Existence and Stability Analysis of Nonlinear Volterra-Fredholm Integral Equations

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Abstract: The aim of this paper is to prove the existence and uniqueness of a solution for a Volterra-Fredholm nonlinear integral equation in two variables under certain conditions. For instance, by using the Lipschitz condition with Banach's contraction principle under assumptions (1)-(4), a unique solution exists in Banach space Z. Moreover, we examine some fundamental characteristics of the solutions for a Volterra-Fredholm nonlinear integral equation in two variables which occur in applications, using the inequality established in [6, Theorem 1]. It has also been demonstrated that the functions and parameters included in the equation under investigation continuously influence the behavior of solutions under perturbations in parameters. This investigation may allow for the extension of findings, and the last section contains an illustrative example for the validity of the obtained results, making the method much more attractive for practical applications. The examples show the method is straightforward and effective, and the approach can also be extended to other nonlinear integral equation problems.

1 INTRODUCTION

The problem under consideration is the general nonlinear Volterra-Fredholm integral equation of the form:

$$\text{ȹ}(\ddot{\text{ϯ}}, \text{ӵ}) = \mathcal{A}\left(\ddot{\text{ϯ}}, \text{ӵ}, \text{ȹ}(\ddot{\text{ϯ}}, \text{ӵ})\right) + \int_0^{\text{ӵ}}\int_{\text{e}} \mathcal{B}\left(\ddot{\text{ϯ}}, \text{ӵ}, \text{ï}, \text{ϐ}, \text{ȹ}(\text{ï}, \text{ϐ}), (\Gamma\text{ȹ})(\text{ï}, \text{ϐ})\right) d\text{ϐ}\, d\text{ï}, \qquad (1)$$

Where:

$$(\Gamma\text{ȹ})(\ddot{\text{ϯ}}, \text{ӵ}) = \int_0^{\text{ӵ}}\int_{\text{e}} \mathcal{H}\left(\ddot{\text{ϯ}}, \text{ӵ}, \ddot{ȥ}, \ddot{\zeta}, \text{ȹ}(\ddot{ȥ}, \ddot{\zeta})\right) d\ddot{\zeta}\, d\ddot{ȥ} \qquad (2)$$

Where $\mathcal{A}, \mathcal{B}$ and $\mathcal{H}$ they are provided, and the unknown function is ȹ .

The name integral equation for any equation involving the unknown function $\varphi(\chi)$ under the integral sign was introduced by du Bois- Reymond in 1888 [1].

In [2] A. M. Wazwaz also presented a reliable strategy for solving a Volterra-Friedholm mixed linear integral equation in the following form:

$$u(x,t) = f(x,t) + \int_0^t \int_\Omega F(x,t,r,s)u(r,s)drds$$
$$(x,t) \in \Omega \times [0,T],$$

where f, F are known functions and u is an unknown function and Ω a closed partial set of $R^n, n = 1,2,3$

The integrodifferential equations that arise in reactor dynamics and mathematical epidemiology are examined while looking at certain initial boundary value problems for parabolic partial differential equations.

In [3] H. R. Thieme considered a model for the spatial spread of an epidemic consisting of a nonlinear integral equation of Volterra-Fredholm type which has an unique solution. The author showed that this solution has a temporally asymptotic limit which describes the final state of the epidemic and is the minimal solution of another nonlinear integral equation.

In [4] O. Diekmann described, derived and analysed a model of spatio-temporal development of an epidemic. The model considered leads to the following nonlinear integral equation of Volterra-Fredholm type:

$$u(t,x) = g(t,x) + \int_0^t \int_\Omega g\big(u(t-\tau,\xi)\big) S_0(\xi) A(\tau,x,\xi)\, d\xi\, d\tau,$$

for all $(t,x) \in [0,\infty) \times \Omega$ where Ω is a bounded domain in R^n. See [5] the form (1) of the equations naturally emerges.

Equation (1) is treated as a mixed Volterra–Fredholm integral equation, as it is of Fredholm type with respect to one variable and of Volterra type with respect to the other. By modifying the method introduced in [6] (see also [7]), an existence result for equation (1) can be formulated.

The primary aim of this paper is to establish the existence and uniqueness of solutions to a nonlinear Volterra–Fredholm integral equation in two variables under suitable assumptions [8].

In [9] B. G. Pachpatte considered the integral equation:

$$u(t,x) = g(t,x) + \int_0^t \int_\Omega g\big(t,x,s,y,u(s,y)\big)\, dy\, ds,$$

for all $(t,x) \in [0,T) \times \Omega = \mathrm{D}$ where Ω is a bounded domain in R^n.

2 PRELIMINARIES

Definition 2.1, [10]. Let T a mapping from a set X into itself. The mapping T has a fixed point if there exists $x_0 \in X$ such that $Tx_0 = x_0$. The most known fixed-point theorem is the Contraction Mapping Principle, due to S.

Definition 2.2, [10]. Let (X,d) be a metric space. A mapping $T : X \to X$ is Lipschitz continuous if there exists a constant $\alpha > 0$ such that $d(Tx,Ty) \le \alpha d(x,y)$ for all $x, y \in X$.

