Mathematical Foundations for Modeling and Scaling of Dynamic Systems

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Abstract: The increasing complexity of electric vehicle (EV) powertrains – integrating multi-domain interactions among traction motors, battery packs, and power electronics – demands mathematically rigorous yet computationally efficient modeling frameworks. This paper presents a structured review of the mathematical foundations for modeling and scaling of dynamic systems, with emphasis on EV applications. Three core analytical tools are examined in depth: 1) state-space representation as a unified multi-domain modeling formalism; 2) similarity transformations for complexity reduction while preserving essential system invariants such as eigenvalues, characteristic polynomials, and transfer functions; and 3) scaling techniques, including dimensionless parameterization and transfer function normalization, for cross-platform simulation efficiency. A structured mapping links these methods to specific EV subsystems: traction motors (4th–8th order systems), battery packs (2nd–10th order equivalent circuits), and power inverters (3rd–6th order systems), identifying the most appropriate scaling strategy for each. The reviewed framework supports scalable simulation, model-based diagnostics, and control system design, bridging high-fidelity physics-based models and computationally efficient representations required for real-time embedded applications. This paper serves as a foundational methodological resource for researchers and practitioners in EV engineering and control systems.

1 INTRODUCTION

The rapid global expansion of electromobility, driven by environmental regulation and the broader transition to sustainable energy, has substantially increased the engineering complexity of electric vehicle (EV) powertrain systems. Modern EVs integrate tightly coupled subsystems – traction motors, power electronic inverters, electrochemical battery packs, and mechanical drivetrains – each operating across different physical domains and timescales [1]. The co-simulation of these subsystems within a single unified framework represents a fundamental challenge in contemporary systems engineering.

Effective analysis, design, and validation of such multi-domain dynamic systems require mathematical modeling tools that are simultaneously expressive, computationally tractable, and scalable across a range of component sizes and operating conditions. Three methodological pillars have emerged as particularly relevant in this context: state-space representation, similarity transformations, and signal scaling techniques.

State-space models provide a compact and general formalism for representing linear and nonlinear dynamic systems of arbitrary order, enabling systematic treatment of controllability, observability, and stability [2]. Unlike transfer-function approaches restricted to single-input single-output (SISO) descriptions, state-space methods naturally accommodate multi-input multi-output (MIMO) configurations typical of EV powertrains, where cross-coupling between flux, torque, voltage, and current dynamics must be explicitly captured [3].

Similarity transformations constitute a second essential tool. By applying invertible coordinate changes in the state space, it is possible to decompose complex high-order systems into canonical or decoupled forms, substantially simplifying analysis and controller synthesis while guaranteeing the preservation of fundamental dynamic invariants [4]. For EV motor models, this allows independent treatment of flux and torque channels; for battery models, it enables the isolation of dominant electrochemical modes.

Scaling and dimensionless parameterization form a third indispensable layer of the framework. When the same control algorithm must be deployed across motors of different rated powers, or when battery models must generalize across different cell chemistries, re-deriving the full mathematical model for each configuration is computationally prohibitive. Scaling techniques allow systematic reformulation of system equations into normalized forms, from which platform-specific models can be recovered by simple parameter substitution [5].

The integration of intelligent monitoring and condition-based diagnostics into EV operation further amplifies the need for scalable dynamic models. Real-time fault detection – whether based on spectral analysis, wavelet decomposition, or data-driven approaches – relies on efficient model representations that can be executed on embedded control hardware [6], [7]. Providing such representations without sacrificing physical fidelity is a central open problem in applied systems modeling.

This paper reviews the mathematical foundations underpinning these three modeling paradigms and demonstrates their structured applicability to EV components. Section 2 develops the theoretical framework, covering state-space models, the matrix exponential solution, transfer functions derived via the Laplace transform, and the formal theory of similarity transformations. Section 3 presents scaling methods and their canonical dimensionless formulations. Section 4 discusses the practical mapping of these methods to specific EV subsystems, supported by a comparative summary table. Section 5 concludes with recommendations for future research directions. This study reviews mathematical approaches to modeling and scaling of dynamic systems, with particular attention to their applicability in electric vehicle (EV) powertrain analysis.

2 MATHEMATICAL FRAMEWORK

2.1 State-Space Representation of Dynamic Systems

A continuous-time linear time-invariant (LTI) dynamic system is described in state-space form as:

$$\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t) + Du(t) \qquad (1)$$

where $x(t) \in \mathbb{R}n$ is the state vector, $u(t) \in \mathbb{R}m$ is the input, $y(t) \in \mathbb{R}p$ is the output, and A, B, C, D are real system matrices of appropriate dimensions. Electrical, mechanical, thermal, and electrochemical subsystems of an EV.

