

# Reliability Estimation of the Alpha Power Fréchet Distribution Using MLE and GA

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**Abstract:** Studying failure data requires a flexible distribution to interpret the probabilistic behavior of these data. Therefore, the reliability function of the Alpha–Power Fréchet (APF) distribution was used to study the failure data. The Alpha power transformation was applied to the classical Fréchet distribution to develop a more flexible model capable of handling heterogeneous datasets. The probability density function, cumulative distribution function, and the corresponding reliability function for the APF distribution were derived. The maximum likelihood method and the genetic algorithm were used to estimate the reliability function of the Alpha–Power Fréchet distribution. A simulation study was conducted with various sample sizes (ranging from n=15 to n=500) and multiple different parameter combinations to evaluate the performance of both estimators. The comparison was based on the Mean Squared Error (MSE) criterion. The results revealed that the maximum likelihood method (MLE) showed a decrease in MSE values directly proportional to the increase in sample size, confirming its consistency and superior accuracy for large samples. Meanwhile, the genetic algorithm (GA) showed a robust performance that outperformed the maximum likelihood method for small and medium samples, making it a viable alternative in such cases, despite a noted tendency to slightly overestimate the reliability value.

## 1 INTRODUCTION

The Fréchet distribution is one of the most important distributions in the exponential family, and it can be used to describe the probabilistic behavior of data containing outliers or extreme values. This distribution is often used in the fields of hydrology, meteorology, finance, and engineering to characterize rare events. It is known as the inverse of the Weibull distribution, and is also known as the extreme value distribution of the second kind. It is used to model the maximum (or minimum) distribution of a large set of identically distributed independent random variables. The probability density function of the Fréchet distribution [1] is

$$f(x, \beta, \lambda) = \frac{\lambda}{\beta} \left(\frac{x}{\beta}\right)^{-(\lambda+1)} e^{-\left(\frac{x}{\beta}\right)^{-\lambda}}, \quad x > 0. \quad (1)$$

The distribution function is

$$F(x, \beta, \lambda) = e^{-\left(\frac{x}{\beta}\right)^{-\lambda}}. \quad (2)$$

The problem of the study was to search for probability distributions that can describe the random behavior of failure data of devices and equipment, which includes heterogeneous data, by finding flexible distributions that are found by searching for ways to expand traditional distributions. The problem of the study was also to search for accurate estimation methods to estimate the reliability function of expanded distributions in light of the development of computer programming languages such as the Python programming language. This study aims to compare two types of estimation methods: parametric estimation methods, represented by maximum likelihood, and artificial intelligence methods.

## 2 ALPHA FRÉCHET POWER DISTRIBUTION

The alpha Fréchet power distribution was proposed by finding an expansion of the Fréchet distribution and applying the formula below [2].

$$f_{APT}(x) = \frac{\log(\alpha)}{\alpha - 1} f(x) \alpha^{F(x)}, \alpha > 0, \alpha \neq 1. \quad (3)$$

If  $f(x)$  represents the density function of the distribution to be expanded,  $F(x)$  represents the cumulative function of the distribution, then by substituting (1) and (2) into (3), the probability density function of the Fréchet alpha power distribution is obtained, as shown in the equation:

$$f_{APT}(x) = \frac{\log(\alpha) \lambda}{\alpha - 1} \frac{1}{\beta} \left(\frac{x}{\beta}\right)^{-(\lambda+1)} e^{-\left(\frac{x}{\beta}\right)^{-\lambda}} \alpha e^{-\left(\frac{x}{\beta}\right)^{-\lambda}}, \quad (4)$$

$$\alpha > 0, \alpha \neq 1.$$

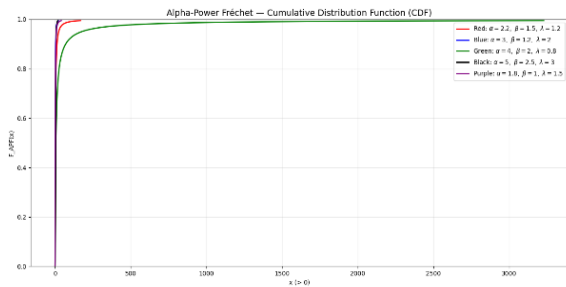


Figure 1: The figure represents the probability density function of the Alpha Power Fréchet distribution.

