Maximum Likelihood Estimation for the Weibull-Burr Distribution

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Abstract: This study introduces a new family of distributions known as the five-parameters Weibull- Burr distribution. This study aims to present a new family of Weibull-Burr distributions by utilizing the maximum likelihood estimation (MLE) method to determine the efficiency of the exponential family through the mean square error (MSE) by utilizing Monte Carlo simulation experiments by using the R program. It is designed to offer greater flexibility in the modeling of the financial, and actuarial data. The distribution in view presents five parameters, being thus capable of describing very peculiar hazard rate behaviors, and extreme values. Among the key statistical properties derived in this paper are the probability density function, cumulative distribution function, and moments of the distribution. Further, the paper concentrates on parameter estimation of the distribution by the method of MLE. The log-likelihood function was formulated, and processed, followed by parameter estimation, and further by the simulation under different sample sizes. Simulation results show that the proposed model can effectively address complex data structures. These results clearly indicate that the Weibull –Burr Model offers a promising alternative to the classical distributions used in modeling in finance, and actuarial domains.

1 INTRODUCTION

In recent years, the growing interest in research has been induced for developing classical probability models to incorporate into its fold, and provide more flexible, and accurate representation of complex real world data [1]. The developments involved expanding the number of parameters or applying some transformation on the original distributions or constructing hybrid models by combining two or more distributions [2]. The foremost feature of the statistical modification is to enhance the precision of inference, and aside from that, changes adapted for asymmetric behavior or extreme values for data, examples of which are financial or actuarial data. This study intends to present a new family of Weibull - Burr distributions through the MLE method for efficient estimation of the parameter based on the Monte Carlo simulation experiments for mean square error (MSE). In this study, R is introduced for the experimental section. In this line of research, the Burr distribution can be considered one of the paramount distributions in representing actual data with extreme behaviors and varied risk functions, forming an extremely flexible new distribution when combined with other distributions such as the Weibull distribution, namely, the five-parameter Weibull-Burr distribution [3]. The proposed distribution combines the properties of two distributions and hence can stand as a better candidate for modeling high-variance data. Some important contributions made in previous research are discussed with the synthesis of those findings, building upon which new developments in this field will be initiated. Some of the relevant previous researches are listed here. A new procedure for the generation of a family of Weibull probability distributions was introduced in (2015) using the logistic Weibull distribution [4]. In (2017), researchers presented a newer generalized distribution that better fits the Weibull and some generalizations, i.e., the alpha-powered Weibull distribution [5]. The next year saw the proposal of a new family of univariate x-Weibull distributions with two sub models that belong to the said family [6]. In (2020), researchers proposed a study titled "A Family of Single-Inverse Weibull Distributions," which provides high flexibility in analyzing life-span data [7]. In (2021), researchers proposed a new class of extreme value statistical distributions specifically designed for modeling complex financial, and actuarial data [8]. In (2023), researchers proposed a Weibull distribution with an estimable shift parameter [9]. In (2025), researchers proposed a study of a new family of Lomax-x-shift distributions: Properties and estimation on financial, and actuarial data [10].

2 WEIBULL –BURR DISTRIBUTION

The new five-parameter Weibull- Burr distribution is an advanced extension of continuous probability distributions, it is designed for modeling the complex three-parameters financial, actuarial, and engineering data with the time interval $(0,\infty)$. This distribution combines the properties of the Burr distribution, known for its flexibility in handling data with the extreme values, and the Weibull distribution, which is popular in modeling data related to reliability, and failure times, providing a powerful framework for analyzing the complex multi-parameter data. The new distribution also aims to improve the data fit by introducing additional parameters that allow controlling over shape, scale, and displacement, making the model capable of representing a wider range of data behaviors compared to traditional distributions. The proposed distribution can represent different types of data, such as the extreme values data or data with a positive or negative skewness. Applying the new distribution on the financial, and actuarial data for the output distribution makes the new distribution more accurate in the modeling of the large losses, which is critical in the financial risk analysis [11], [12]. It also allows processing the complex data while maintaining a clear statistical interpretation of the distribution parameters [13], [14]. The general formula for the distribution as below:

Regarding a random variable T, let h(t) be its pdf, where $T \in [a, b]$ for $-\infty \leq a < b < \infty$. Similarly, let $W[G(x; \zeta)]$ be its CDF, contingent upon the vector parameter ζ meeting the following requirements:

- $W[G(x; \zeta)] \in [a, b]$;
- In addition to being monotonically rising, $W[G(x; \zeta)]$ is differentiable;
- $W[G(x; \zeta)] \to a$ as $x \to -\infty$ and $W[G(x; \zeta)] \to b$ as $x \to +\infty$.