2.2 Solution of the State Equation

The homogeneous solution $\dot{x} = Ax$ with initial condition $x(0) = x_0$ is derived from the power-series ansatz:

$$x(t) = \sum_{i=0}^{\infty} K_i t^i x_0 \qquad (2)$$

Substituting into the differential equation and matching coefficients yields $K_i = A^i/i!$, so that the solution takes the form of the matrix exponential:

$$x(t) = e^{At}x_0 = \lim_{k\to\infty}\left(I + \frac{At}{k}\right)^k x_0 \qquad (3)$$

In the discrete-time approximation with sampling interval h, this becomes:

$$x(t+h) \approx (I + Ah), x(t), x(0) = x_0 \qquad (4)$$

Equation (4) constitutes the basis for numerical integration of state-space models and is directly implemented in simulation environments such as MATLAB/Simulink for real-time EV control applications [2].

2.3 Transfer Function Derivation via the Laplace Transform

Applying the Laplace transform to (1) under zero initial conditions yields the algebraic system:

$$sX(s) = AX(s) + BU(s), Y(s) = CX(s) + DU(s)$$

Solving for $X(s)$ and substituting:

$$\mathrm{Y}(s) = [C(sI - A)^{-1}B + D]\mathrm{U}(s) = \mathrm{G}(s), \mathrm{U}(s) \qquad (5)$$

The matrix $G(s) = C(sI - A)^{-1}B + D$ is the transfer function matrix of the system. For a stable system driven by a harmonic input $u(t) = ae^{j\omega t}$, the Cauchy integral formula yields the steady-state frequency response:

$$\tilde{y}(t) = C(j\omega I - A)^{-1}B, \tilde{u}(t) \qquad (6)$$

so that the matrix frequency transfer function takes the form:

$$G(j\omega) = C(j\omega I - A)^{-1}B \qquad (7)$$

Each scalar entry $g_{ki}(j\omega)$ of this matrix establishes the relationship between the i-th harmonic input and the k-th output component of the steady-state response:

$$\tilde{y}k(t) = |gki(j\omega)|, a\cos(\omega t + \phi), \quad \phi = \arg g_{ki}(j\omega) \tag{8}$$

This frequency-domain characterization is the analytical basis for vibro-acoustic and spectral diagnostics of EV drivetrain components [6].

2.4 Similarity Transformations

A similarity (coordinate) transformation $x = T\tilde{x}$, where $T \in \mathbb{R}^{m\times m} with \det T \neq 0$, maps the original system matrices to an equivalent representation:

$$\tilde{A} = T^{-1}AT, \qquad \tilde{B} = T^{-1}B, \qquad \tilde{C} = CT, \qquad \tilde{D} = D \tag{9}$$

The following four properties are invariant under any similarity transformation [4]:

1) Eigenvalues of A: $\lambda_i(\tilde{A}) = \lambda_i(A)$;
2) Determinant: $\det(\tilde{A}) = \det(A)$;
3) Trace: $(\tilde{A}) = (A)$
4) Transfer function: $\tilde{G}(s) = C(sI - A)^{-1}B + D = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}$.

These invariance properties ensure that similarity transformations yield a physically equivalent – but mathematically simpler – description of the system. A canonical choice of T leads to the companion (Frobenius) form, in which matrix A takes the structure:

$$A = \begin{matrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lfloor -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \rfloor \end{matrix} \tag{10}$$

The characteristic polynomial of A_c is:

$$\phi(\lambda) = \det(\lambda I - A_c) = \lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a = \sum_{i=0}^{n} a_i\lambda^{n-i}, \qquad a_0 = 1 \tag{11}$$

The companion form (10) is particularly useful for order reduction: by truncating states associated with fast, weakly observable or weakly controllable modes, a reduced-order model is obtained that preserves the dominant dynamics of the original system [5].

3 SCALING METHODS FOR DYNAMIC SYSTEM MODELS

3.1 Motivation for Scaling

When a mathematical model of an EV subsystem must be reused across configurations that differ in rated power, voltage, or capacity, direct reuse of the original model is generally not possible without re-identification of all parameters. Scaling provides a systematic methodology for normalizing system variables and equations, enabling the derivation of dimensionless models that are configuration-independent [5].