To obtain the cumulative function of the proposed distribution, we can apply the expansion equation for the cumulative function with the power of alpha (5):

$$F_{APT}(x) = \frac{\alpha^{F(x)} - 1}{\alpha - 1}, \alpha > 0, \alpha \neq 1. \quad (5)$$

After applying the expansion and using the cumulative function of the original distribution, the cumulative function of the proposed distribution is obtained.

$$F_{APT}(x) = \frac{\alpha e^{-\left(\frac{x}{\beta}\right)^{-\lambda}} - 1}{\alpha - 1}, \alpha > 0, \alpha \neq 1. \quad (6)$$

The reliability function for the alpha power distribution is [3].

$$R_{APT}(x) = 1 - F_{APT}(x).$$

$$R_{APT}(x) = 1 - \frac{\alpha e^{-\left(\frac{x}{\beta}\right)^{-\lambda}} - 1}{\alpha - 1}. \quad (7)$$

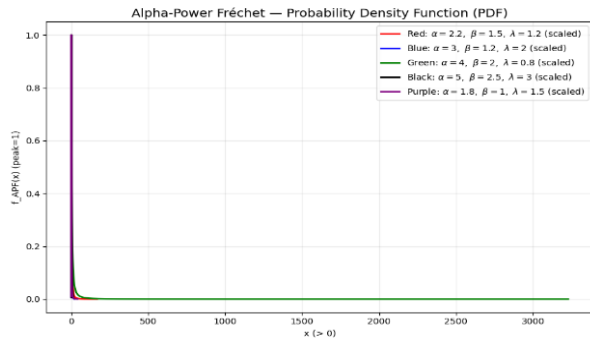


Figure 2: The figure represents the cumulative function of the Alpha Power Fréchet distribution.

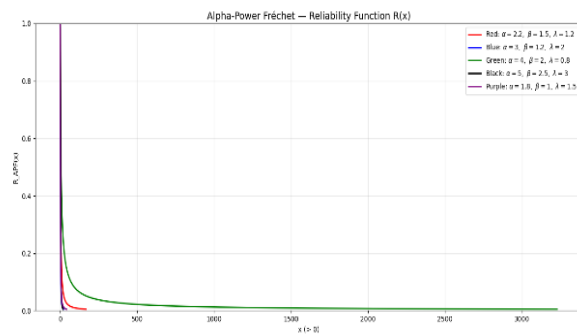


Figure 3: The figure represents the Reliabilityfunction of the Alpha Power Fréchet distribution.

### 3 ESTIMATING THE FRÉCHET ALPHA POWER DISTRIBUTION

The parameters and the reliability function of the Fréchet alpha power distribution were estimated using two methods: the maximum likelihood method and the genetic algorithm method, where the function was taken.

#### 3.1 Estimation by the Maximum Likelihood Method

This method is one of the most common and widely used estimation methods among researchers, as it produces estimators that possess the characteristics of a good estimator, such as unbiasedness, stability [4], and consistency. This method estimates the parameters by maximizing the logarithm of the maximum likelihood function.

