An Enhanced Fréchet Distribution: Properties, Computational Methods and Applications in Cancer Survival and Material Strength

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Abstract: This study is devoted to the development of the Enhancing Fréchet distribution (EF) through the integration of hybrid Weibull-G (HWG) family with Fréchet distribution. The resulting EF model demonstrates greater flexibility in modelling datasets characterized by heavy tails and pronounced positive skewness. A comprehensive investigation of its statistical properties is conducted. As verified by the graphs of the different functions, which showed a variety of shapes depending on parameter values. Moments, variance, skewness, and kurtosis were also calculated, confirming the presence of heavy tails and strong positive skewness in EF distribution. In addition, the parameters EF distribution was estimated employing different methods, followed by a simulation that showed that the MLE method was the most accurate and least biased compared to the other methods, especially with increasing sample size. On the applied side, the performance of the new distribution was tested on two real data sets where it yielded the lowest values for information criteria and the highest p-values, confirming its high fit to the data.

1 INTRODUCTION

Statistical distributions have long formed the backbone of probability theory and mathematical statistics, providing mathematical frameworks for describing the behavior of random variables. The journey began with simple distributions such as the Bernoulli and binomial distributions, which were sufficient to describe specific phenomena. As data complexity increased in the 20th century, the need arose for more flexible distributions capable of representing diverse properties such as over-variance, asymmetry, and heavy tails.

To overcome these limitations, statisticians began developing extended distribution families by adding new parameters. One of the most important of these methods, introduced in the last two decades, is the T-X method, introduced by researchers Alzaatreh in 2013 [1] as a general framework for generating new distributions, this approach is based on transforming a random variable T through an appropriate generating function, and combining it with X (any random variable) that follows a known distribution. Resulting preserves the fundamental properties of baseline distribution while introducing additional flexibility in modeling diverse data behaviors. Examples of statistical families that have used this method include can see sources ([2] - [11]). Among these families emerged the Hybrid Weibull-G (H.W.G) family with CDF and pdf function defined as follows [12], [13]:

$$F(x) = 1 - e^{-\omega\left[-G(x).\log(1-G(x))\right]^{\delta}} \quad (1)$$

$$f(x) = \frac{\omega\delta\, g(x)\left[\frac{G(x)}{1-G(x)} - \log\left(1-G(x)\right)\right]}{\left[-G(x).\log\left(1-G(x)\right)\right]^{1-\delta}} \times e^{-\omega\left[-G(x).\log(1-G(x))\right]^{\delta}} \quad (2)$$

where $G(x)$ and $g(x)$ are CDF and pdf for any baseline distribution, respectively, with $x \in \mathbb{R}^{+}$ and $\omega, \delta > 0$ are shape parameters for H.W.G.

This study aims to provide a qualitative addition to the field of statistical modeling by enhancing the traditional Fréchet distribution and expanding its scope of practical applications. The study integrates the Fréchet distribution with the hybrid Weibull-G (H.W.G) family. The work will focus on exploring the in-depth statistical properties of EF, encompassing deriving a number of properties of the enhancing distribution. On the applied side, the parameters of EF distribution were estimated using three distinct estimation techniques. Their performance was rigorously assessed through an extensive Monte Carlo simulation study. The procedure demonstrates the feasibility and practical applicability of EF distribution in accurately modeling real data characterized by skewness and heavy tails, it will be applied to two real-world data sets: cancer patient survival data and engineering materials durability data. The performance of the new distribution will be compared with a set of competitive distributions using accuracy criteria.

2 ENHANCING FRÉCHET DISTRIBUTION

Take X be a positive random variable $X \in (0, \infty)$, The functions of Fréchet distribution are defined respectively, by the formula:

$$G(x) = e^{-\alpha x^{-\theta}}, x \in (0, \infty) \tag{3}$$

$$g(x) = \alpha\theta x^{-(\alpha+1)} e^{-\alpha x^{-\theta}} \tag{4}$$

where α, θ respectively are shape parameter and scale parameter.

As previously defined, the CDF of the Enhancing Fréchet (EF) distribution is found by combining Equation 1 with (3) by replace every $G(x)$in equation 1 by in (3) as follows:

Let ω, δ be the shape parameters of the HWG generator, and let α, θ be the scale and shape parameters of the baseline Fréchet distribution, then the CDF function for EF distribution has a form:

$$F_{EF}(x) = 1 - e^{-\omega\left[-e^{-\alpha x^{-\theta}} .\log\left(1-e^{-\alpha x^{-\theta}}\right)\right]^{\delta}} \tag{5}$$

By deriving the above equation, the distribution function pdf is obtained as follows:

$$f_{EF}(x) = \frac{\omega\delta\, \alpha\theta e^{-\alpha x^{-\theta}}\left[\frac{e^{-\alpha x^{-\theta}}}{1-e^{-\alpha x^{-\theta}}} - \log\left(1-e^{-\alpha x^{-\theta}}\right.\right.}{x^{\alpha+1}\left[-e^{-\alpha x^{-\theta}} .\log\left(1-e^{-\alpha x^{-\theta}}\right)\right]^{1-\delta}} \tag{6}$$

$$\times\ e^{-\omega\left[-e^{-\alpha x^{-\theta}} .\log\left(1-e^{-\alpha x^{-\theta}}\right)\right]^{\delta}}$$

The first step in demonstrating the elasticity of a distribution is to plot the CDF and PDF functions with different parameter values, which are shown in Figures 1 and 2.

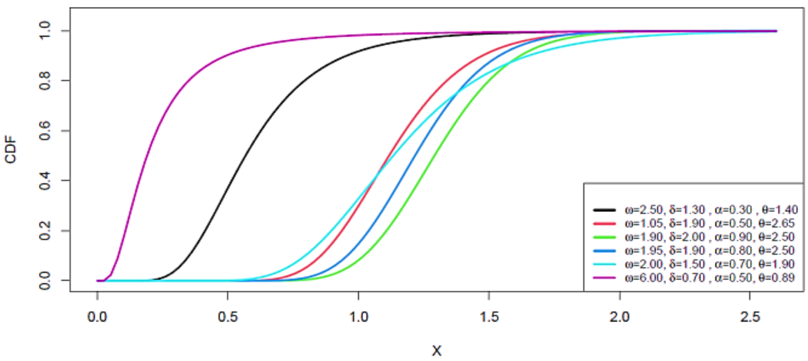

Figure 1: Plot the CDF for EF Distribution.

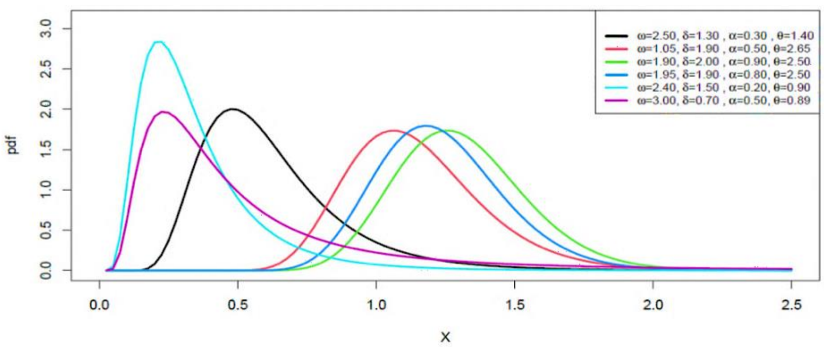

Figure 2: Plot the pdf for EF Distribution.