The CDF associated with the T-X family of distributions was recently defined by [5]:

$$F(x) = \int_a^{w[G(x);\xi]} h(t)dt \quad (1)$$

In this article, we propose the following staggered transformation:

$$W[G(x; \zeta)] = x\,(G(x) + m),$$

where $x \geq 0$, $G(x; \zeta)$ is the CDF of one of the classical models, and $m \geq 0$ is a constant representing the shift parameter. $W[G(x; \zeta)]$ on $[0, +\infty]$ satisfies the aforementioned requirements. We study here with h(t) the density function of the Weibull function whose support is for $x \geq 0$. This gives us a new family of shifted Weibull distributions whose CDF is defined b:

$$F(x)=\int_a^{x[G(x)+m]} h(t)dt, \quad (2)$$

$$F(x)=1- e^{-\left[\frac{xG(x)+mx}{\lambda}\right]^{\theta}}. \quad (3)$$

where $x \geq 0$, $\theta = (\zeta, \alpha, \lambda, m) \geq 0$.

This distribution function is differentiated based on x. As a result, we get:

$$F(x) = 1 - e^{-\left[\frac{x-x(1+x^{\alpha})^{-\beta}+mx}{\lambda}\right]^{\theta}}, \quad (4)$$

$$f(x)=\frac{\theta}{\lambda^{\theta}}\,[xG(x) + mx]^{\theta-1}\,[xg(x) + G(x) + m]\; e^{-\left[\frac{xG(x)+mx}{\lambda}\right]^{\theta}}, \quad (5)$$

$$f(x)= \frac{\theta}{\lambda^{\theta}}[x - x(1 + x^{\alpha})^{-\beta}+mx]^{\theta-1}[1+\alpha\beta x^{\alpha}(1 + x^{\alpha})^{-\beta-1} + (1 + x^{\alpha})^{-\beta} + m] \quad e^{-\left[\frac{x-x(1-x^{\alpha})^{-\beta}+mx}{\lambda}\right]^{\theta}}, \quad (6)$$

$$S(t)= e^{-\left[\frac{x-x(1+x^{\alpha}]^{-\beta}+mx}{\lambda}\right]^{\theta}}, \quad (7)$$

$$h(x) = \frac{\theta}{\lambda^{\theta}}[x-x(1+x^{\alpha})^{-\beta}+mx]^{\theta-1}[1+\alpha\beta x^{\alpha}(1+x^{\alpha})^{-\beta-1}+ (1+x^{\alpha})^{-\beta} +m], \quad (8)$$

The purpose of the distribution function: Weibull-Burr distribution as a statistical model can be used in the modeling of the financial and actuarial data. The probability density function (PDF) “has a condition” that must be satisfied, as below [15]:

1) Non negative;
2) The condition for the cumulative distribution function (CDF) is that it does not exceed one [16], [17], (Table 1).

$$\lim_{x-0} F(x) = 0, \quad (9)$$

$$\lim_{x-\infty} F(x) = 1. \quad (10)$$

Table 1: The properties of the Weibull- Burr distribution.