The equation below represents the logarithm of the maximum likelihood function for the Fréchet alpha power distribution [5], [6].

$$L(\theta) = \prod_{i=1}^n f(x_i; \theta), \quad (8)$$

$$L(\theta) = \prod_{i=1}^n \left( \frac{\ln(\alpha) \lambda}{\alpha - 1} \left( \frac{x_i}{\beta} \right)^{-(\lambda+1)} e^{-\left(\frac{x_i}{\beta}\right)^{-\lambda}} \alpha e^{-\left(\frac{x_i}{\beta}\right)^{-\lambda}} \right), \quad (9)$$

$$L(\theta) = \left( \frac{\ln(\alpha)}{\alpha - 1} \right)^n \left( \frac{\lambda}{\beta} \right)^n \cdot \left( \sum_{i=1}^n \left( \frac{x_i}{\beta} \right)^{-(\lambda+1)} e^{-\sum_{i=1}^n \left(\frac{x_i}{\beta}\right)^{-\lambda}} \alpha e^{-\sum_{i=1}^n \left(\frac{x_i}{\beta}\right)^{-\lambda}} \right). \quad (10)$$

Taking the natural logarithm of the function factorial

$$\begin{aligned} \log(L(\theta)) &= \\ &= n \ln(\ln(\alpha)) - n * \ln(\alpha - 1) + n \ln(\lambda) - n \\ &\quad * \ln(\beta) - (\lambda + 1) \ln \left( \sum_{i=1}^n \frac{x_i}{\beta} \right) \\ &\quad + \sum_{i=1}^n \left( \frac{x_i}{\beta} \right)^{-\lambda} \\ &\quad + \ln(\alpha) e^{-\left(\sum_{i=1}^n \left(\frac{x_i}{\beta}\right)^{-\lambda}\right)}. \end{aligned} \quad (11)$$

To obtain the estimators of the parameters of the probability density function for the alpha-force distribution  $(\alpha, \beta, \lambda)$ , the partial derivative of the logarithm of the probability function is taken as follows [7]:

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \frac{n - n\alpha - \sum e^{-\left(\frac{x_i}{\beta}\right)^{-\lambda}}}{\alpha}, \quad (12)$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \beta} &= \frac{\lambda}{\beta} \left( \frac{x_i}{\beta} \right)^{-\lambda} + \frac{\lambda \ln(\alpha)}{\beta} \sum \left( \left( \frac{x_i}{\beta} \right)^{(-\lambda)} e^{-\left(\frac{x_i}{\beta}\right)^{-\lambda}} \right) \\ &\quad - \frac{n(\beta - \lambda - 1)}{\beta}, \end{aligned} \quad (13)$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \lambda} &= \\ &= n - \left( \frac{x_i}{\beta} \right)^{-\lambda} \ln \left( \frac{x_i}{\beta} \right) - \sum \left( \ln \left( \frac{x_i}{\beta} \right) \right) \\ &\quad - \ln(\alpha) \sum \left( \left( \frac{x_i}{\beta} \right)^{-\lambda} \ln \left( \frac{x_i}{\beta} \right) e^{-\left(\frac{x_i}{\beta}\right)^{-\lambda}} \right). \end{aligned} \quad (14)$$

### 3.2 Estimation Using the Genetic Algorithm

The genetic algorithm is described as a heuristic method that simulates the idea of adapting to complex data and equations [8]. Its philosophy is based on the

fact that, instead of relying on a single solution, a community of hypotheses is built, and these mechanisms compete through selection [9], mating, and mutation. With this approach, randomness is systematically used not only to break the stability of the results, but also to open new paths in the solution space and avoid falling into weak local optimization. The success of the GA also depends on the quality of the coding and the suitability of the fitness function to the statistical objective [10]. The problem is formulated by estimating the parameters of the APF distribution, i.e.,  $\theta = (\alpha, \beta, \lambda)$ , through a positive simulation sample  $x_1, x_n$ . With the parameters  $\alpha > 1$ ,  $\beta > 0$ , and  $\lambda > 0$ , we measure the quality of the parameters  $(\alpha, \beta, \lambda)$  using a criterion, which is the logarithm of the likelihood function for the data under the APF model, where the fitness function (Fitness =  $-\log$ -likelihood), with the goal of minimizing the cost. Initial values for the parameters  $(\alpha, \beta, \lambda)$  are generated using a hybrid method that considers diversity and objectivity in the form of seeds. We place at least two widely spaced seeds, such as  $(\alpha=2, \beta=2, \lambda=2)$  and  $(\alpha=12, \beta=4, \lambda=3)$  [10], to cover different gates in the solution space. An initial population is then generated around these individuals, adding small differences for each parameter while clipping within the bounds. The goal is to start the search from known but diverse points, ensuring exploratory power in the first generations. All the parameters that have been taken go through the testing stage using the fitness function taken from the logarithm of the factorial of the probability function of the proposed distribution, as in (11) above.