In Figure 1 notice that all curves start from zero and gradually converge toward 1 as the x value increases. This reflects the fundamental nature of cumulative distribution functions, as the rate of convergence toward 1 varies with different parameters. Some curves converge faster, while others converge more slowly, demonstrating the flexibility of the distribution to represent different data patterns.

Figure 2 shows the behavior of the probability density function (PDF) for the same set of parameters. The curves exhibit a variety of shapes, ranging from heavy-tailed curves with sharp peaks to more elongated curves with less sharp peaks. This diversity of distributional shapes underscores the enhanced capacity of EF to accommodate various data structures, particularly those characterized by extreme values.

Now we find the survival function by form [14], [15]:

$$S_{EF}(x) = 1 - F_{EF}(x)$$

$$S_{EF}(x) = e^{-\omega\left[-e^{-\alpha x^{-\theta}}.\log\left(1-e^{-\alpha x^{-\theta}}\right)\right]^{\delta}} \tag{7}$$

And the hazard function can be finding as [16], [17]:

$$h_{EF}(x) = \frac{f_{EF}(x)}{S_{EF}(x)}$$

$$h_{EF}(x) = \frac{\left[\frac{e^{-\alpha x^{-\theta}}}{1-e^{-\alpha x^{-\theta}}} - \log\left(1-e^{-\alpha x^{-\theta}}\right)\right]}{x^{\alpha+1}\left[-e^{-\alpha x^{-\theta}}.\log\left(1-e^{-\alpha x^{-\theta}}\right)\right]^{1-\delta}} \times \omega\delta\,\alpha\theta e^{-\alpha x^{-\theta}} \tag{8}$$

Plotting the survival function and hazard of an EF distribution is not just a visual representation; it is a powerful analytical tool for understanding when events occur (from the survival function) and how their risks change over time (from the hazard function). This makes it essential for decision-making in sensitive areas that rely on the time-course analysis of data. Figures 3 and 4 represent the plots of these functions.

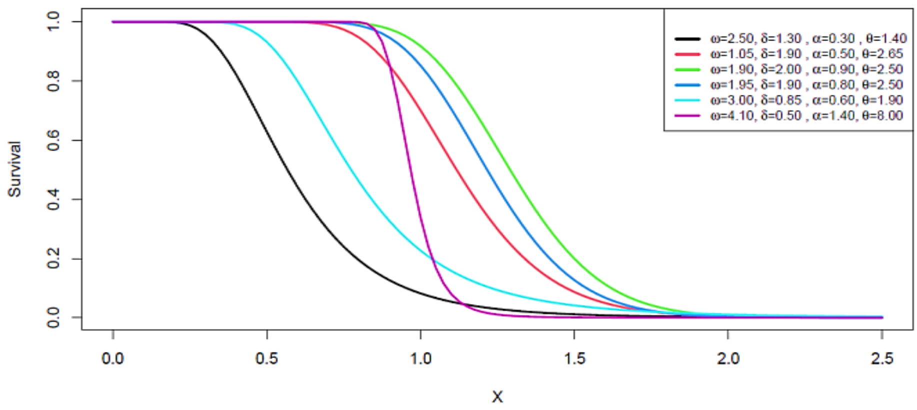

Figure 3: Plot the survival for EF Distribution.

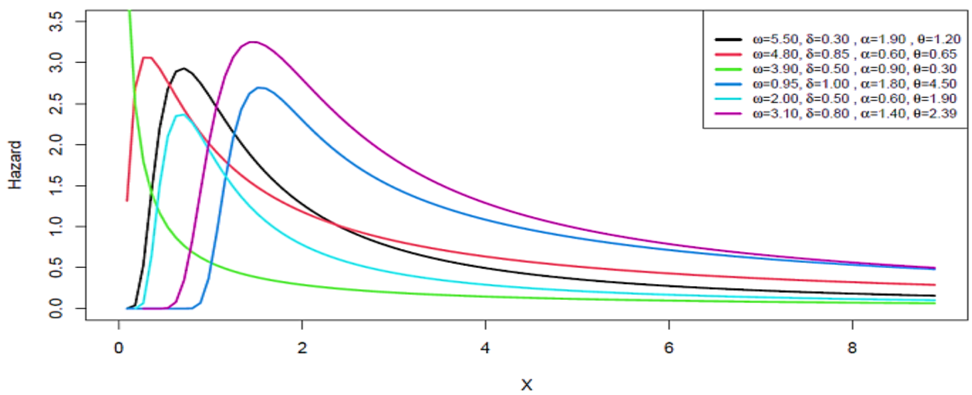

Figure 4: Plot the hazard for EF Distribution.

Table 1: Some special sub models of EF distribution.

Model	Fixed Parameters	$G(x)$	Applications
Classical Fréchet	$\omega = \delta = 1$	$1 - e^{-\alpha x^{-\theta}}$	Modeling extreme values
Hybrid Weibull Fréchet	$\delta = 1$	$1 - e^{-\omega\left[-e^{-\alpha x^{-\theta}}.\log\left(1-e^{-\alpha x^{-\theta}}\right)\right]}$	Survival and reliability analysis
Exponential Asymptotic Limit	$\theta \to \infty^{+}$	$1 - e^{-\omega y e^{-y}}$	Modeling failure rates
Heavy Tail	$\omega = \delta = \alpha = 1$	$1 - e^{\left[e^{-x^{-\theta}}.\log\left(1-e^{-x^{-\theta}}\right)\right]}$	Financial and physical data
Separate representation	$\theta \to \infty^{+}$	$1 - e^{-\omega x^{-\delta}}$	Correct meters and data

Table 1 illustrates some special cases of distribution when parameter values are modified or rounded.

Figure 3 illustrates the probability of values remaining for more than a specific time (x). We note that all curves start from a value of 1 and gradually decrease toward zero, with different rates of decrease depending on the parameter values.

Figure 4 reveals the instantaneous rate of occurrence (such as failure or death) subject to survival until that time. The curves show diverse hazard patterns: curves that start with high risk and then decrease, curves whose risk gradually increases over time, and curves that show a peak at medium hazard before stabilizing. This diversity confirms the ability of the boosted distribution to simulate complex risk behavior in real-world applications.

Some exceptional cases of the distribution can be obtained by substituting values for its parameters in CDF, and pdf distribution, as shown in Table 1 as follows.