Properties/Formula
Mean
$\mathbf{E}(x)= Ab\,\frac{\Gamma_{(1)}}{\theta+\alpha j}+\ Ab^{-\frac{1}{\theta+\alpha j}+1}$
Variance
$V(x)= Ab\,\frac{\Gamma_{(2)}}{\theta+\alpha j}+\ Ab^{\left(-\frac{2}{\theta+\alpha j}+1\right)}-(Ab(\frac{\Gamma_{(1)}}{\theta+\alpha j}+\ Ab^{(-\frac{1}{\theta+\alpha j}+1)})^2$
Moment
$Ab\,\frac{\Gamma_{(r)}}{\theta+\alpha j}+1\ Ab^{(-\frac{r}{\theta+\alpha j}+1)}$
Skewness
$\frac{Ab(\frac{\Gamma_{(3)}}{\theta+\alpha j}+\ Ab^{(-\frac{3}{\theta+\alpha j}+1)}-3\ Ab(\frac{\Gamma_{(1)}}{\theta+\alpha j}+\ Ab^{(-\frac{1}{\theta+\alpha j}+1)}Ab(\frac{\Gamma_{(3)}}{\theta+\alpha j}+\ Ab^{(-\frac{3}{\theta+\alpha j}+1)}+2(\ Ab(\frac{\Gamma_{(1)}}{\theta+\alpha j}+\ Ab^{(-\frac{1}{\theta+\alpha j}+1)})^3}{[\ Ab(\frac{\Gamma_{(2)}}{\theta+\alpha j}+\ Ab^{\left(-\frac{2}{\theta+\alpha j}+1\right)}-(Ab(\frac{\Gamma_{(1)}}{\theta+\alpha j}+\ Ab^{(-\frac{1}{\theta+\alpha j}+1)})^2]^{\frac{3}{2}}}$
Kurtosis
$\frac{Ab(\frac{\Gamma_{(4)}}{\theta+\alpha j}+\ Ab^{\left(-\frac{4}{\theta+\alpha j}+1\right)}-4Ab(\frac{\Gamma_{(1)}}{\theta+\alpha j}+\ Ab^{\left(-\frac{1}{\theta+\alpha j}+1\right)}Ab(\frac{\Gamma_{(3)}}{\theta+\alpha j}+\ Ab^{\left(-\frac{3}{\theta+\alpha j}+1\right)}+6(Ab(\frac{\Gamma_{(1)}}{\theta+\alpha j}+\ Ab^{(-\frac{1}{\theta+\alpha j}+1)})^2-\ Ab(\frac{\Gamma_{(2)}}{\theta+\alpha j}+1\ Ab^{\left(-\frac{2}{\theta+\alpha j}+1\right)}-3(Ab(\frac{\Gamma_{(1)}}{\theta+\alpha j}+1\ Ab^{(-\frac{1}{\theta+\alpha j}+1)}}{(Ab(\frac{\Gamma_{(2)}}{\theta+\alpha j}+\ Ab^{\left(-\frac{2}{\theta+\alpha j}+1\right)}-(Ab(\frac{\Gamma_{(1)}}{\theta+\alpha j}+\ Ab^{(-\frac{1}{\theta+\alpha j}+1)})^2)^2}$
Moment generating function
$\sum_{k=0}^{\infty}\frac{(t)^k}{k!}\ \Gamma(\frac{k}{\theta+j\alpha})+Ab^{-\frac{k}{\theta+j\alpha}}$

$$\frac{\partial lnL}{\partial\theta}=\frac{n}{\theta}-n\ \ln(\lambda)+\sum_{i=1}^{n}\ln\left[xi-xi(1+xi^{\alpha})^{-\beta}+mxi\right]-\sum_{i=1}^{n}[xi-xi(1+xi^{\alpha})^{-\beta}+mxi]^{\theta}.\ln\left[xi-xi(1+xi^{\alpha})^{-\beta}+mxi\right]+Ln\lambda.\lambda^{-\theta}\sum_{i=1}^{n}[x_i-x_i(1+x_i^{\alpha})^{-\beta}+mx_i]^{\theta} \quad (13)$$

$$\frac{\partial lnL}{\partial\alpha}=\ \sum_{i=1}^{n}\left[xi(1+xi^{\alpha})^{-\beta}\ln(1+xi^{\alpha})\right)\sum_{i=1}^{n}\frac{-(\beta^2\mp\beta)\alpha^2xi^{\alpha+1}(1+xi^{\alpha})^{-\beta-2}+(1+xi^{\alpha})^{-\beta}\left(\beta\alpha^2xi^{\alpha-1}+xi\beta\right)-\beta xi^{\alpha}\ln(xi)(1+xi^{\alpha})^{-\beta-1}}{\left[1+\alpha\beta xi^{\alpha}(1+xi^{\alpha})^{-\beta-1}-(1+xi^{\alpha})^{-\beta}+m\right]} \quad (14)$$