$$\begin{aligned} \log(L(\theta)) &= n * \ln(\ln(\alpha)) - n * \ln(\alpha - 1) + n * \\ &\quad \ln(\lambda) - n * \ln(\beta) - (\lambda + 1) * \ln \left( \sum_{i=1}^n \frac{x_i}{\beta} \right) + \\ &\quad \sum_{i=1}^n \left( \frac{x_i}{\beta} \right)^{-\lambda} + \ln(\alpha) * e^{-\left(\sum_{i=1}^n \left(\frac{x_i}{\beta}\right)^{-\lambda}\right)}. \end{aligned}$$

The above equation is used to find the fitness function.

$$Fitness(\alpha, \beta, \lambda) = - \sum_{i=1}^n \log f_{APF}(x_i; \alpha, \beta, \lambda). \quad (15)$$

Next, we move to the selection step, where we use tournament selection ( $k = 3$ ) to select the best parents by taking three random hypotheses and choosing the values with the highest fit. We repeat the process twice to select two parents. We apply simple elitism by transferring the best one or two types of individuals as they are to the next generation. Next comes the crossover, which takes two forms. The first is a single cut on the vector  $[\alpha, \beta, \lambda]$  by swapping the positions of the parameters after the cut that fall on

the parents, for example, the right of the first with the left of the second and the left of the first with the right of the second. The second type of crossover is linear crossover, which is done by constructing the two offspring as a linear mixture of the parents with a mixing coefficient  $r \sim U(0,1)$  according to the following two equations [11], [12]:

$$C_1 = r p_1 + (1 - r)p_2. \tag{16}$$

$$C_2 = r p_2 + (1 - r)p_1. \tag{17}$$

Where:

- C1, C2: children,
- P1, P2: parents
- r: mixing coefficient.

This type maintains continuity in the parameter space and allows for smooth jumps between solutions. Next, we move to the most important section, which adds the concept of randomness to the genetic algorithm: mutation. A random value is added to each gene by a small amount, for example, 0.001, to move from broad exploration to fine-tuning. After mating and mutation, we move to the repair and constrain section, where we return  $\alpha$ ,  $\beta$ , and  $\lambda$  to the allowed range. We then examine which numerical states ultimately lead to the stopping criterion, which could be a maximum number of generations (such as 100 generations), or a random error difference between the actual and estimated values, which is small enough to allow for the best-known solution at the stopping moment. The estimation is evaluated according to the criteria used in this research, the mean error standard (MSE) for reliability, using Monte Carlo iterations [9], [10], [11].

## 4 SIMULATION

The simulation in this research presents the experimental methodology adopted to compare the accuracy and reliability of the Alpha Power Fréchet distribution parameter estimation methods at (n= 15, 25, 35, 50, 75, 100, 500) observations, each segment representing a fixed time period used to calculate the number of operating units at the beginning of the period and the number of errors or failures (faults) recorded during it. Based on this information, the reliability function was estimated. All the above operations and default parameter values were included in Table 1 to calculate the potential and genetic algorithm [12]. The objective of this section is to evaluate the performance of each method using the mean square error (MSE) criterion via Monte Carlo simulation.

Table 1: Default values for the parameters ( $\alpha$ ,  $\beta$ ,  $\lambda$ ) for the (Alpha Power Fréchet) distribution.

