Parameter Estimation of the Semicircular Lomax Distribution

Zahraa Majeed and Wafaa Hussein
Department of Statistics, College of Administration and Economics, Wasit University, 52001 Wasit, Iraq
wjaffer@uowasit.edu.iq. std2023zahraa@uowasit.edu.iq

Keywords: Inverse Stereographic Projection, Stereographic Semicircular Lomax Distribution, Survival Function, Cramer Von Mises Method.

Abstract: This study aims to convert a traditional Lomax distribution into the Stereographically Projected Semicircular Lomax (SSL) distribution by making use of the inverse stereographic projection. The main focus is to semicircularize gathering of data from various applied disciplines to impart flexibility at a higher operational level and, therefore, allow appropriate estimation to be carried out through a flexible model for parameters related to SSL distribution. Further, the entire collection of statistical properties of the proposed distribution has been completely laid out from first principles, including those for the CDF, survival function S(t), and the hazard rate function h(t) in addition to the PDF. There have been two methods adopted for the parameter estimation: Maximum Likelihood Estimation and Cramér von Mises methods. Monte Carlo simulation studies were employed in which several fixed sample sizes were considered in order to analyze the performance of the MLE and CVM methods of parameter estimation. The comparison has been made under the Mean Squared Error (MSE) criterion. The results disclosed that MLE does better than CVM in giving smaller MSE values and hence for estimation.

1 INTRODUCTION

Circular data is crucial since natural and social phenomena during which cyclic repetition is involved are researched. Such data could be measured along axes that involve angles, directions, or positions along circles or semicircles. Hence, for example, one may consider wind direction or phenomenal geographic paths.

Such data have completely different statistical properties as compared to the traditional linear data, as the values are constrained in a closed interval normally taken as $(0, 2\pi)$ depending on the nature of the phenomenon under research. Linear distributions may not be suitable to analyze circular data because of the specific structure that circular data have. In modeling such data, special distributions must be considered since they correspond to the circular nature of the data.

One special subclass lying within the semicircle of the unit circle is known as semi-circular data restricted to the interval $(0, \pi)$. To cater for this issue, the Semi-Circular Lomax Distribution has been put forward. It finds applications in economic modeling and is defined in the interval $(0, \pi)$. Nevertheless, its classical linear approach is generally inappropriate for data with circular or semi-circular characteristics.

This motivated the use of transformation techniques such as the Inverse Stereographic Projection to develop new, more flexible semi-circular distributions suited to this type of data. The primary objective behind developing such a new distribution is to integrate the properties of the Lomax distribution with circular transformation methods, allowing for enhanced applications in risk analysis, prediction, and the research of relationships among periodic variables.

Moreover, this approach helps bridge the methodological gap caused by the limitations of classical linear distributions in handling circular or semi-circular data. Several relevant studies have contributed to this area of research:

- Byoung and his colleagues provided a set of semicircular Laplace distributions in 2008 to enable one to fit particular data sets by simple projection [1].
- In the year 2014, Girija et al. formulated the Double Exponential Semi-Circular Distribution using stereographic projection methods [2].

- In 2019, Yedlapalli and collaborators proposed the Stereographic Semi-Circular Lindley Distribution [3].
- In 2022, Ayesha Iftikhar introduced the Modified Third Burr-III(hcMB-III) Stereographic Semi-Circular Distribution [4].
- In 2024, Oleiwi et al. introduced the Stereographic Semi-Circular Rayleigh Distribution [5].

Building on these three attempts, there is a semi-circular model- Stereographically Projected Semicircular Lomax Distribution which results from the application of inverse stereographic projection to the traditional Lomax distribution. Using varying parameter values, the probability density function (PDF) and cumulative distribution function (CDF) were plotted using the proposed distribution. The first two trigonometric moments were then computed in order to investigate the statistical properties of the new model.

2 INVERSE STEREOGRAPHIC PROJECTION METHOD

One of the critical methods transferred by inverse stereographic projection is that continuous distributions defined on the real number line are transformed into distributions defined on the circumference of the unit circle. The angular domain created by this transformation does not cover the entire circular range but is restricted to an open semicircle. In turn, the resulting distribution does not possess complete rotational symmetry.

The limitation has been discussed in past literature, including those by Yedlapalli, and Radhika [6], [7] and another study by Srinivas, Phani, and Girija (2019), which focused on how the limited angular coverage affects circular data modeling.