3 SOME PROPERTIES FOR ENHANCING FRÉCHET DISTRIBUTION

3.1 CDF and PDF Expansion

Given the control role of CDF and pdf of EF distribution in deriving its statistical properties, and in light of the inherent complexity of these functions, series expansions are employed to facilitate their handling. As binomial, exponential, and logarithm expansions [17], [18]. Accordingly, the CDF expansion for EF distribution expressed in form:

$$F_{EF}(x) = 1 - He^{-\alpha(j+2i\delta)x^{-\theta}} \tag{9}$$

where $H = \sum_{i=j=0}^{\infty} \frac{(-1)^{i+\delta+j}}{i!} d_{i\delta,j}\, \omega^{i}$, $d_{i\delta,j} = j^{-1}\sum_{s=1}^{j} \frac{[s(i\delta+1)-j]}{s+1}$ for $j \geq 0$ and $d_{i\delta,0} = 1$

In the same way to expand pdf to get (10):

$$\begin{aligned} f_{EF}(x) &= Ne^{-\alpha(k+2i\delta+2\delta+z)x^{-\theta}}\, x^{-(\alpha+1)} \\ &\quad - Me^{-\alpha(k+2i\delta+2\delta+1)x^{-\theta}}\, x^{-(\alpha+1)} \end{aligned} \tag{10}$$

where $N = \sum_{i=k=z=0}^{\infty} \frac{(-1)^{i+2\delta+k+z-1}}{i!} d_{i\delta+\delta-1,k}\, \omega^{i+1}\delta\, \alpha\theta$, and $M = \sum_{i=j=0}^{\infty} \frac{(-1)^{i+2\delta+j-1}}{i!} d_{i\delta+\delta,j}\, \omega^{i+1}\delta\, \alpha\theta$, such as $d_{i\delta+\delta-1,k} = k^{-1}\sum_{s=1}^{k} \frac{[s(i\delta+\delta-1+1)-k]}{s+1}$ for $k \geq 0$ and $d_{i\delta+\delta-1,0} = 1$, and $d_{i\delta+\delta,j} = j^{-1}\sum_{l=1}^{j} \frac{[l(i\delta+\delta+1)-j]}{l+1}$ for $j \geq 0$ and $d_{i\delta+\delta,0} = 1$

The $F_{EF}^{t}(x)$ has a form:

$$F_{EF}^{t}(x) = \left(1 - e^{-\omega\left[-e^{-\alpha x^{-\theta}}.\log\left(1-e^{-\alpha x^{-\theta}}\right)\right]^{\delta}}\right)^{t} \tag{11}$$

Also can be expanded to form:

$$F_{EF}^{t}(x) = \Theta e^{-\alpha(u+2l\delta)x^{-\theta}} \tag{12}$$

where $\Theta = \sum_{q=l=0}^{\infty} \frac{(-1)^{q+l+\delta+u}}{i!} \binom{t}{q} d_{l\delta,u}\omega^{l}q^{l}$, and $d_{i\delta,u} = u^{-1}\sum_{s=1}^{u} \frac{[s(l\delta+1)-u]}{s+1}$ for $u \geq 0$ and $d_{l\delta,0} = 1$

3.2 Moments for EF distribution

Moments provide fundamental numerical measures that characterize the shape and structure properties of a probability distribution associated with a random variable. They play a crucial role in describing characteristics. Formally, let X any random variable, the n^{th}moment of a distribution is defined as follows [19],[20]:

$$\mathcal{M}_n = E(x^n) = \int_{-\infty}^{\infty} x^n f(x)dx \tag{13}$$

To obtain the moment function of EF distribution, substitute (10) into (13) as follows:

$$\mathcal{M}_n = N\int_0^\infty e^{-\alpha(k+2i\delta+2\delta+z)x^{-\theta}}\, x^{n-(\alpha+1)}dx$$

$$-M\int_0^\infty e^{-\alpha(k+2i\delta+2\delta+1)x^{-\theta}}\, x^{n-(\alpha+1)}dx$$

$$\mathcal{M}_n = \frac{1}{\theta}\Gamma\left(\frac{\alpha-n}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-n}{\theta}} - M\big(\alpha(k+2im+2m+1)\big)^{\frac{\alpha-n}{\theta}}\right\} \tag{14}$$

From (14) can be get:

$$\mathcal{M}_1 = \frac{1}{\theta}\Gamma\left(\frac{\alpha-1}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-1}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-1}{\theta}}\right\} \tag{15}$$

$$\mathcal{M}_2 = \frac{1}{\theta}\Gamma\left(\frac{\alpha-2}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-2}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-2}{\theta}}\right\} \tag{16}$$

$$\mathcal{M}_3 = \frac{1}{\theta}\Gamma\left(\frac{\alpha-3}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-3}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-3}{\theta}}\right\} \tag{17}$$

$$\mathcal{M}_4 = \frac{1}{\theta}\Gamma\left(\frac{\alpha-4}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-4}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-4}{\theta}}\right\} \tag{18}$$

$$\sigma^2 = \frac{1}{\theta}\Gamma\left(\frac{\alpha-2}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-2}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-2}{\theta}}\right\}$$
$$-\left(\frac{1}{\theta}\Gamma\left(\frac{\alpha-1}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-1}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-1}{\theta}}\right\}\right)^2 \tag{19}$$

$$S = \frac{\frac{1}{\theta}\Gamma\left(\frac{\alpha-3}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-3}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-3}{\theta}}\right\}}{\left(\frac{1}{\theta}\Gamma\left(\frac{\alpha-2}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-2}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-2}{\theta}}\right\}\right)^{\frac{3}{2}}} \tag{20}$$

$$K = \frac{\frac{1}{\theta}\Gamma\left(\frac{\alpha-4}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-4}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-4}{\theta}}\right\}}{\left(\frac{1}{\theta}\Gamma\left(\frac{\alpha-2}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-2}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-2}{\theta}}\right\}\right)^{2}} - 3 \tag{21}$$

Table 2: The moments for the EF distribution with different values of the parameters.

ω	δ	α	θ	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	σ^2	S	K
1.8	1.6	1.4	0.1	0.021554	0.018802	0.016632	0.014887	0.018337	6.451481	42.11197
			0.2	0.021816	0.019215	0.01713	0.015429	0.018739	6.431403	41.79026
		1.6	0.3	0.011337	0.010179	0.009221	0.008418	0.01005	8.979007	81.24768
			0.4	0.011429	0.010331	0.009412	0.008634	0.0102	8.963453	80.89674
	1.9	1.8	0.5	0.002102	0.001949	0.001815	0.001698	0.001945	21.10036	447.065
			0.6	0.002112	0.001967	0.001839	0.001726	0.001962	21.08359	446.1635
		1.9	0.7	0.001431	0.001341	0.001262	0.00119	0.001339	25.68288	661.6519
			0.8	0.003014	0.002819	0.002646	0.002491	0.00281	17.67929	313.5805

So the variance, skewness and kurtosis functions of EF distribution can be obtained as follows [21], [22]:
Table 2 shows the first four moments along with the torsion and kurtosis values for different sets of parameter values.

From Table 2, all moments are positive but very small (less than 1), indicating that the data are concentrated near zero with long tails (typical of extreme value increase distributions, such as Fréchet). The effect of θ is greatest, as increasing (holding other parameters constant), the moments decrease, indicating an increasing concentration of data near zero. The variance is relatively small but increases as the parameters change, reflecting the flexibility of the EF distribution in representing heterogeneous data. All skewness values are positive and significant, indicating strong positive skewness (longer right tails), and larger. θ implies greater skewness, confining increasing asymmetry. The kurtosis values are very high, indicating the presence of heavy tails (i.e., more extreme values than in a normal distribution).

Therefore, EF distribution can be said to have long, heavy tails and strong positive skewness (the data are concentrated on the left, with rare large values).