If $0 \le \alpha < 1$, then T is called a contraction mapping. α is called the contractively factor of T.

A (not necessarily Lipschitz) mapping $T : X \to X$ is said to have a fixed point if there exists $x \in X$ such that Tx = x.

It is not hard to see that a (nonzero) Lipschitz continuous mapping $T : X \to X$ with Lipschitz constant q is continuous. Indeed, for $\varepsilon > 0$, if we choose $0 < \delta < \frac{\varepsilon}{q}$, then

$$x, y \in X, d(x,y) < \delta \Rightarrow d(Tx,Ty) \le q d(x,y) < q.\frac{\varepsilon}{q} = \varepsilon.$$

Definition 2.3, [11]. Let $f(n), g(n)$ be known functions, we say that $f(n) = O(g(n))$ when $n \to 0$, if there is a positive constant A then

$$|f(n)| \le A|\mathrm{g}(n)|, \forall\ |n| > n_0.$$

3 MAIN RESULTS

Let $\mathcal{R}$ represent the set of real numbers, $\mathcal{R}_+ = [0,\infty)$, and let $\mathrm{e} = \Pi_{i=1}^{n}[a_i, b_i]$ and $(a_i < b_i)$ be the specified subsets of $\mathcal{R}$. Define $C(X_1, X_2)$ as the category of continuous functions spanning the sets X_1, X_2 and $\omega = \mathrm{e} \times \mathcal{R}_+$.

Examine the following two-variable nonlinear integral problem of the Volterra-Fredholm type.

$$\begin{aligned} &\varphi(\ddot{\dagger}, \ddot{ч}) \\ &= \mathcal{A}\left(\ddot{\dagger}, \ddot{ч}, \varphi(\ddot{\dagger}, \ddot{ч})\right) \\ &+ \int_0^{\ddot{ч}} \int_{\mathrm{e}} \mathcal{B}\left(\ddot{\dagger}, \ddot{ч}, \ddot{\imath}, ϐ, \varphi(\ddot{\imath}, ϐ), (\Gamma\varphi)(\ddot{\imath}, ϐ)\right) dϐ\, d\ddot{\imath}, \end{aligned} \quad (3)$$

for all $(\ddot{\dagger}, \ddot{ч}) \in \omega$ where

$$\begin{aligned} &(\Gamma\varphi)(\ddot{\dagger}, \ddot{ч}) \\ &= \int_0^{\ddot{ч}} \int_{\mathrm{e}} \mathcal{H}\left(\ddot{\dagger}, \ddot{ч}, \ddot{z}, \ddot{\zeta}, \varphi(\ddot{z}, \ddot{\zeta})\right) d\ddot{\zeta}\, d\ddot{z}, \end{aligned} \quad (4)$$

and $\mathcal{A} \in C(\omega \times \mathcal{R}, \mathcal{R})$, $\mathcal{B} \in C(\omega^2 \times \mathcal{R}^2, \mathcal{R})$, $\mathcal{H} \in C(\omega^2 \times \mathcal{R}, \mathcal{R})$.

In this paper, we will demonstrate the presence and uniqueness of the Volterra-Fredholm integral equation solution in two variables. Using Banach's contraction principle, we investigate the existence and uniqueness of a solution to the Volterra-Fredholm integral problem. Additionally, a number of the solution's characteristics are identified.

Let $\mathcal{Z}$ denote the space of functions that satisfy the constraint $\ddot{\phi} \in C(\omega, \mathcal{R})$.

$$\left|\ddot{\phi}(\ddot{\dagger}, \ddot{ч})\right| = O\left(exp\left(\lambda(\ddot{\dagger} + |\ddot{ч}|)\right)\right) \quad (5)$$

where a constant is $\lambda > 0$. We define the norm "The norms ||v|| of a vector v and ||A|| of a matrix A are nonnegative real numbers that measure the size of v and A. In the case of a vector, norms are generalizations of the Euclidean lengths of vectors" in space $\mathcal{Z}$.

$$\left|\ddot{\phi}\right|_{\mathcal{Z}} = \sup_{(\ddot{\dagger}, \ddot{ч}) \in \omega} \left[\left|\ddot{\phi}(\ddot{\dagger}, \ddot{ч})\right| exp\left(-\lambda(\ddot{\dagger} + |\ddot{ч}|)\right)\right]. \quad (6)$$

It is easy to see that $\mathcal{Z}$ is a Banach space when the norm specified in (2.4 is used. And

$$|\ddot{\phi}|_{\mathcal{Z}} \le \mathcal{M} \quad (7)$$

where a constant is $\mathcal{M} \ge 0$. Presenting some qualitative elements of (3) solutions discovered in [12], [13] ,is the primary goal of this part.

The existence and uniqueness of the solution to problem (3) are established using the Banach fixed-point theorem.