$\alpha$	2	4	12	25	3
$\beta$	2	1	4	10	0.2
$\lambda$	2	6	3	1	5

The Invers method was used to obtain the tracking variables of the Alpha Power Fréchet distribution according to the following formula.

$$X = \beta * \left( - \ln \left( \frac{\ln(R*\alpha - R + 1)}{\ln(\alpha)} \right) \right)^{-\frac{1}{\lambda}}. \tag{18}$$

Where  $\alpha$  is the location parameter,  $\beta$  is the scaling parameter, and  $\lambda$  is the shape parameter. The simulation was repeated 2000 times.

Table 2: Results of simulation of the maximum likelihood method for the reliability of the Alpha Power Fréchet distribution for default parameter.

values at ( $\alpha=2$ ; $\beta=2$ ; $\lambda=2$ )				
$(\alpha=2; \beta=2; \lambda=2)$				
	MLA		GA	
n	Reliability	MSE	Reliability	MSE
15	0.53116	0.00320	0.2633	0.0795
25	0.52787	0.00268	0.2601	0.0742
35	0.52269	0.00202	0.2589	0.0721
50	0.52095	0.00156	0.2593	0.0708
75	0.51819	0.00092	0.2609	0.0708
100	0.51482	0.00077	0.2594	0.0693
500	0.50656	0.00014	0.2611	0.0689

As shown in Table 2, which displays the reliability values and their Mean Squared Error (MSE) for both the maximum likelihood method and the genetic algorithm under the parameter set ( $\alpha=2$ ,  $\beta=2$ ,  $\lambda=2$ ), it is observed that the MSE for the maximum likelihood estimator is lower than that of the genetic algorithm for all sample sizes.

Table 3: Results of simulation of the maximum likelihood method for the reliability of the Alpha Power Fréchet distribution for default parameter.

values at ( $\alpha=4$ ; $\beta=1$ ; $\lambda=6$ )				
$(\alpha = 4; \beta = 1; \lambda = 6)$				
	MLA		GA	
n	Reliability	MSE	Reliability	MSE
15	0.54027	0.00434	0.58601	0.34355
25	0.53034	0.00308	0.59008	0.34823
35	0.52443	0.00239	0.59142	0.34979
50	0.51956	0.00136	0.59210	0.35059
75	0.51705	0.00077	0.59248	0.35104
100	0.51377	0.00063	0.59261	0.35118
500	0.50540	0.00009	0.59278	0.35139

As shown in Table 3, which displays the reliability values and their Mean Squared Error (MSE) for both the maximum likelihood method and the genetic algorithm under the parameter set ( $\alpha=4, \beta=1, \lambda=6$ ), it is noted that the MSE for the maximum likelihood estimator is substantially lower than that of the genetic algorithm across all sample sizes.

Table 4: Results of simulation of the maximum likelihood method for the reliability of the Alpha Power Fréchet distribution for default parameter.

values at ( $\alpha=12; \beta=4; \lambda=3$ )				
( $\alpha = 12; \beta = 4; \lambda = 3$ )				
	MLA		GA	
n	Reliability	MSE	Reliability	MSE
15	0.50928	0.00611	0.2566	0.0763
25	0.51102	0.00372	0.2586	0.0736
35	0.51291	0.00269	0.2621	0.0737
50	0.51379	0.00201	0.2592	0.0712
75	0.51467	0.00138	0.2607	0.0706
100	0.50928	0.00611	0.2566	0.0763
500	0.51102	0.00372	0.2586	0.0736

As illustrated in Table 4, which compares the reliability estimates and their Mean Squared Error (MSE) for the maximum likelihood and genetic algorithm methods using the parameters ( $\alpha=12, \beta=4, \lambda=3$ ), it is evident that the MSE of the maximum likelihood estimator is consistently lower than that of the genetic algorithm across all sample sizes.