Moments Generating function for EF distribution. The MGF of the EF distribution can be found using the moments function in (14) and expanding the exponential function to get [23]:

$$M_x(y)_{\text{EF}} = \sum_{s=0}^{\infty} \frac{y^s}{s!}\left[\frac{1}{\theta}\Gamma\left(\frac{\alpha-n}{\theta}\right)\left\{NA_1^{\frac{\alpha-n}{\theta}} - MA_2^{\frac{\alpha-n}{\theta}}\right\}\right] \quad (22)$$

Characteristic function for EF distribution.

The Characteristic function of EF is derived by employing moments in (14) and exponential expansion, yielding the form [24]:

$$C_x(y)_{EF} = Q_x(t) = \sum_{v=0}^{\infty} \frac{(it)^v}{v!}\left[\left[\frac{1}{\theta}\Gamma\left(\frac{\alpha-n}{\theta}\right)\left\{NA_1^{\frac{\alpha-n}{\theta}} - MA_2^{\frac{\alpha-n}{\theta}}\right\}\right]\right] \quad (23)$$

where $A_1 = \alpha(k + 2i\delta + 2\delta + z)$, and $A_2 = \alpha(k + 2im + 2m + 1)$.

3.3 Incomplete Moments for EF distribution

The incomplete moment of any distribution is defined by [19], [25]:

$$I_n = \int_{-\infty}^{y} x^n f(x)dx \quad (24)$$

To obtain the incomplete moment function of EF distribution, substitute (10) into (24) as follows:

$$I_N = \frac{N}{\theta}A_1^{\frac{n-\alpha}{\theta}}\Gamma\left(\frac{\alpha-n}{\theta},\frac{A_1}{y^\theta}\right) - \frac{M}{\theta}A_2^{\frac{n-\alpha}{\theta}}\Gamma\left(\frac{\alpha-n}{\theta},\frac{A_2}{y^\theta}\right) \quad (25)$$

where $y \to \infty$ then $I_N \to \mathcal{M}_n$ (i.e. (25) converges to (13) or the incomplete moments converge to moments).

3.4 Probability Weighted Moments

To calculate the probability weighted moments of EF by applying the equation [26]:

$$\tau_{m,t} = E\left(x^m F_{EF}^t(x)\right) = \int_{-\infty}^{\infty} x^m f_{EF}(x)F_{EF}^t(x)dx$$

By substituting (10) and (12) into above expression, the following formula is obtained:

$$\tau_{m,t} = \int_0^\infty \left(e^{-\alpha(k+2i\delta+2\delta+zu+2l\delta)x^{-\theta}}\, x^{m-(\alpha+1)} - Me^{-\alpha(k+2i\delta+2\delta+1+u+2l\delta)x^{-\theta}}\, x^{m-(\alpha+1)}\right)dx$$

Then, integrate the preceding equation to obtain the final formula:

$$\tau_{m,t} = \frac{\Theta}{\theta}\Gamma\left(\frac{\alpha-m}{\theta}\right)\left(NC_1^{\frac{m-\alpha}{\theta}} - MC_2^{\frac{m-\alpha}{\theta}}\right) \quad (26)$$

where $C_1 = \alpha(k + 2i\delta + 2\delta + zu + 2l\delta)$, and $C_2 = \alpha(k + 2i\delta + 2\delta + 1 + u + 2l\delta)$

3.5 Quantile Function of EF Distribution

The quantile function, also known as the inverse CDF , is a fundamental concept in statistics and probability theory. This function is used to determine the value that a random variable will not exceed with a given probability. Its function $Q(p)$ for a random variable X is defined as follows [27], [28]:

$$Q(p) = F^{-1}(x)$$

Where $F(x)$ for each $p \in (0,1)$. Then the Quantile function of the EF distribution by form:

$$Q(x) = \left[\frac{1}{\alpha}\log\left\{\frac{\pi - W_{-1}(-\delta e^{\pi})}{\pi}\right\}\right]^{-\frac{1}{\theta}}, \pi = \left(-\frac{\log(1-u)}{\omega}\right)^{\frac{1}{\delta}} \quad (27)$$

where $W_{-1}(-\delta e^{\pi})$ is the secondary real branch of the Lambert W function, defined for $-\delta e^{\pi} \in [-e^{-1}, 0)$.

The measurements of skewness and kurtosis based on the Quantile function were defined as follows:

$$S = \frac{Q\left(\frac{6}{8}\right) - 2Q\left(\frac{4}{8}\right) + Q\left(\frac{2}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \quad (28)$$

$$K = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \quad (29)$$

Table 3 presents a random variable values using some parts of Quantile function for various parameter values.

From the above table, it is noted that the large. u, the significant value of the quantile, as expected (the quantile is an increasing function). This demonstrates the flexibility of EF in capturing a wide range of data behaviors, including its capability to model extreme values characterized by long-tails especially at small values of θ.

The Median of the EF distribution can be found by substituting $u = 0.5$ into (27), to get:

$$Q(x) = \left[\frac{1}{\alpha}\log\left\{\frac{\pi - W_{-1}(-\delta e^{\pi})}{\pi}\right\}\right]^{-\frac{1}{\theta}}, \pi = \left(\frac{0.6931}{\omega}\right)^{\frac{1}{\delta}} \quad (30)$$

4 ESTIMATION PARAMETERS

4.1 Maximum Likelihood Estimation - MLE

The technique known as MLE involves maximizing the likelihood function, which expresses the likelihood of witnessing the provided data under the suggested model, to find the values of unknown parameters of a statistical model. Mathematically, the MLE is expressed as [29], [30] if we have a probability distribution $f(x/\vartheta)$ with ϑ being parameters and a sample of data $X = (x_1, x_2, \dots, x_n)$:

$$L(\vartheta, x) = \prod_{i=1}^{n} f_{EF}(x)$$

$$L(\vartheta, x) = \prod_{i=1}^{n} \frac{\omega\delta\,\alpha\theta e^{-\alpha x_i^{-\theta}}\left[\frac{e^{-\alpha x_i^{-\theta}}}{1-e^{-\alpha x_i^{-\theta}}} - \log\left(1-e^{-\alpha x_i^{-\theta}}\right)\right]}{x^{\alpha+1}\left[-e^{-\alpha x_i^{-\theta}}.\log\left(1-e^{-\alpha x_i^{-\theta}}\right)\right]^{1-\delta}} e^{-\omega\left[-e^{-\alpha x_i^{-\theta}}.\log\left(1-e^{-\alpha x_i^{-\theta}}\right)\right]^{\delta}}$$

Compute the log-likelihood in the form:

$$L = n\log\omega + n\log\delta + n\log\theta + n\log\alpha - \alpha\sum_{i=1}^{n} x_i^{-\theta} - (\alpha+1)\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} log\left\{\frac{e^{-\alpha x_i^{-\theta}}}{1-e^{-\alpha x_i^{-\theta}}} - log\left(1-e^{-\alpha x_i^{-\theta}}\right)\right\} - \omega\sum_{i=1}^{n}\left[-e^{-\alpha x_i^{-\theta}}.\log\left(1-e^{-\alpha x_i^{-\theta}}\right)\right]^{\delta} + \sum_{i=1}^{n} log\left\{\left[-e^{-\alpha x_i^{-\theta}}.\log\left(1-e^{-\alpha x_i^{-\theta}}\right)\right]^{1-\delta}\right\} \quad (31)$$

Table 3: Explanation of the quantile function for a specific EF distribution with different parameter values.