The Inverse Stereographic Projection is defined as follows:

$$T(\theta) = x = vtan\left(\frac{\theta}{2}\right). \quad x \in (-\infty, \infty), \theta \in (-\pi, \pi), v > 0, \tag{1}$$

And then

$$\theta = 2\tan^{-1}\left(\frac{x}{v}\right). \tag{2}$$

The CDF of the semicircular distribution is given by

$$G(\theta) = F(x) = F\left(vtan\left(\frac{\theta}{2}\right)\right), \tag{3}$$

$$g(\theta) = \frac{v}{2} f\left(vtan\left(\frac{\theta}{2}\right)\right) sec^2\left(\frac{\theta}{2}\right). \tag{4}$$

The method discussed here views the inverse stereographic projection as applied to a probability distribution in order to produce a new semicircular distribution on the interval (0, π) which could be conveniently used for modeling semicircular data.

3 THE SEMICIRCULAR LOMAX DISTRIBUTION (SSL)

This segment introduces another member of the family of transformations of the semicircular distributions, such as the semicircular lomax distribution under stereographic projections. Definition: We say that a random variable X in the real line is lomax distributed with scale parameter α>0, and location parameter λ>0, if its probability density function and cumulative distribution functions of X are respectively given by [8]:

$$f(x, \alpha, \lambda) = \frac{\alpha}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-(\alpha+1)}, \tag{5}$$

where x>0 and

$$F(x, \alpha, \lambda) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}. \tag{6}$$

3.1 The PDF and CDF of SSL Distribution

Note [6] that the Probability Density Functions and Cumulative Distribution Functions corresponding to the Stereographically Projected Semicircular Lomax Distribution can be respectively written as:

$$g(\theta) = \frac{v}{2}\frac{\alpha}{\lambda}\left(1 + \frac{\mathrm{vtan}\left(\frac{\theta}{2}\right)}{\lambda}\right)^{-(\alpha+1)} \sec^2\left(\frac{\theta}{2}\right), \tag{7}$$

The shape of the PDF for different parameter values is illustrated in Figure 1.

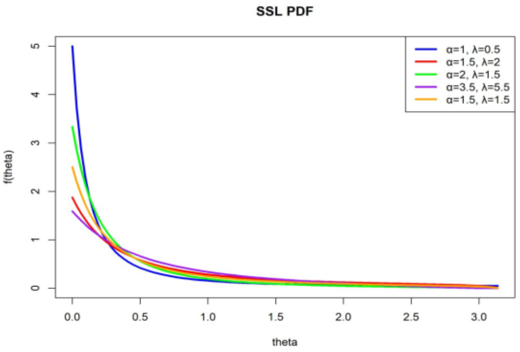

Figure 1: PDF of the Stereographic Lomax distribution.

The CDF of the Stereographically Projected Semicircular Lomax Distribution (SSL), or the nonreliability function, gives the probability that a unit fails before time t. It is defined as:

$$\mathrm{G}(\theta,\alpha,\lambda) = 1-\left(1+\frac{vtan\frac{\theta}{2}}{\lambda}\right)^{-\alpha} . \qquad (8)$$

The corresponding CDF curves are shown in Figure 2.

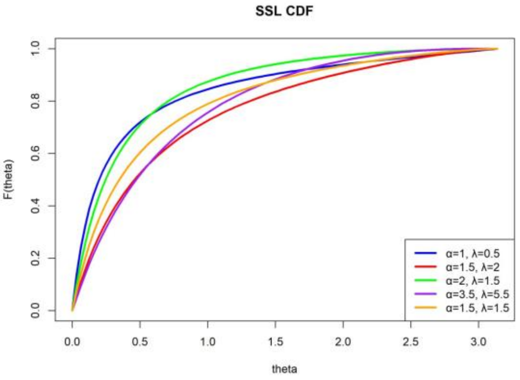

Figure 2: CDF of the Stereographic Lomax distribution.