3.2 Signal Scaling and Dimensionless Parameterization

Consider the original (dimensional) input-output description of a controlled process:

$$\hat{y} = \widehat{G\hat{u}} + \widehat{G_d\hat{d}}, \hat{e} = \hat{y} - \hat{r} \tag{12}$$

where $\hat{u}, \hat{d}, \hat{y}, \hat{e}, \hat{r}$ are, respectively, the control input, disturbance, output, error, and reference signals, all in physical (dimensional) units. Introducing diagonal scaling matrices $D_, D_, D$ such that the normalized signals

$$d = D_d^{-1}\hat{d}, u = D_u^{-1}\hat{u}, e = D_e^{-1}\hat{e}, r = D_u^{-1}\hat{r} \tag{13}$$

have unit magnitude bounds, the system is transformed into the dimensionless form:

$$y = Gu + G_d d, e = y - r \tag{14}$$

where the scaled transfer matrices are given by:

$$G = D_e^{-1}\hat{G}, D_u, G_d = D_e^{-1}\widehat{G_d}, D_d \tag{15}$$

The dimensionless formulation (14)–(15) is particularly advantageous for MIMO control system design, since all variables are of comparable magnitude, improving the numerical conditioning of matrix operations such as singular value decomposition and H_∞ norm computation [3], [5].

3.3 Scaling of Elementary Dynamic Links

Any n-th order LTI system with a single input and single output can be decomposed into a block diagram containing only amplifying (gain), integrating, and summing elements. The state-space matrices of such a decomposition can be obtained directly from the companion form (10) via similarity transformation, allowing systematic construction of scaled block diagrams for EV subsystems of arbitrary order.

For systems with serial, parallel, or feedback interconnections of elementary dynamic links Σ_1 and Σ_2, the composite state-space matrices can be derived analytically [2]:

Series connection:

$$A = \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix}$$

Parallel connection:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

Feedback connection: composed via the standard well-posedness conditions on the feedback loop.

These compositional rules enable modular construction of multi-domain EV powertrain models from independently validated subsystem models.

Scaling is especially effective when designing control systems for complex dynamic objects of the MIMO class.

4 APPLICATION TO ELECTRIC VEHICLE SYSTEMS

4.1 Structured Mapping of Methods to EV Components

Table 1 provides a structured mapping of the mathematical methods reviewed in Sections 2 and 3 to the principal dynamic subsystems of an electric vehicle powertrain. For each subsystem, the typical model order, the recommended scaling approach, and supporting references are indicated.

Table 1: Application of scaling and transformation methods to EV powertrain components.

EV Component	System Order	Scaling Method	Reference
Traction motor	4–8	State-space similarity transformation	[7]
Battery pack	2–10	Dimensionless parameterization	[20]
Power inverter	3–6	Transfer function scaling	[21]

4.2 Traction Motor Modeling

A permanent magnet synchronous motor (PMSM) or induction motor (IM) is typically represented in the synchronously rotating $d - q$ reference frame as a 4th to 8th order LTI system, with state variables comprising the d- and q-axis flux linkages (or currents), rotor speed, and, in higher-order formulations, mechanical oscillation modes of the drivetrain [7]. The state-space form is:

$$\dot{x_m} = A_m x_m + B_m u_m, y_m = C_m x_m$$

where $x_m = \left[i_d,, i_q,, \omega_r,, \theta_r\right]^\top$ for a 4th-order PMSM model. The application of a similarity transformation T that block-diagonalizes A_m allows independent tuning of the flux-weakening and torque-production channels, forming the analytical basis of field-oriented control (FOC) [7].

4.3 Battery Pack Modeling

Battery packs are commonly represented by equivalent circuit models (ECM) consisting of an open-circuit voltage source, an ohmic resistance, and one or more RC parallel branches capturing diffusion dynamics. A second-order (1-RC) ECM takes the state-space form with $x_b = [V_{RC},,\mathrm{SoC}]^\top$, while higher-order (up to 10th order) models include additional RC branches and thermal states. Dimensionless parameterization is obtained by normalizing voltages by the nominal open-circuit voltage V_0 and currents by the rated capacity Q_n, yielding a normalized model in the form of (14)–(15) that is independent of cell chemistry and applicable across battery chemistries via parameter re-identification alone [20].

4.4 Power Inverter and Control Unit

A three-phase voltage source inverter (VSI) with its PWM controller is typically characterized by a 3rd to 6th order transfer function model, capturing the effects of switching delays, LC filter dynamics, and current control loop bandwidth. Transfer function scaling – achieved by normalizing frequencies to the inverter switching frequency ω_{sw} and gains to the DC-link voltage V_{dc} – enables a single normalized controller design to be deployed across inverters of different power ratings by straightforward rescaling of gains, without altering the control structure [21].

