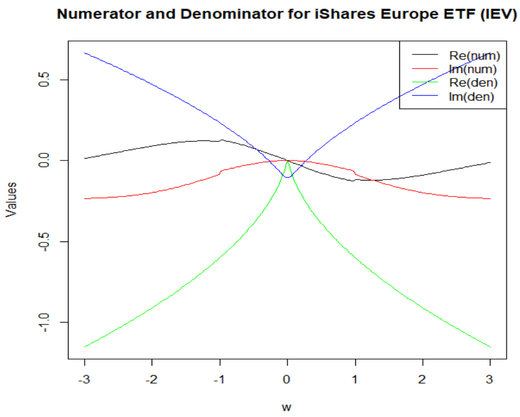

Figure 2: Reference: R statistical software by the researcher.

6.3 Numerical Simulation and Results

The S&P 500 index over the period from January 1, 2014, to December 31, 2019, is considered for analysis.

Figure 3 illustrates the behavior of the numerator and denominator functions within the framework of the fractional diffusion equation with Lévy processes applied to the S&P 500 index. Specifically, it presents the real and imaginary components of both functions over the range of w values from −3 to 3.

The real and imaginary parts of the numerator function are represented by the blue and red lines, respectively, while the green and purple lines correspond to the real and imaginary parts of the denominator function. The results reveal smooth and continuous behavior of both functions across the specified range.

Notably, neither the numerator nor the denominator functions exhibit abrupt changes or discontinuities. This suggests that, within the considered representation, the model produces a continuous approximation of the underlying financial dynamics rather than explicitly capturing jump behavior.

Overall, the plot provides insight into the mathematical properties of the fractional diffusion equation with Lévy processes and its application to financial time series data, highlighting its ability to represent stable and continuous dynamics in the S&P 500 index.

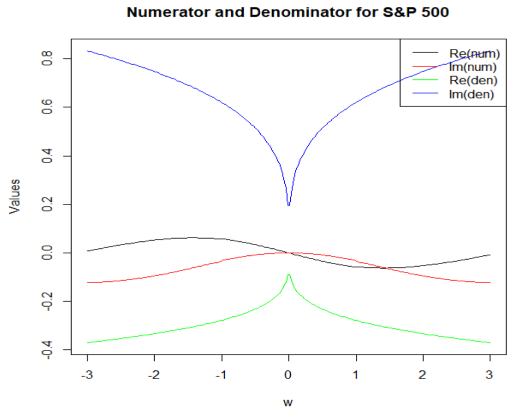

Figure 3: Reference: R statistical software by the researcher.

7 CONCLUSIONS

This paper has presented a novel framework that integrates fractional diffusion dynamics with Lévy-driven jumps, extending the classical Heston model to better capture the complexities of financial markets. By embedding fractional Brownian motion within a stochastic volatility setting, we have addressed significant empirical features such as volatility clustering, heavy tails, and sudden market jumps, which are often inadequately modeled by traditional approaches. The incorporation of fractional Brownian motion introduces long-memory effects, allowing the model to reflect persistent behavior observed in financial time series. This is crucial for accurately modeling asset prices, as it acknowledges the reality of market dynamics where past price movements influence future behavior. Moreover, the use of Lévy processes enables the model to account for discontinuous jumps, thereby providing a more realistic representation of extreme events that frequently occur in financial markets. Through the derivation of a Fourier transform representation for option pricing, we have established a robust mathematical foundation for the proposed model. The Monte Carlo simulation scheme further enhances the practical applicability of our approach, allowing for the numerical evaluation of complex stochastic integrals. Additionally, the implementation of maximum likelihood estimation (MLE) for parameter calibration ensures that our model can be effectively tailored to fit empirical financial data, enhancing its relevance for practitioners. The application of the model to real financial data, including the S&P 500 index and the iShares Europe ETF, demonstrates its potential to improve the accuracy of option valuation and the measurement of market risk. The results suggest that our framework provides a more comprehensive understanding of market dynamics compared to classical models, particularly in explaining the observed phenomena of volatility clustering and the heavy-tailed nature of financial returns. Despite the advancements presented in this paper, it is essential to acknowledge the limitations and areas for future research. While our model offers a significant step forward, further exploration is needed to refine the integration of fractional dynamics and jump processes. Future work could focus on extending the framework to incorporate additional market factors, such as interest rates and macroeconomic indicators, which could further enhance predictive capabilities. Moreover, the robustness of the model across different asset classes and market conditions should be investigated. Conducting stress tests and scenario analyses will be vital in assessing the model's performance under extreme market conditions, thus ensuring its applicability in risk management and derivative pricing. In conclusion, the fractional diffusion–Lévy process framework proposed in this paper represents a meaningful advancement in financial modeling. By addressing both long memory and discontinuities, the model not only enhances theoretical understanding but also provides practical tools for market practitioners. As financial markets continue to evolve, the need for sophisticated models that can accurately capture their complexities becomes increasingly critical. This research contributes to that objective, paving the way for future innovations in financial modeling and analysis.

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