3.2 Survival Measures and Hazard Function of SSL Distribution

References [9] provide us with the information that from (7) and (8), we can generalize the survival function S(t) and hazard function h(t) for the Stereographically Projected Semicircular Lomax Distribution.

$$S(\theta) = 1-G(\theta) = \left(1-\left(1-\left(1+\frac{vtan\frac{\theta}{2}}{\lambda}\right)^{-\alpha}\right)\right).$$

$$S(\theta) = \left(1+\frac{vtan\frac{\theta}{2}}{\lambda}\right)^{-\alpha} . \qquad (9)$$

The behavior of the survival function is illustrated in Figure 3.

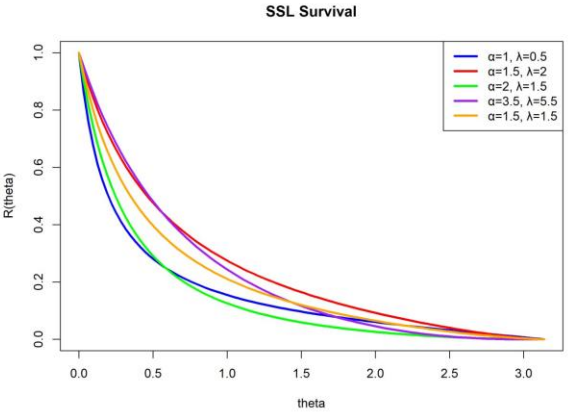

Figure 3: Survival function of the Stereographic Lomax distribution.

$$h(\theta) = \frac{g(\theta)}{1-G(\theta)} = \frac{\frac{v}{2}\frac{\alpha}{\lambda}\left(1+\frac{vtan\left(\frac{\theta}{2}\right)}{\lambda}\right)^{-(\alpha+1)} sec^2\left(\frac{\theta}{2}\right)}{\left(1+\frac{vtan\frac{\theta}{2}}{\lambda}\right)^{-\alpha}} \qquad (10)$$

The corresponding hazard function is presented in Figure 4.

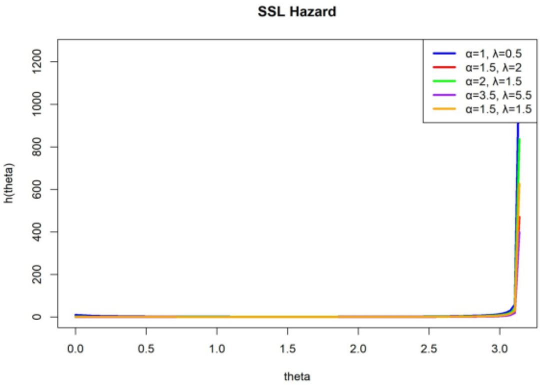

Figure 4: Graph of Hazard function of the Stereographic lomax distribution.

3.3 The r^{th} Moment of SSL Distribution

The r^{th} non-central moment of the SSL distribution, $(\theta^{\,r})$ SSL, can be obtained through the pdf in (7) with the following form:

$$E(\theta)^r = \int_0^{\pi} \theta^r \; f(\theta,\alpha,\lambda)\, d\theta \; = \int_0^{\pi} \frac{v}{2}\,\frac{\alpha}{\lambda} (\theta)^r \left(1 + \frac{vtan\frac{\theta}{2}}{\lambda}\right)^{-(\alpha+1)} \sec^2 \mathrm{T}\left(\frac{\theta}{2}\right) d\theta\,, \qquad (11)$$

$$= \frac{v}{2}\,\frac{\alpha}{\lambda} \int_0^{\pi} (\theta)^r \left(1 + \frac{vtan\frac{\theta}{2}}{\lambda}\right)^{-(\alpha+1)} \sec^2\left(\frac{\theta}{2}\right) d\theta.$$

Since $x = v\tan\left(\frac{\theta}{2}\right)$, $\theta = 2\tan^{-1}\left(\frac{x}{v}\right)$, $d\theta = \left(\frac{2v}{v^2+x^2}\right) dx = \frac{v}{2}\,\frac{\alpha}{\lambda}\int_0^{\infty}\left(2\tan^{-1}\left(\frac{x}{v}\right)\right)^r \left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)} \sec^2\left(\frac{2\tan^{-1}\left(\frac{x}{v}\right)}{2}\right)\left(\frac{2v}{v^2+x^2}\right) dx.$