$$\frac{\partial lnL}{\partial\beta}=\sum_{i=1}^{n}\frac{xi(1+xi^{\alpha})^{-\beta}\ln(1+xi^{\alpha})}{[xi-xi(1+xi^{\alpha})^{-\beta}+mxi]}+\sum_{i=1}^{n}\frac{\beta\alpha xi^{\alpha}(1+xi^{\alpha})^{-\beta-1}\ln(1+xi^{\alpha})-(1+xi^{\alpha})^{-\beta-1}\alpha xi^{\alpha}+(1+xi^{\alpha})^{-\beta}\ln(1+xi^{\alpha})}{[1+\beta\alpha xi^{\alpha}(1+xi^{\alpha})^{-\beta-1}-(1+xi^{\alpha})^{-\beta}+m]}, \quad (16)$$

$$\frac{\partial\ Ln\ L}{\partial m}=(\theta-1)\sum_{i=1}^{n}\frac{xi}{[xi-xi(1+x_i^{\alpha})^{-\beta}+mxi]}+\sum_{i=1}^{n}\frac{1}{[1-\beta\alpha x_i^{\alpha}(1+x_i^{\alpha})^{-\beta-1}-(1+x_i^{\alpha})^{-\beta}+m}-\theta\lambda^{-\theta}\sum_{i=1}^{n}[xi-xi(1+x_i^{\alpha})^{-\beta}+mxi]^{\theta-1}(xi) \quad (18)$$

2.1 Maximum Likelihood Method (MLE)

It is considered one of the most widely used methods in statistics for estimating parameters [18], [19]. It was contributed by Joseph Lagrange, and Daniel Bernoulli, and was proposed by the British scientist Ronald Fisher in the early twentieth century. He was first presented it in a research paper, due to its useful properties, including Sufficiency, impartiality, and reliability. This method aims to find estimated values for the parameters to be estimated, thus maximizing the probability function. If the random variable (x) has the probability density function (PDF) of the proposed distribution, then the maximum probability function for the independent random variables (xi) with the same distribution is as follows, [20].

$$\text{lnL}=n\ln(\theta)-n\,\theta\ln(\lambda)+(\theta-1)\sum_{i=1}^{n}\ln\left[xi-xi(1+x_i^{\alpha})^{-\beta}+mx_i\right]+\sum_{i=1}^{n}\ln\left[1+\alpha\beta x_i^{\alpha}\left(1+x_i^{\alpha}\right)^{-\beta-1}+\left(1+x_i^{\alpha}\right)^{-\beta}+m\right]\right]-\lambda^{-\theta}\sum_{i=1}^{n}\left[xi-xi(1+xi^{\alpha})^{-\beta}+mxi\right]^{\theta}. \quad (11)$$

To obtain the MLE of the parameters of the Weibull-Burr distribution [21], [22], it can be deriving the (7) as follows (12) – (18):

$$\frac{\partial lnL}{\partial\lambda}=-\frac{n\theta}{\lambda}+\theta\lambda^{-\theta-1}\sum_{i=1}^{n}[xi-xi(1+xi^{\alpha})^{-\beta}+mxi]^{\theta}, \quad (12)$$

$$-\sum_{i=1}^{n}\theta\lambda^{-\theta}\left[xi-xi\left(1+xi^{\alpha}\right)^{-\beta-1}+mxi\right]^{\theta-1}\beta xi^{\alpha+1}\ln(xi)(1+xi^{\alpha})^{-\beta-1}\ldots \quad (15)$$

$$-\lambda^{\theta}\theta\sum_{i=1}^{n}\left[xi-xi(1-xi^{\alpha})^{-\beta}+mxi\right]^{\theta-1}(xi(1-xi^{\alpha})^{-\beta}\ln(1-xi^{\alpha})) \quad (17)$$

2.2 The Experimental Simulation

The behavior of the probability density function (PDF) and cumulative distribution function (CDF) of the Weibull–Burr distribution for different parameter values is illustrated in Figure 1. Different default values were used for each of the five distribution parameters to complete five simulation experiments, with repetition once, and default values for each parameter in the five models and sample sizes (n=25, 50, 75, 125, 200) as shown in the Table 2 and default values. The number of times the simulation experiments were 1000 times. Accordingly, the estimated Weibull- Burr distribution parameters obtained from the simulation study using the MLE are presented in Tables 3, 5, 7, 9, and 11, while the corresponding MSE values are presented in Tables 4, 6, 8, 10 and 12.