	$(\omega, \delta, \alpha, \theta)$				
u	(1.8,1.9,1.4,1.7)	(1.6,1.4,1.4,1.5)	(1.5,1.6,1.5,1.6)	(1.7,1.5,1.8,1.7)	(1.7,1.9,1.7,1)
0.1	1.317960	1.194348	1.332529	1.386355	1.970648
0.2	1.491886	1.407464	1.545657	1.592880	2.439520
0.3	1.631171	1.590061	1.723840	1.764469	2.845417
0.4	1.759989	1.769166	1.894843	1.928387	3.244388
0.5	1.889081	1.959118	2.072376	2.097946	3.666552
0.6	2.027135	2.174631	2.269273	2.285399	4.142399
0.7	2.185507	2.43857	2.504351	2.508552	4.718758
0.8	2.386139	2.800418	2.816750	2.804250	5.494824
0.9	2.694769	3.423055	3.330425	3.289162	6.787325

Table 4: Monte Carlo simulations conducted for EF distribution.

ω =1.6, δ =1.3, α =1.9, θ =1.1											
N	Est.	Ess. Par.	MLE	LSE	WLSE	N	Est.	Ess. Par.	MLE	LSE	WLSE
70	Mean	$\hat{\omega}$	1.4752145	1.8614104	1.65949808	210	Mean	$\hat{\omega}$	1.603106833	1.56756871	1.61401493
		$\hat{\delta}$	1.22835499	1.1135785	1.0718747			$\hat{\delta}$	1.5193209	1.0933770	1.1856953
		$\hat{\alpha}$	2.3646745	2.0241325	2.1300302			$\hat{\alpha}$	2.1132552	2.0918893	2.1095845
		$\hat{\theta}$	1.11962212	1.08169871	1.07847996			$\hat{\theta}$	1.090619255	1.08959176	1.06341091
	MSE	$\hat{\omega}$	0.4345895	0.6392675	0.51416972		MSE	$\hat{\omega}$	0.233121709	0.12152551	0.12865889
		$\hat{\delta}$	2.04734115	0.6896355	0.4155742			$\hat{\delta}$	1.6409932	0.4489576	0.4801535
		$\hat{\alpha}$	0.8640154	0.3044784	0.3516827			$\hat{\alpha}$	0.5356266	0.1887299	0.2365791
		$\hat{\theta}$	0.12778147	0.04918305	0.04569176			$\hat{\theta}$	0.072765749	0.04564394	0.04383683
	RMSE	$\hat{\omega}$	0.6592340	0.7995421	0.71705629		RMSE	$\hat{\omega}$	0.482826789	0.34860509	0.35869053
		$\hat{\delta}$	1.43085330	0.8304430	0.6446505			$\hat{\delta}$	1.2810126	0.6700430	0.6929311
		$\hat{\alpha}$	0.9295243	0.5517956	0.5930284			$\hat{\alpha}$	0.7318651	0.4344305	0.4863940
		$\hat{\theta}$	0.35746534	0.22177252	0.21375631			$\hat{\theta}$	0.269751273	0.21364444	0.20937246
	Bias	$\hat{\omega}$	0.1247855	0.2614104	0.05949808		Bias	$\hat{\omega}$	0.003106833	0.03243129	0.01401493
		$\hat{\delta}$	0.07164501	0.1864215	0.2281253			$\hat{\delta}$	0.2193209	0.2066230	0.1143047
		$\hat{\alpha}$	0.4646745	0.1241325	0.2300302			$\hat{\alpha}$	0.2132552	0.1918893	0.2095845
		$\hat{\theta}$	0.01962212	0.01830129	0.02152004			$\hat{\theta}$	0.009380745	0.01040824	0.03658909
140	Mean	$\hat{\omega}$	1.53952724	1.7250935	1.61235302	280	Mean	$\hat{\omega}$	1.61721954	1.64095786	1.57583611
		$\hat{\delta}$	1.4086616	1.1287	1.1367819			$\hat{\delta}$	1.4972093	1.20968389	1.21383508
		$\hat{\alpha}$	2.2042555	2.0975041	2.2004221			$\hat{\alpha}$	2.0856541	2.1042104	2.0366132
		$\hat{\theta}$	1.09014082	1.06284354	1.02318118			$\hat{\theta}$	1.106398438	1.04535746	1.08562392
	MSE	$\hat{\omega}$	0.25843474	0.3274072	0.18176731		MSE	$\hat{\omega}$	0.31906997	0.18466328	0.08248827
		$\hat{\delta}$	1.6269457	0.4979029	0.4210810			$\hat{\delta}$	2.0465212	0.46596600	0.32846800
		$\hat{\alpha}$	0.6301536	0.2780136	0.3535798			$\hat{\alpha}$	0.4463212	0.2124728	0.1252380
		$\hat{\theta}$	0.06320890	0.06203874	0.03729340			$\hat{\theta}$	0.056805371	0.03206397	0.02236085
	RMSE	$\hat{\omega}$	0.50836477	0.5721951	0.42634178		RMSE	$\hat{\omega}$	0.56486279	0.42972466	0.28720771
		$\hat{\delta}$	1.2755178	0.7056224	0.6489075			$\hat{\delta}$	1.4305667	0.68261702	0.57312128
		$\hat{\alpha}$	0.7938221	0.5272699	0.5946258			$\hat{\alpha}$	0.6680727	0.4609478	0.3538898
		$\hat{\theta}$	0.25141380	0.24907578	0.19311498			$\hat{\theta}$	0.238338775	0.17906414	0.14953546
	Bias	$\hat{\omega}$	0.06047276	0.1250935	0.01235302		Bias	$\hat{\omega}$	0.01721954	0.04095786	0.02416389
		$\hat{\delta}$	0.1086616	0.1713000	0.1632181			$\hat{\delta}$	0.1972093	0.09031611	0.08616492
		$\hat{\alpha}$	0.3042555	0.1975041	0.3004221			$\hat{\alpha}$	0.1856541	0.2042104	0.1366132
		$\hat{\theta}$	0.00985918	0.03715646	0.07681882			$\hat{\theta}$	0.006398438	0.05464254	0.01437608

4.2 Ordinary Least Squares - LSE

The equation for this method is given by the form [31], [12]:

$$\varphi(\vartheta)=\sum_{i=1}^{m}\left[F(x_i)-\frac{i}{n+1}\right]^2$$

$$\varphi(\vartheta)=\sum_{i=1}^{n}\left[1-e^{-\omega\left[-e^{-\alpha x^{-\theta}}.\log\left(1-e^{-\alpha x^{-\theta}}\right)\right]^{\delta}}-\frac{i}{n+1}\right]^2 \quad (32)$$

4.3 Weighted Least Squares - WLS

The equation for this method is given by the form [31], [12]:

$$W(\vartheta)=\sum_{i=1}^{n}W_i\left[F(x_i)-\frac{i}{n+1}\right]^2$$

$$W(\vartheta)=\sum_{i=1}^{n}W_i\left[1-e^{-\omega\left[-e^{-\alpha x^{-\theta}}.\log\left(1-e^{-\alpha x^{-\theta}}\right)\right]^{\delta}}-\frac{i}{n+1}\right]^2 \quad (33)$$

Equations (31), (32), and (33) are derived for the four parameters $(\omega,\delta,\alpha,\theta)$ and then set equal to zero to obtain an estimate of parameters for the EF model.