Since $\sec^2 \tan^{-1}\left(\frac{x}{v}\right) = \frac{v^2+x^2}{v^2} = \frac{v}{2}\,\frac{\alpha}{\lambda}\int_0^{\infty}\left(2\tan^{-1}\left(\frac{x}{v}\right)\right)^r \left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)} \left(\frac{v^2+x^2}{v^2}\right)\left(\frac{2v}{v^2+x^2}\right) dx = \frac{\alpha}{\lambda}\int_0^{\infty}\left(2\tan^{-1}\left(\frac{x}{v}\right)\right)^r \left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)} dx.$

Using the power series of $\tan^{-1}(x)$.

$$\text{Let } \tan^{-1} = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} \left(\frac{\left(\frac{x}{v}\right)^2}{\left(\frac{x}{v}\right)^2+1}\right)^{k+\frac{1}{2}}$$

$$= \frac{\alpha}{\lambda}\int_0^{\infty}\left(2\sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} \left(\frac{\left(\frac{x}{v}\right)^2}{\left(\frac{x}{v}\right)^2+1}\right)^{k+\frac{1}{2}}\right)^r \left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)} dx.$$

$$\text{let } u = \frac{\left(\frac{x}{v}\right)^2}{\left(\frac{x}{v}\right)^2+1} \qquad x = v\left(\frac{1}{u}-1\right)^{\frac{-1}{2}}$$

$$dx = \frac{v}{2u^2}\left(\frac{1}{u}-1\right)^{\frac{-3}{2}} du$$

$$= \frac{\alpha}{\lambda}\int_0^1 \left(2\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}\, u^{k+\frac{1}{2}}\right)^r \left(1+\frac{v\left(\frac{1}{u}-1\right)^{\frac{-1}{2}}}{\lambda}\right)^{-(\alpha+1)} \frac{v}{2u^2}\left(\frac{1}{u}-1\right)^{\frac{-3}{2}} du.$$

Using Newton Binomial formula we obtain

$$\left(1+\frac{v\left(\frac{1}{u}-1\right)^{\frac{-1}{2}}}{\lambda}\right)^{-(\alpha+1)} = \sum_{j=0}^{\infty}\binom{-(\alpha+1)}{j}\left(\frac{v}{\lambda}\right)^j \left(\left(\frac{1}{u}-1\right)^{\frac{-1}{2}}\right)^j$$

$$= \frac{\alpha}{\lambda}\int_0^1 \frac{v}{2u^2}\left(2\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}\, u^{k+\frac{1}{2}}\right)\sum_{j=0}^{\infty}\binom{-(\alpha+1)}{j}\left(\frac{v}{\lambda}\right)^j\left(\frac{1}{u}-1\right)^{\frac{-3}{2}-\frac{j}{2}} du\,,$$

$$(1-u)^{\frac{-3}{2}-\frac{j}{2}} = \sum_{m=0}^{\infty}\binom{\frac{-3}{2}-\frac{j}{2}}{m}(-1)^m\left(\frac{1}{u}\right)^{-m} (u)^{\frac{3}{2}+\frac{j}{2}}$$

$$= \frac{\alpha}{\lambda}\int_0^1 \frac{v}{2u^2}\left(2\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}\, u^{k+\frac{1}{2}}\right)^r \sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\binom{-(\alpha+1)}{j}\binom{\frac{-3}{2}-\frac{j}{2}}{m}\left(\frac{v}{\lambda}\right)^j(-1)^m\left(\frac{1}{u}\right)^{\frac{-3}{2}-\frac{j}{2}-m} du$$

$$= \frac{\alpha 2^r}{\lambda}\frac{v}{2}\sum_{j,m=0}^{\infty}\binom{-(\alpha+1)}{j}\left(\frac{v}{\lambda}\right)^j\binom{\frac{-3}{2}-\frac{j}{2}}{m}(-1)^m\int_0^1\left(\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}\, u^k\right)^r (u)^{m+\frac{r}{2}+\frac{3}{2}+\frac{j}{2}-2} du$$