The purpose is to demonstrate the efficiency of the MLE. The comparison criterion, the mean square error (MSE), is used according to the following (19):

$$\text{MSE} = \frac{\sum_{i=1}^{r}(\widehat{\theta i}-\theta)^2}{r}. \quad (19)$$

Table 2: The values of the default parameters (α, β, λ, θ, m).

α	9.8	4.5	3.5	1.5	0.5
β	0.1	0.35	0.5	0.5	0.5
λ	0.04	0.1	1.4	1.6	1.5
θ	1	1	2	0.5	0.5
m	0.06	0.15	0.08	1.25	0.1

Table 3: The estimated values of the parameters (α, β, λ, θ, m,). for the first experiment.

n	Parameter	Estimator
25	α	10.17533
50	β	0.112606
75	λ	0.044613
125	θ	1.004373
200	m	0.066892

Table 4: The (MSE) method for the first experiment.

n	Parameter	MSE
25	α	1.700772
50	β	0.003185
75	λ	0.00378
125	θ	0.29549
200	m	0.009087

Table 5: The estimated values of the parameters (α, β, λ, θ, m,). for the second experiment.

n	Parameter	Estimator
25	α	4.916392
50	β	0.445059
75	λ	0.127591
125	θ	1.029991
200	m	0.194455

Table 6: The (MSE) method for the second experiment.

n	Parameter	MSE
25	α	1.396555
50	β	0.21675
75	λ	0.145187
125	θ	1.462275
200	m	0.408576

Table 7: The estimated values of the parameters (α, β, λ, θ, m,). for the third experiment.

n	Parameter	Estimator
25	α	3.535887
50	β	0.500019
75	λ	1.425153
125	θ	2.027436
200	m	0.091558

Table 8: The MSE method for the third experiment.

n	Parameter	MSE
25	α	0.027641
50	β	0.002642
75	λ	0.34137
125	θ	0.433154
200	m	0.060002

Table 9: Shows the estimated values of the parameters (α, β, λ, θ, m,). for the fourth experiment.

n	Parameter	Estimator
25	α	1.522018
50	β	0.512781
75	λ	1.700785
125	θ	0.504265
200	m	1.34772

Table 10: The MSE method for the fourth experiment.

n	Parameter	MSE
25	α	0.013857
50	β	0.18419
75	λ	0.238929
125	θ	0.006866
200	m	0.193461

Table 11. The estimated values of the parameters (α, β, λ, θ, m,). for the fifth experiment.

n	Parameter	Estimator
25	α	0.510069
50	β	0.540635
75	λ	1.701895
125	θ	0.511138
200	m	0.130527

Table 12: The MSE method for the fifth experiment.

n	Parameter	MSE
25	α	0.013565
50	β	1.545329
75	λ	1.589458
125	θ	0.104269
200	m	0.033266