5 SIMULATION

Monte Carlo techniques are used to estimate complex distributions through random sampling approaches. The accuracy with which three estimators (MLE, LSE, and WLSE) product parameters EF distribution over sample size $N = 70, 140, 210, 280, \ldots$ up to 100 was evaluated for this work using these techniques. In addition to bias calculations [30], the results were assessed using MSE [31] and the root its RMSE [32]. The samples $N = 70, 140, 210$, and 280 yielded the most accurate estimates of parameters ω, δ, α, and θ for the compared parameters. The sample sizes are shown in Table 4, the simulation was performed using R programming language.

Table 4 indicates that the parameter estimates approach true values as the sample size increasing as the sample size increases. Bias decreases with increasing sample size, especially for MLE, confirming the unbiasedness of this method compared to LSE and WLSE. MLE exhibits superior performance in most cases, especially for parameters δ and α, Where MSE and RMSE values are smaller compared to LSE and WLSE, thus, MLE has the best accuracy and consistency, followed by WLSE and LSE.

6 APPLICATION

To demonstrate the effectiveness of EF distribution in practical applications, it was applied to two data sets. The first set represented medical data, the first dataset corresponds to the remission times of a random sample of 128 bladder cancer patients. The second dataset pertains to engineering reliability data and consists of the failure times of compressive carbon fibers, measured for 69 components each with a length of 50 mm. The descriptive statistics for the two sets of data are shown as follows:

The results of the proposed distribution were compared with six other distributions. Table 5 presents the CDF functions of these comparative distributions, as follows:

Table 5: CDF function for comparative distributions.

Distribution	CDF
Beta Fréchet distribution (BeF)	$\beta\left(e^{-\alpha x^{-\theta}},\omega,\delta\right)$
Kumaraswamy Fréchet distribution (New) (KuF)	$1-\left(1-\left(e^{-\alpha x^{-\theta}}\right)^{\omega}\right)^{\delta}$
Exponential Generalized Exponential Fréchet distribution (New) (EGF)	$\left(1-\left(e^{-\alpha x^{-\theta}}\right)^{k}\right)^{c}$
Log Gamma Fréchet	$\frac{1-e^{1-\left(1-\omega e^{-\alpha x^{-\theta}}\right)^{\delta}}}{1-e^{1-(1-\omega)^{\delta}}}$
[0,1] Truncated Exponentiated Exponential Fréchet (New) ([0,1] TEEF)	$\frac{\left(1-e^{-\alpha\omega x^{-\theta}}\right)^{\delta}}{(1-e^{-k})^{c}}$
Fréchet distribution (F)	$e^{-\alpha x^{-\theta}}$

Table 6: Results of the criteria for the distributions.

	Dist.	-L	AIC	CAIC	BIC	HQIC
Data set 1	EF	411.7591	831.5182	831.8461	842.895	836.1405
	BeF	415.4692	838.9383	839.2662	850.3151	843.5605
	KuF	414.0446	836.0892	836.4171	847.466	840.7114
	EGF	435.8316	879.7564	880.0842	891.1331	884.3786
	N[0,1] NHF	413.961	835.9219	836.2498	847.2987	840.5442
	N[0,1] TEEF	415.7867	839.5735	839.9014	850.9502	844.1957
	F	460.964	925.928	926.0248	931.6164	928.2391
Data set 2	EF	34.51252	77.02519	77.69186	85.72274	80.45693
	BeF	38.0213	84.0682	84.73487	92.76575	87.49994
	KuF	36.99326	81.99025	82.65692	90.6878	85.42199
	EGF	37.90646	83.81551	84.48218	92.51306	87.24725
	N[0,1] NHF	40.77596	89.55381	90.22048	98.25136	92.98555
	N[0,1] TEEF	35.51778	79.03631	79.70298	87.73386	82.46806
	F	43.86001	91.72003	91.91358	96.0688	93.4359

Table 7: Results of the criteria for the distributions.

Data	Dist.	W	A	K-S	p-value
Data set 1	EF	0.1287589	0.9580693	0.0769401	0.4397636
	BeF	0.1902141	1.505737	0.08963973	0.259238
	KuF	0.1571864	1.27655	0.07946541	0.3989351
	EGF	0.7006894	4.782829	0.1407645	0.01303947
	N[0,1] NHF	0.1736346	1.304028	0.09233958	0.2289807
	N[0,1] TEEF	0.194154	1.545229	0.08825249	0.2759158
	F	1.404492	8.756085	0.1898896	0.000210589
Data set 2	EF	0.0336067	0.2259957	0.06545795	0.9433906
	BeF	0.1380418	0.8230522	0.09955464	0.5398611
	KuF	0.1104121	0.6573354	0.09323381	0.6243478
	EGF	0.1306284	0.7766481	0.09202611	0.640751
	N[0,1] NHF	0.173459	1.071343	0.1308811	0.2154578
	N[0,1] TEEF	0.0288085	0.2971873	0.04868391	0.9978748
	F	0.275234	1.651979	0.1249851	0.2618771

This comparison uses eight measures. The Good-of-fit: Kolmogorov-Smirnov (KS), Anderson-Darling (A), Cramer-von-Mises (W) and p-value with HQIC, BIC, AIC, and CAIC information criteria [32]. These measures, which are commonly employed to evaluate the goodness of fit of statistical models, are reported in Tables 5 and 6, respectively Table 6 shows EF distribution achieves the lowest values for all metrics in both datasets, indicating a clear advantage in data fitting. EF outperforms all comparable distributions in modeling medical and engineering data, making it the optimal model according to information metrics.

Table 7 presents the results of statistical goodness-of-fit tests. EF achieves the lowest K-S value in both datasets, 0.0769 (dataset 1) and 0.0654 (dataset 2). At the same time, the p-value for EF is high (>0.43 and >0.94), failing to reject the distribution hypothesis (i.e., good fit). EF passes all statistical goodness-of-fit tests with the best performance, supporting its use in modeling survival and material durability data [33].

As a visual evaluation to illustrate the results of Tables 6 and 7, the fitted densities for both types of data are plotted, as well as the empirical CDF for the data, which is shown in the Figures 5-8.

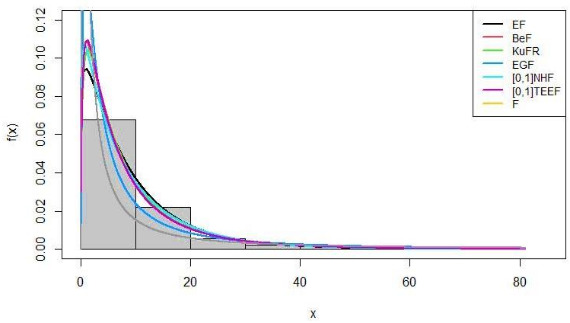

Figure 5: Fitted densities for Data 1.

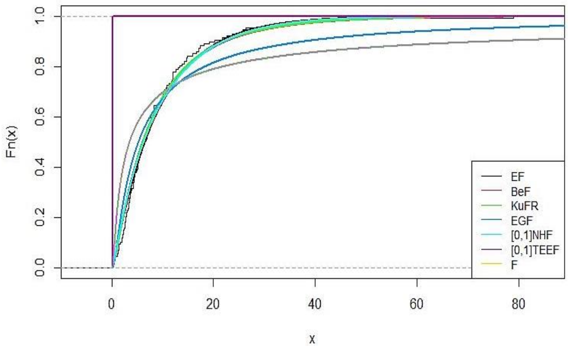

Figure 6: Empirical CDF for Data 1.