According to the circumstance $\left(\sum_{k=0}^{\infty} a_k\, u^k\right)^r = \sum_{k=0}^{\infty} b_k\,(u)^k$; r is a natural number $b_0 = a_0^r$, and $b_h = \frac{1}{a_0\, h}\sum_{k=1}^{h}(kr-h+k)\; a_k\, b_{h-k}$; $h \geq 1$, the $E(\theta)^r$ with $a_k = \frac{(2k)!}{2^{2k}(k!)^2(2k+1)}$ *will be*

$$E(\theta)^r = \frac{b_k\,\alpha 2^r}{\lambda}\frac{v}{2}\sum_{j,m,k=0}^{\infty}\binom{-(\alpha+1)}{j}\left(\frac{v}{\lambda}\right)^j\binom{\frac{-3}{2}-\frac{j}{2}}{m}(-1)^m\int_0^1 (u)^{k+m+\frac{r}{2}+\frac{j}{2}+\frac{3}{2}-2} du,$$

$$E(\theta)^r = \frac{b_k\,\alpha 2^r}{\lambda}\frac{v}{2}\sum_{j,m,k=0}^{\infty}\binom{-(\alpha+1)}{j}\left(\frac{v}{\lambda}\right)^j\binom{\frac{-3}{2}-\frac{j}{2}}{m}(-1)^m\frac{1}{k+m+\frac{r}{2}+\frac{j}{2}+\frac{3}{2}-1}. \qquad (12)$$

3.4 Trigonometric Moments

To study the population characteristic, the first and second trigonometric moments of the Stereographic semicircular Lomax distribution are derived. The moments of the distribution, being the b^th; b= 0, ±1, ±2…, can be introduced from [3] [10].

$$\emptyset_b = \alpha_b + i\delta_b \quad , \emptyset_b = b, \mp 1, \mp 2 \ldots\ldots$$
$$\alpha_b = E(\cos b\theta)$$
$$\lambda_b = E(\sin b\theta)$$

$$E(\cos b\theta) = \int_0^{\pi}\cos(b\theta)\, f(\theta,\alpha,\lambda) d\theta\,, \qquad (13)$$

$$E(\cos b\theta) = \int_0^{\pi}\cos(b\theta)\frac{v}{2}\frac{\alpha}{\lambda}\left(1+\frac{v\tan\frac{\theta}{2}}{\beta}\right)^{-(\alpha+1)} \sec^2\left(\frac{\theta}{2}\right) d\theta,$$

$$\text{let } \cos(b\theta) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(b\theta)^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(b)^{2n}(\theta)^{2n},$$

$$E(\cos b\theta) = = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(b)^{2n}\int_0^{\pi}(\theta)^{2n}\frac{v}{2}\frac{\alpha}{\lambda}\left(1+\frac{v\tan\frac{\theta}{2}}{\beta}\right)^{-(\alpha+1)} \sec^2\left(\frac{\theta}{2}\right) d\theta,$$

$$E(\cos b\theta) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(b)^{2n}\; E(\theta)^{2n},$$

$$\alpha_1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(1)^{2n}\, E(\theta)^{2n}, \qquad (14)$$

$$\alpha_2 = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(2)^{2n}\, E(\theta)^{2n}, \qquad (15)$$

$$E(\sin b\theta) = \int_0^{\pi} \sin(b\theta)\ \frac{v}{2}\frac{\alpha}{\lambda}\left(1+\frac{v\tan\frac{\theta}{2}}{\lambda}\right)^{-(\alpha+1)} \sec^2\left(\frac{\theta}{2}\right)\,d\theta, \quad (16)$$

$$\text{let} \sin(b\theta) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\,(b\theta)^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\,(b)^{2n+1}(\theta)^{2n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\,(b)^{2n+1}\int_0^{\pi}(\theta)^{2n+1}\,\frac{v}{2}\frac{\alpha}{\lambda}\left(1+\frac{v\tan\frac{\theta}{2}}{\lambda}\right)^{-(\alpha+1)} \sec^2\left(\frac{\theta}{2}\right)\,d\theta$$

$$E(\sin b\theta) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\,(b)^{2n+1}\ E(\theta)^{2n+1}$$

$$\lambda_1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\,(1)^{2n+1}\ E(\theta)^{2n+1}, \quad (17)$$

$$\lambda_2 = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\,(2)^{2n+1}\ E(\theta)^{2n+1}. \quad (18)$$

Here $(\theta^{2n+1})\,SSl\ and\ \ E(\theta^{2n})SSL$ as in (8) respectively.

With r = 2n + 1 and r = 2n.