4.5 Interconnection and Scalable Simulation

The modular interconnection rules described in Section 3.3, combined with the scaled subsystem models of Sections 4.2–4.4, enable the construction of a full-vehicle powertrain model of controllable complexity. By selecting the appropriate truncation level in the companion form (10) for each subsystem, a hierarchy of models – from detailed design models (high-order) to real-time embedded models (low-order) – can be derived within a unified mathematical framework. This approach is directly supported by MATLAB Control System Toolbox and Simulink environments [2], [20].

5 DISCUSSION

5.1 Comparative Evaluation of Scaling Approaches

The three scaling paradigms reviewed in this paper – similarity transformation, dimensionless parameterization, and transfer function normalization – are complementary rather than competing. Similarity transformations operate on the internal representation of the system and are best suited for analytical tasks such as decoupling, controllability decomposition, and order reduction. Dimensionless parameterization, by contrast, operates on the external signal variables and is most effective when the same model structure must be reused across a family of physically similar but differently sized components, as in battery pack scaling across different cell counts. Transfer function normalization occupies an intermediate position, making it particularly appropriate for controller synthesis tasks where frequency-domain performance specifications (bandwidth, gain margin, phase margin) are defined in normalized units.

5.2 Implications for Model-Based Diagnostics

The framework presented here has direct implications for real-time condition monitoring and fault diagnosis in EV systems. Scalable state-space models provide a natural basis for observer design (Kalman filters, Luenberger observers) that can track the evolution of internal states – such as battery state-of-charge (SoC), state-of-health (SoH), or motor winding temperature – without direct measurement [6]. Deviations of the observer-estimated states from the actual measured outputs serve as residuals for fault detection [12]. The use of dimensionless parameterization ensures that the same observer algorithm can be applied across EV variants of different rated capacities, which is a significant practical advantage in fleet-level diagnostics.

Furthermore, when combined with spectral and wavelet analysis of motor current or vibration signals, the state-space framework supports the extraction of diagnostic features that are directly linked to the physical parameters of the mathematical model – such as changes in inductance due to winding faults or changes in the RC time constants of a battery equivalent circuit due to aging [17], [18].

5.3 Limitations and Future Directions

Several limitations of the present framework merit acknowledgment. First, the analysis is restricted to linear time-invariant systems. In practice, EV components exhibit significant nonlinearities: magnetic saturation in motors, nonlinear open-circuit voltage characteristics in batteries, and switching nonlinearities in inverters. Extensions to linear parameter-varying (LPV) or bilinear state-space formulations are required to address these effects rigorously.

Second, the scaling methods discussed assume that the structural form of the model (number of states, interconnection topology) is fixed across the scaling range. For large-scale systems, topology changes may be necessary, requiring more advanced model order reduction techniques such as balanced truncation or Krylov subspace methods.

Third, the integration of data-driven and machine learning methods with the physics-based state-space framework represents a promising avenue for future research. Hybrid models that combine first-principles structure with data-driven parameter identification [19] can potentially overcome the limitations of purely analytical approaches, particularly for aging battery packs or mechanically worn motor bearings, where first-principles parameter values are difficult to maintain accurately over the component lifetime.

6 CONCLUSIONS

This paper has presented a structured review of the mathematical foundations for modeling and scaling of dynamic systems, with application to electric vehicle powertrains. The following conclusions are drawn:

1) State-space representation provides a unified and computationally tractable framework for multi-domain EV modeling. The matrix exponential solution, companion canonical form, and transfer function derivation via the Laplace

transform collectively constitute the analytical core of this approach.

2) Similarity transformations enable complexity reduction while rigorously preserving four fundamental system invariants: eigenvalues, determinant, trace, and transfer function. Their application to traction motor models supports field-oriented control design and dynamic decoupling.
3) Scaling methods – including dimensionless parameterization and transfer function normalization – facilitate the systematic reuse of validated models across component families of different sizes, eliminating the need for full re-identification. A structured mapping of these methods to EV subsystems (Table 1) demonstrates their practical applicability.
4) The reviewed framework directly supports scalable simulation, model-based diagnostics, and embedded control design, bridging the gap between high-fidelity physics-based models and computationally efficient representations required for real-time operation.

Future work should address the extension of these methods to nonlinear and linear parameter-varying (LPV) system classes, as well as their integration with machine learning-based parameter identification and fault prognosis algorithms. Practical implementation can be carried out within the MATLAB/Simulink environment using the Control System Toolbox and Simscape Electrical libraries.

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