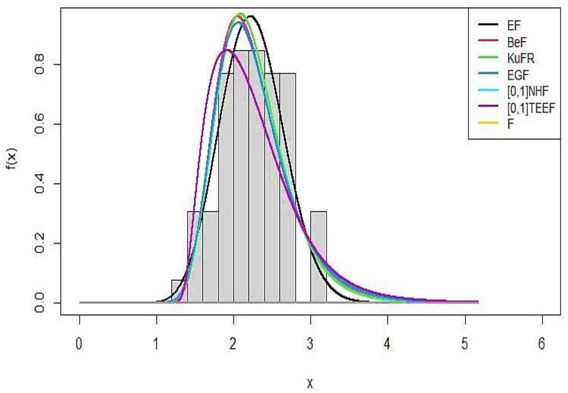

Figure 7: Fitted densities for Data 2.

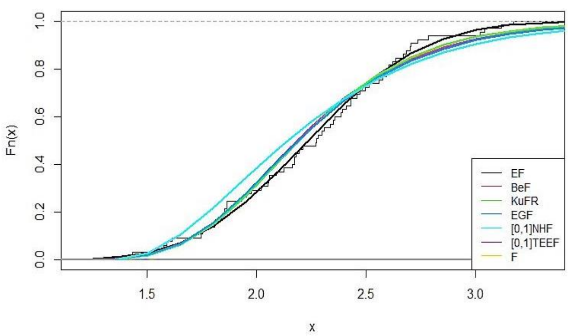

Figure 8: Empirical CDF for Data 2.

Figure 5 shows that the EF curve accurately captures the long tail of the data, reflecting its ability to model extreme values. This is consistent with the heavy-tailed distribution characteristic of EF, as shown in the previous tables. The superiority of EF in this context supports its use in analyzing survival data, where data are often asymmetric and contain extreme values.

Figure 6 shows a high degree of convergence between the two curves, especially in the low and median regions, confirming the goodness of fit of EF to the data. This is consistent with the results of the K-S test, which showed a low value of 0.0769 and a high p-value of 0.439, supporting the hypothesis that the data follow an EF distribution.

Figure 7 shows EF's flexibility in fitting low-variance data concentrated around the median values, as demonstrated by the convergence between the experimental and theoretical curves. This is consistent with the low standard deviation of 0.42 in the engineering data, confirming EF's ability to model engineering data reliability. Figure 8 highlights the accuracy of EF in fitting the empirical CDF, especially in the central range. EF's superior performance here mirrors the results from the previous tables. These results confirm that EF is capable of modeling not only extreme data but also more regular data.

7 CONCLUSIONS

Based on the results obtained in this study, the EF distribution has a high capacity to represent data with heavy tails and positive skewness (it was also confirmed to have high skewness and kurtosis values), and numerical results of the statistical measures. This trait makes it suitable for modeling data with extreme values without changing the underlying data structure, making it applicable in scenarios such as medical survival and engineering reliability. The numerical results proved that the maximum likelihood method was the most accurate and stable for modeling the distribution parameters of the data, with the lowest bias and MSE values compared with the least squares and weighted methods, whose model performance showed a significant improvement with increasing sample size. This type of behavior is consistent with the accuracy of the estimation even on small and medium-sized samples. In applications, EF's distribution was obviously better than the available (competing) distributions on information criteria and fit tests, with the lowest criteria values and the highest statistical significance values. The graphical findings further confirmed these findings by displaying a good agreement between the conceptual model and the practical data and that EF is a faithful and flexible approximation of the real-life data.

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REFERENCES

[1] A. Alzaatreh, C. Lee and F. Famoye, “A new method for generating families of continuous distributions,” Metron, pp. 63-79, 71 2013, [Online]. Available: https://doi.org/10.1007/978-3-319-49094-6_27.

[2] G. A. Mahdi, M. A. Khaleel, A. M. Gemeay, M. Nagy, A. H. Mansi, M. M. Hossain and E. Hussam, “A new hybrid odd exponential-Φ family: Properties and applications,” AIP Advances, vol. 4, 14 2024, [Online]. Available: https://doi.org/10.1063/1.5141712.

[3] M. Aslam, Z. Asghar, Z. Hussain and S. F. Shahc, “A modified TX family of distributions: classical and Bayesian analysis,” Journal of Taibah University for Science, pp. 254-264, 1 14 2020, [Online]. Available: https://doi.org/10.1080/16583655.2020.1712089.

[4] M. A. Khaleel, P. Oguntunde, J. N. Al abbasi, N. A. Ibrahim and M. H. AbuJarad, “The Marshall-Olkin Topp Leone-G family of distributions: A family for generalizing probability models,” Scientific African, vol. 8, p. e00470, 2020, [Online]. Available: https://doi.org/10.1016/j.sciaf.2020.e00470.

[5] N. A. Noori, A. A. Khalaf and M. A. Khaleel, “A New Generalized Family of Odd Lomax-G Distributions Properties and Applications,” Advances in the Theory of Nonlinear Analysis and Its Application, pp. 1-16, 4 7 2023, [Online]. Available: https://doi.org/10.1007/s40065-023-00355-0.

[6] Y. Zhenwu, Z. Ahmad, Z. Almaspoor and S. K. Khosa, “On the Genesis of the Marshall-Olkin Family of Distributions via the T-X Family Approach,” Statistical Modeling, pp. 753-760, 1 67 2021, [Online]. Available: https://doi.org/10.1177/1471082X211011121.

[7] Z. Shah, D. M. Khan, Z. Khan, N. Faiz, S. Hussain, A. Anwar, T. Ahmad and K.-I. Kim, “A new generalized logarithmic–X family of distributions with biomedical data analysis,” Applied Sciences, p. 3668, 6 13 2023, [Online]. Available: https://doi.org/10.3390/app13113668.

[8] Z. Ahmad, E. Mahmoudi, S. Dey and S. K. Khosa, “Modeling Vehicle Insurance Loss Data Using a New Member of T-X Family of Distributions,” Journal of Statistical Theory and Applications, pp. 133-147, 2 19 2020, [Online]. Available: https://doi.org/10.1080/15598608.2020.1727126.

[9] A. Mahdavi and S. Giovana, “A method to expand family of continuous distributions based on truncated distributions,” Journal of Statistical Research of Iran, pp. 231-247, 2 13 2017, [Online]. Available: https://doi.org/10.22067/jsri.v13i2.6275.

[10] M. H. Tahir, M. Zubair, M. Mansoor, G. M. Cordeiro, M. Alizadeh and G. G. Hamedani, “A new Weibull-G family of distributions,” Hacettepe Journal of Mathematics and Statistics, pp. 629-647, 2 45 2016, [Online]. Available: https://doi.org/10.15672/hjms.2016.2.45.

[11] H. S. Klakattawi and W. H. Aljuhani, “A new technique for generating distributions based on a combination of two techniques: Alpha power transformation and exponentiated TX distributions family,” Symmetry, p. 412, 3 13 2021, [Online]. Available: https://doi.org/10.3390/sym13030412.