Simulated data and a quantile function for *S*SL distribution. By means of inversion of the cdf in (8), the quantile function of the *S*SL distribution is obtained as in [11].

$$\theta_{(q)SSL} = 2\tan^{-1}\left(\frac{\lambda}{v}\left[(1-u)^{\frac{-1}{\alpha}} - 1\ \right]\right). \quad (19)$$

The median of the SSL random variable can be gained from (20) by setting $q = \frac{1}{2}$ as

$$\theta_{med} = 2\tan^{-1}\left(\frac{\lambda}{v}\left[\left(1-\frac{1}{2}\right)^{\frac{-1}{\alpha}} - 1\ \right]\right). \quad (20)$$

By replacing q with u a random variable that follows *S*SL distribution can be simulated as:

$$\theta_{SSL} = 2\tan^{-1}\left(\frac{\lambda}{v}\left[(1-u)^{\frac{-1}{\alpha}} - 1\ \right]\right), \quad (21)$$

where u is an interval based uniform random number (0,1).

4 ESTIMATION METHODS

4.1 Maximum Likelihood Method

It is a method most often used for estimation because it possesses a set of characteristics that distinguish it from others. These features include consistency, which is mostly present, stability, and unbiasedness [12] - [14].

$$f(\theta,\alpha,\lambda) = \frac{v}{2}\,\frac{\alpha}{\lambda}\left(1+\frac{v\tan\frac{\theta}{2}}{\lambda}\right)^{-(\alpha+1)} sec^2\left(\frac{\theta}{2}\right),$$

$$Lf(\theta,\alpha,\lambda) = \prod_{i=1}^{n}\left[\frac{v}{2}\frac{\alpha}{\lambda}\left(1+\frac{v\tan\frac{\theta_i}{2}}{\lambda}\right)^{-(\alpha+1)} sec^2\left(\frac{\theta_i}{2}\right)\right],$$

$$= \left(\frac{v}{2}\right)^n\left(\frac{\alpha}{\lambda}\right)^n \prod_{i=1}^{n}\left(1+\frac{v\tan\frac{\theta_i}{2}}{\lambda}\right)^{-(\alpha+1)} \prod_{i=1}^{n} sec^2\left(\frac{\theta_i}{2}\right), \quad (22)$$

$$Ln\,L = n\,\mathrm{Ln}\,v + n\,Ln\,\alpha - n\,Ln2 - n\,Ln\,\lambda - (\alpha + 1)\sum_{i=1}^{n} Ln\left(1+\frac{v\tan\frac{\theta_i}{2}}{\lambda}\right) + 2\sum_{i=1}^{n} Ln\sec\left(\frac{\theta_i}{2}\right),$$

$$\frac{\partial\,Ln\,L(\alpha,\lambda)}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} Ln\left(1+\frac{v\tan\frac{\theta_i}{2}}{\lambda}\right) = 0. \quad (23)$$

The partial derivative with respect to the parameter λ.

$$\frac{\partial Ln\,L(\alpha,\lambda)}{\partial\lambda} = -\frac{n}{\lambda} + (\alpha+1)\sum_{i=1}^{n}\frac{\frac{v\tan\frac{\theta_i}{2}}{\lambda^2}}{1+\frac{v\tan\frac{\theta_i}{2}}{\lambda}} = 0. \quad (24)$$

4.2 Cramer–von Mises Method

Such a minimum distance estimator constructed from the difference between the theoretical and empirical cumulative distribution functions is realizable as follows: By taking partial derivatives with respect to the parameters and setting the results equal to zero, we find the parameter estimates [15], [16].

$$k(\alpha,\lambda) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(\theta,\alpha,\lambda) - \frac{2i-1}{2n}\right]^2. \quad (25)$$

$$k(\alpha,\lambda) = \frac{1}{12\mathrm{n}} + \sum_{i=1}^{n}\left[1-\left(1+\frac{v\tan\frac{\theta}{2}}{\lambda}\right)^{-\alpha} - \frac{2i-1}{2n}\right]^2,$$

$$\frac{\partial\,k}{\partial\,\alpha} = 2\sum_{i=1}^{n}\left[F(\theta,\alpha,\lambda) - \frac{2i-1}{2n}\right]\ \frac{\partial F(\theta,\alpha,\lambda)}{\partial\alpha},$$