Table 5: Results of simulation of the maximum likelihood method for the reliability of the Alpha Power Fréchet distribution for default parameter.

values at ( $\alpha=25; \beta=10; \lambda=1$ )				
( $\alpha = 25; \beta = 10; \lambda = 1$ )				
	MLA		GA	
n	Reliability	MSE	Reliability	MSE
15	0.46535	0.00385	0.82066	0.67429
25	0.47263	0.00223	0.83063	0.69012
35	0.47636	0.00143	0.83370	0.69512
50	0.47978	0.00118	0.83542	0.69794
75	0.48469	0.00090	0.83675	0.70015
100	0.48600	0.00061	0.83709	0.70071
500	0.49409	0.00011	0.83762	0.70161

The results presented in Table 5, which details the reliability estimates and their corresponding Mean Squared Error (MSE) for the maximum likelihood and genetic algorithm methods under the parameter configuration ( $\alpha=25, \beta=10, \lambda=1$ ), demonstrate that the MSE for the maximum likelihood estimator is consistently lower than that of the genetic algorithm for every sample size examined.

Table 6: Results of simulation of the maximum likelihood method for the reliability of the Alpha Power Fréchet distribution for default parameter.

values at ( $\alpha=3; \beta=0.2; \lambda=5$ )				
( $\alpha = 3; \beta = 0.2; \lambda = 5$ )				
	MLA		GA	
n	Reliability	MSE	Reliability	MSE
15	0.53845	0.00467	0.63422	0.40342
25	0.53362	0.00404	0.64661	0.41846
35	0.52127	0.00179	0.65139	0.42444
50	0.52134	0.00159	0.65472	0.42869
75	0.51747	0.00082	0.65634	0.43079
100	0.51372	0.00064	0.65690	0.43152
500	0.50633	0.00013	0.65770	0.43257

Table 6 presents the reliability estimates and their corresponding Mean Squared Error (MSE) for the maximum likelihood and genetic algorithm methods, given the parameters ( $\alpha=3, \beta=0.2, \lambda=5$ ). The results indicate that the MSE for the maximum likelihood estimator is lower than that of the genetic algorithm across all examined sample sizes.

## 5 CONCLUSIONS

Based on the comprehensive simulation study, the following key conclusions can be drawn:

- 1) The Maximum Likelihood (MLE) method demonstrates superior performance over the Genetic Algorithm (GA) in estimating the reliability function of the Alpha Power Fréchet distribution. This conclusion is consistently supported by lower Mean Squared Error (MSE) values across all tested sample sizes and parameter sets ( $\alpha, \beta, \lambda$ ).
- 2) The MLE method shows the property of consistency: its estimation error (MSE) decreases systematically as the sample size increases (from  $n=15$  to  $n=500$ ). This indicates that the estimates converge to the true parameter values with more data.
- 3) The reliability values estimated by GA are consistently higher than those obtained via MLE for the same parameters, indicating a slight upward bias in the GA estimates. Nevertheless, GA shows robust performance, particularly at small to medium sample sizes, making it a viable alternative when classical methods like MLE are difficult to implement due to computational or analytical complexities.

For estimating the reliability of the Alpha Power Fréchet distribution, the Maximum Likelihood

method is recommended due to its higher accuracy and consistency. The genetic algorithm serves as a competent complementary tool, especially in challenging scenarios where traditional optimization fails.

## RECOMMENDATIONS

Based on the results, the maximum likelihood (MLE) method is recommended as the primary approach for estimating the reliability function of the Alpha Power Fréchet distribution, particularly for large sample sizes ( $n \geq 50$ ), due to its high accuracy and consistency. The genetic algorithm (GA) is suggested as an alternative estimation method, offering robustness and effectiveness, especially for small or medium sample sizes or when the likelihood function is complex. A hybrid approach can also be developed, combining the advantages of both methods, where the GA is used to determine optimal parameter values that are then employed as initial values for the MLE algorithm to achieve high optimization accuracy. It is further recommended to apply both methods to real data from engineering and machinery fields to validate these results and compare performance in practical applications. Additionally, the study could be expanded to include other performance evaluation metrics, such as bias (BIAS), and the performance of the genetic algorithm can be enhanced by fine-tuning parameters like mutation rate, population size, and breeding method to achieve higher accuracy and lower mean square error (MSE).

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