[12] Z. Ahmad, M. Elgarhy and G. G. Hamedani, “A new Weibull-X family of distributions: properties, characterizations and applications,” Journal of Statistical Distributions and Applications, vol. 5, 1 5 2018, [Online]. Available: https://doi.org/10.1186/s40488-018-0085-7.

[13] N. A. Noori, M. A. Khaleel and A. M. Salih, “Some Expansions to The Weibull Distribution Families with Two Parameters: A Review,” Babylonian Journal of Mathematics, pp. 61-87, 2025, [Online]. Available: https://doi.org/10.18021/bjm.2025.19.

[14] A. M. Abd El-latif, F. A. Almulhim, N. A. Noori, M. A. Khaleel and B. S. Alsaedi, “Properties with application to medical data for new inverse Rayleigh distribution utilizing neutrosophic logic,” Journal of Radiation Research and Applied Sciences, p. 101391, 2 18 2025, [Online]. Available: https://doi.org/10.1016/j.jrras.2025.101391.

[15] Hassan, M. Sabry and A. Elsehetry, “A new probability distribution family arising from truncated power Lomax distribution with application to Weibull model,” Pakistan Journal of Statistics and Operation Research, pp. 661-674, 2020, [Online]. Available: https://doi.org/10.18187/pjsor.v16i3.2827.

[16] E. E. Akarawak, S. J. Adeyeye, M. A. Khaleel, A. F. Adedotun, A. S. Ogunsanya and A. A. Amalare, “The inverted Gompertz-Fréchet distribution with applications,” Scientific African, p. e01769, 2023, [Online]. Available: https://doi.org/10.1016/j.sciaf.2023.e01769.

[17] K. H. Habib, A. M. Salih, M. A. Khaleel and M. K. Abdal-hammed, “OJCA Rayleigh distribution, Statistical Properties with Application,” Tikrit Journal of Administration and Economics Sciences, 19 2023, [Online]. Available: https://doi.org/10.31272/tjaes.v19i1.54321.

[18] K. H. Al-Habib, A. M. Salih, M. K. Abdal-Hammed, M. A. Khaleel and Z. Y. Algamal, “Estimating the parameters of [0, 1] Truncated Nadarajah–Haghighi inverse Weibull distribution,” in AIP Conference Proceedings, vol. 3264, no. 1, p. 050051, Mar. 2025, [Online]. Available: https://doi.org/10.1063/5.0123456.

[19] K. H. Habib, M. A. Khaleel, H. Al-Mofleh, P. E. Oguntunde and S. J. Adeyeye, “Parameters Estimation for the [0, 1] Truncated Nadarajah Haghighi Rayleigh Distribution,” Scientific African, p. e02105, 2024, [Online]. Available: https://doi.org/10.1016/j.sciaf.2024.e02105.

[20] H. Sharqa, M. Ahsan-ul-Haq, J. Zafar and M. A. Khaleel, “Unit Xgamma Distribution: Its Properties, Estimation and Application: Unit-Xgamma Distribution,” Proceedings of the Pakistan Academy of Sciences: A. Physical and Computational Sciences, pp. 15-28, 1 59 2022, [Online]. Available: https://doi.org/10.5353/paascs.2022.59.01.

[21] H. J. Gómez, K. I. Santoro, I. B. Chamorro, O. Venegas, D. I. Gallardo and H. W. Gómez, “A Family of Truncated Positive Distributions,” Mathematics, p. 4431, 21 11 2023, [Online]. Available: https://doi.org/10.3390/math21114431.

[22] K. N. Abdullah, N. A. Noori and M. A. Khaleel, “Data Modelling and Analysis Using Odd Lomax Generalized Exponential Distribution: an Empirical Study and Simulation,” Iraqi Statisticians Journal, pp. 146-162, 1 2 2025, [Online]. Available: https://doi.org/10.31272/isj.2025.2.12.

[23] K. H. Al-Habib, M. A. Khaleel and H. Al-Mofleh, “A new family of truncated Nadarajah-Haghighi-G properties with real data applications,” Tikrit Journal of Administrative and Economic Sciences, p. 2, 61 19 2023, [Online]. Available: https://doi.org/10.31272/tjaes.2023.61.02.

[24] N. S. Khalaf, A. Hameed, K. Moudher, M. A. Khaleel and Z. M. Abdullah, “The Topp Leone flexible Weibull distribution: an extension of the flexible Weibull distribution,” International Journal of Nonlinear Analysis and Applications, pp. 2999-3010, 1 13 2022, [Online]. Available: https://doi.org/10.22075/ijnaa.2022.13.1.2999.

[25] S. Abid and R. Abdulrazak, “[0, 1] truncated Frechet-Weibull and Frechet distributions,” International Journal of Research in Industrial Engineering, pp. 106-135, 1 7 2018, [Online]. Available: https://doi.org/10.20469/ijrie.2018.10006-7.

[26] S. Rezaei, A. K. Marvasty, S. Nadarajah and M. Alizadeh, “A new exponentiated class of distributions: Properties and applications,” Communications in Statistics-Theory and Methods, pp. 6054-6073, 12 46 2017, [Online]. Available: https://doi.org/10.1080/03610926.2016.1248467.

[27] Z. Ahmad, M. Elgarhy and G. G. Hamedani, “A new Weibull-X family of distributions: properties, characterizations and applications,” Journal of Statistical Distributions and Applications, p. 5, 1 5 2018, [Online]. Available: https://doi.org/10.1186/s40488-018-0085-7.

[28] Y. Wang, Z. Feng and A. Zahra, “A new logarithmic family of distributions: Properties and applications,” CMC-Comput. Mater. Contin, pp. 919-929, 66 2021, [Online]. Available: https://doi.org/10.32604/cmc.2021.016314.

[29] R. M. Mahabubur, B. Al-Zahrani and M. Q. Shahbaz, “A general transmuted family of distributions,” Pakistan Journal of Statistics and Operation Research, pp. 451-469, 2018, [Online]. Available: https://doi.org/10.18187/pjsor.v14i2.1732.

[30] S. Naz, L. A. Al-Essa, H. S. Bakouch and C. Chesneau, “A transmuted modified power-generated family of distributions with practice on submodels in insurance and reliability,” Symmetry, p. 1458, 7 15 2023, [Online]. Available: https://doi.org/10.3390/sym15091458.

[31] P. E. Oguntunde, M. A. Khaleel, H. I. Okagbue and O. A. Odetunmibi, “The Topp–Leone Lomax (TLLo) distribution with applications to airborne communication transceiver dataset,” Wireless Personal Communications, pp. 349-360, 2019, [Online]. Available: https://doi.org/10.1007/s11277-019-06659-2.

[32] R. A. Bantan, F. Jamal, C. Chesneau and M. Elgarhy, “A new power Topp–Leone generated family of distributions with applications,” Entropy, vol. 21, 12 2019, [Online]. Available: https://doi.org/10.3390/e21121177.

[33] F. Khubaz, M. K. Abdal-Hameed, N. H. Mohamood and M. A. Khaleel, “Gompertz Inverse Weibull Distribution, some statistical properties with Application Real Dataset,” Tikrit Journal of Administration and Economics Sciences, 19 2023, [Online]. Available: https://doi.org/10.31272/tjaes.2023.19.01.