$$\frac{\partial k}{\partial\alpha} = 2\sum_{i=1}^{n}\left[1-\left(1+\frac{v\tan\frac{\theta}{2}}{\lambda}\right)^{-\alpha} - \frac{2i-1}{2n}\right].\left(1+\frac{v\tan\frac{\theta}{2}}{\lambda}\right)^{-\alpha} ln\left(1+\frac{v\tan\frac{\theta}{2}}{\lambda}\right) = 0. \quad (26)$$

$$\frac{\partial\,k}{\partial\,\lambda} = 2\sum_{i=1}^{n}\left[F(\theta,\alpha,\lambda) - \frac{2i-1}{2n}\right]\ \frac{\partial F(\theta,\alpha,\lambda)}{\partial\lambda},$$

$$\frac{\partial k}{\partial\lambda} = 2\sum_{i=1}^{n}\left[1-\left(1+\frac{v\tan\frac{\theta}{2}}{\lambda}\right)^{-\alpha} - \frac{2i-1}{2n}\right].\,\alpha\left(1+\frac{v\tan\frac{\theta}{2}}{\lambda}\right)^{-\alpha-1}\ \frac{v\tan\frac{\theta}{2}}{\lambda^2} = 0. \quad (27)$$

5 SIMULATION STUDY

Empirical behaviors of Maximum likelihood and Cramer–von Mises of the SSL parameters had to be detected at various sample sizes (10, 25, 50, 75, and 100) through simulation studies via R. The SSL value is simulated based on (21) by plugging in the parameters' default values shown in the table, where λ=0.5; α=1; ν=1. For the experiment, K=1000. The tests are compared and determined through the use of the mean square error criterion.

$$\text{MSE}(\Omega) = \frac{1}{1000}\sum_{i=1}^{1000}\left(\widehat{\Omega}_i - \Omega\right)^2. \qquad (28)$$

Accordingly, the estimates of the parameters of the semicircular stereographic Lomax distribution resulting from the simulation study using the (MLE) and (CVM) methods are shown in Table 1, and the mean squared error (MSE) values of these estimates are shown in Table 2.

Table 1: The estimated parameter values at the assumed values (α = 1, λ = 0.5).

n	Method	α_est.	λ_est.
10	MLE	1.21577	0.63889
	CVM	1.11423	0.60357
25	MLE	1.12402	0.60136
	CVM	1.10399	0.59211
50	MLE	1.10891	0.59126
	CVM	1.10164	0.58811
75	MLE	1.07310	0.56215
	CVM	1.08787	0.57085
100	MLE	1.05928	0.54983
	CVM	1.08009	0.56395

Table 2: The MSE values of the parameters at the assumed values (α = 1, λ = 0.5).

n	Method	α_MSE	λ_MSE.
10	MLE	0.29276	0.13052
	CVM	0.26990	0.13643
25	MLE	0.15802	0.09489
	CVM	0.18548	0.10513
50	MLE	0.11017	0.06915
	CVM	0.14159	0.08352
75	MLE	0.07047	0.04753
	CVM	0.11346	0.06440
100	MLE	0.05437	0.03830
	CVM	0.08901	0.05245

The results of Tables 1 and 2 indicate that the Maximum Likelihood Estimation (MLE) method yields better performance in estimating the parameters (α, λ) compared to the Cramer–von Mises (CVM) method. There are smaller MSEs across smaller to larger sample sizes; hence, this conclusion. The larger the sample size, the smaller the MSE values, which in turn show increasing estimation accuracy; this property is known as consistency.

6 CONCLUSIONS

The Stereographic Semicircular Lomax Distribution (SSL) was introduced through the application of the Inverse Stereographic Projection method. The main contribution of this study is to extend the classical Lomax model to a semicircular framework, which has not been considered in previous studies. The proposed distribution is flexible and capable of handling the varied kinds of semicircular data observed in several fields. The basic statistical properties, including the probability density function and cumulative distribution function, survival function, hazard rate, along with others.

Simulations indicated that the Maximum Likelihood Estimation technique is superior for deriving parameter estimates compared to the Cramer–von Mises approach, particularly as the sample size expands. This conclusion was further substantiated by The MSE values of the MLE estimators are lower compared to the Cramer Von Mises estimators, especially with increasing sample size. Although the study is based on simulated data, the results demonstrate the effectiveness of the proposed distribution in accurately reflecting the statistical behavior of semicircular data. it also confirms its suitability and ability to represent semicircular data.