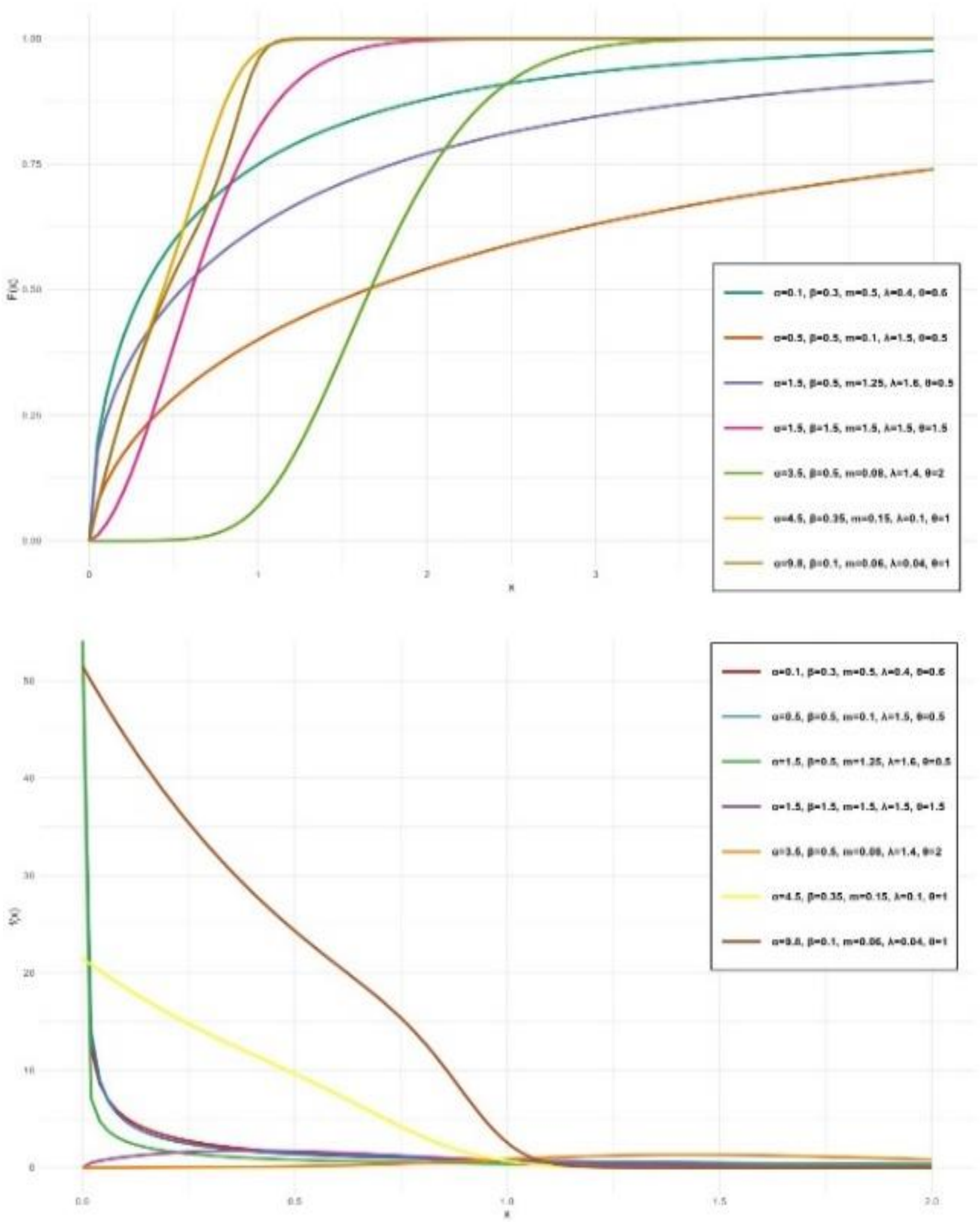

Figure 1: The behavior of (CDF and PDF) functions for Weibull-Burr Distribution with various parameters.

3 CONCLUSIONS

In this research, it provided a new statistical distribution is called Weibull - Burr distribution. The efficiency of the MLE was proven on the five-parameter Weibull- Burr distribution. The results were showed that the parameters estimate approach the true values were increasing sample size (n), and the MSE values were decrease with increasing sample size n. For some parameters, such as (λ, m and θ), the estimates were increased at small sample sizes, and the MSE was initially high, then stabilizes with increasing sample size (n). However, (α, β) was more stabled at medium sample sizes. The results were confirmed that increasing in the sample size leaded to a reduction in the error in the estimates for all of the parameters. From sample size (n≥125) and above, the MSE becomes lower than it is.

Also, the simulation results were showed that the accuracy of the parameters estimation for the Weibull - Burr distribution improved significantly with the increasing sample size. Therefore, it is recommended to use a sample size of no less than n=125 to ensure accurate estimates. Estimated the parameters by using the MLE by minimizing the MSE. Also, should be attention to paid to the displacement parameter (m) and the scale parameter (λ), as they may require additional estimations to improve their accuracy in the small samples. In addition, should be monitored the MSE to assess the quality of the estimations in the experiments, which in turn helps reveal the direct effect of sample size on the accuracy of the parameters. Finally, as a future work, it is possible - for example- to use the proposed statistical distribution on the huge dataset of the real-world or mixing the proposed statistical distribution with another statistical distribution and applying it on the real-world dataset.

4 RECOMMENDATIONS

We recommend using the Transformation (T-X) method with other distributions to obtain a new family based on the Weibull-Burr distribution, due to its high flexibility in modeling heavy tailed financial and actuarial data or generating a new family from other distributions. We also recommend using other methods, such as Bayesian methods, to estimate distribution parameters when working with relatively large datasets, as they are more stable and accurate. We also recommend using real data representing larger and more diverse samples from different insurance companies or across multiple time periods to obtain better results.

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