7 RECOMMENDATIONS

The study therefore suggests that MLE should be central to parameter estimation for any SSL distribution on account of its perceived increased adeptness. Due to the very general capability to model any semi-circular data, the proposed distribution finds application in many areas. The focal points of the coming researchers should be as follows: the complete comparison of the performance of the alternative estimators such as Least Squares or Bayesian Estimation against MLE and CVM methods. It is further strongly recommended to try this distribution on empirical data from practical applications to critically test its practical usefulness.

REFERENCES

[1] B. J. Ahn and H. M. Kim, “A new family of semicircular models: The semicircular Laplace distributions,” Communications for Statistical Applications and Methods (CSAM), vol. 15, no. 5, pp. 775-781, 2008.

[2] S. V. S. Girija, A. D. Rao, and P. Yedlapalli, “New circular model induced by inverse stereographic projection on double exponential model: Application to birds migration data,” Journal of Applied Mathematics, Statistics and Informatics, vol. 10, no. 1, pp. 5-17, 2014.

[3] P. Yedlapalli, K. L. N. Sastry, and A. D. Rao, “On stereographic semicircular quasi Lindley distribution,” Journal of New Results in Science, vol. 8, no. 1, pp. 6-13, 2019.

[4] A. Iftikhar, A. Ali, and M. Hanif, “Half circular modified Burr-III distribution: Application with different estimation methods,” PLOS ONE, vol. 17, no. 5, e0261901, 2022.

[5] N. A. Oleiwi, E. F. Muhi, and N. A. Al-Khairullah, “A new family of semicircular generalized Rayleigh distribution,” in AIP Conference Proceedings, vol. 3229, no. 1, p. 080026, Nov. 2024.

[6] P. Yedlapalli, S. Girija, A. D. Rao, and K. Sastry, “A new family of semicircular and circular arc tan-exponential type distributions,” Thai Journal of Mathematics, vol. 18, no. 2, pp. 775-781, 2020.

[7] P. Yedlapalli, A. Radhika, S. Girija, and A. D. Rao, “On trigonometric moments of the stereographic semicircular gamma distribution,” European Journal of Pure and Applied Mathematics, vol. 10, no. 5, pp. 1124-1134, 2017.

[8] A. J. Lemonte and G. M. Cordeiro, “An extended Lomax distribution,” Statistics, vol. 47, no. 4, pp. 800-816, 2013.

[9] R. A. Fisher, “On the mathematical foundations of theoretical statistics,” Philosophical Transactions of the Royal Society of London. Series A, vol. 222, no. 594-604, pp. 309-368, 1922.

[10] S. H. Abid, H. N. Al-Noor, and N. A. Oleiwi, “Kumaraswamy semicircular exponentiated Weibull distribution: Properties and simulation,” 2023.

[11] S. H. Abid, N. H. Al-Noor, and M. A. A. Boshi, “On the generalized inverse Weibull distribution,” AIP Conference Proceedings, vol. 2086, p. 030002, 2019.

[12] K. K. Das and T. D. Roy, “Applicability of length biased weighted generalized Rayleigh distribution,” Advances in Applied Science Research, vol. 2, no. 4, pp. 320-327, 2011.

[13] N. Al Dhumad, A. Issa, and A. Abbas, “A new approach for moments estimator and percentages method for estimating the parameters of truncated exponential Marshall-Olkin exponential distribution,” IET Conference Proceedings, vol. 2024, no. 34, pp. 533-537, Mar. 2025, doi: 10.1049/icp.2025.0139.

[14] S. S. Akram and A. H. Yousif, “Bayesian estimation of reliability for multicomponent stress-strength model based on Topp-Leone distribution,” WJPS, vol. 1, no. 3, pp. 90-104, Dec. 2022, doi: 10.31185/wjps.70.

[15] H. Al-Mofleh, A. Z. Afify, and N. A. Ibrahim, “A new extended two-parameter distribution: Properties, estimation methods, and applications in medicine and geology,” Mathematics, vol. 8, no. 9, p. 1578, 2020.

[16] P. D. M. Macdonald, “Comments and queries: Comment on ‘An estimation procedure for mixtures of distributions’ by Choi and Bulgren,” Journal of the Royal Statistical Society: Series B (Methodological), vol. 33, no. 2, pp. 326-329, 1971.