Theorem 1. Assume that

1) (3) are functions $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{H}$ meet the requirements.

$$\left|\mathcal{A}(\ddot{\dagger}, \ddot{ч}, \varphi) - \mathcal{A}(\ddot{\dagger}, \ddot{ч}, \ddot{v})\right| \le g(\ddot{\dagger}, \ddot{ч})|\varphi - \ddot{v}|, \quad (8)$$

$$\left|\mathcal{B}(\ddot{t},\ddot{u},\ddot{\imath},\beta,\varphi,\bar{\varphi})-\mathcal{B}(\ddot{t},\ddot{u},\ddot{\imath},\beta,\ddot{\upsilon},\bar{\ddot{\upsilon}})\right| \le r(\ddot{t},\ddot{u},\ddot{\imath},\beta)[|\varphi-\ddot{\upsilon}| + |\bar{\varphi}-\bar{\ddot{\upsilon}}|] \quad (9)$$

And

$$\left|\mathcal{H}(\ddot{t},\ddot{u},\ddot{z},\ddot{\zeta},\varphi)-\mathcal{H}(\ddot{t},\ddot{u},\ddot{z},\ddot{\zeta},\ddot{\upsilon})\right| \le m(\ddot{t},\ddot{u},\ddot{z},\ddot{\zeta})|\varphi-\ddot{\upsilon}| \quad (10)$$

where $g \in C(\omega,\mathcal{R}_+)$ and $r, m \in C(\omega^2,\mathcal{R}_+)$

2) for λ as in (5);
3) there exists a nonnegative constant $\alpha < 1$ such that:

$$g(\ddot{t},\ddot{u})e^{\lambda(\ddot{t}+|\ddot{u}|)} + \int_0^{\ddot{u}}\int_{\mathrm{e}} r(\ddot{t},\ddot{u},\ddot{\imath},\beta)\left[e^{\lambda(\ddot{\imath}+|\beta|)} + \int_0^{\ddot{u}}\int_{\mathrm{e}} m(\ddot{\imath},\beta,\ddot{z},\ddot{\zeta})e^{\lambda(\ddot{z}+|\ddot{\zeta}|)}\,d\ddot{\zeta}\,d\ddot{z}\right]d\beta\, d\ddot{\imath} \le \alpha e^{\lambda(\ddot{t}+|\ddot{u}|)} \quad (11)$$

4) A nonnegative constant β exists such that

$$\left|\mathcal{A}(\ddot{t},\ddot{u},0)\right| + \int_0^{\ddot{u}}\int_{\mathrm{e}} \left|\mathcal{B}\left(\ddot{t},\ddot{u},\ddot{\imath},\beta,0,(\Gamma 0)(\ddot{\imath},\beta)\right)\right| d\beta\, d\ddot{\imath} \le \beta e^{\lambda(\ddot{t}+|\ddot{u}|)} \quad (12)$$

where $\mathcal{A}$, $\mathcal{B}$ are as defined in (3). Then, $\varphi(\ddot{t},\ddot{u})$ *in* ω *on* $\mathcal{Z}$ is the only solution to (3).

proof. Define the operator Ψ by letting $\varphi \in \mathcal{Z}$.

$$(\Psi\varphi)(\ddot{t},\ddot{u}) = \mathcal{A}\left(\ddot{t},\ddot{u},\varphi(\ddot{t},\ddot{u})\right) + \int_0^{\ddot{u}}\int_{\mathrm{e}} \mathcal{B}\left(\ddot{t},\ddot{u},\ddot{\imath},\beta,\varphi(\ddot{\imath},\beta),(\Gamma\varphi)(\ddot{\imath},\beta)\right) d\beta\, d\ddot{\imath} \quad (13)$$

for all $(\ddot{t},\ddot{u}) \in \omega$

To prove the theorem, from (13) and applying the theories, we have

$$\left|(\Psi\varphi)(\ddot{t},\ddot{u})\right| \le \left|\mathcal{A}\left(\ddot{t},\ddot{u},\varphi(\ddot{t},\ddot{u})\right)\right| + \int_0^{\ddot{u}}\int_{\mathrm{e}} \left|\mathcal{B}\left(\ddot{t},\ddot{u},\ddot{\imath},\beta,\varphi(\ddot{\imath},\beta),(\Gamma\varphi)(\ddot{\imath},\beta)\right)\right| d\beta\, d\ddot{\imath} \quad (14)$$

$$\begin{aligned}
&\le \left|\mathcal{A}\left(\ddot{t},\ddot{u},\varphi(\ddot{t},\ddot{u})\right)-\mathcal{A}(\ddot{t},\ddot{u},0)+\mathcal{A}(\ddot{t},\ddot{u},0)\right| \\
&+ \int_0^{\ddot{u}}\int_{\mathrm{e}} \Big\{\Big|\mathcal{B}\Big(\ddot{t},\ddot{u},\ddot{\imath},\beta,\varphi(\ddot{\imath},\beta), \\
&[\int_0^{\ddot{u}}\int_{\mathrm{e}} \left|\mathcal{H}\left(\ddot{t},\ddot{u},\ddot{z},\ddot{\zeta},\varphi(\ddot{z},\ddot{\zeta})\right)-\mathcal{H}(\ddot{t},\ddot{u},\ddot{z},\ddot{\zeta},0) + \mathcal{H}(\ddot{t},\ddot{u},\ddot{z},\ddot{\zeta},0)\right| d\ddot{\zeta}\, d\ddot{z}]\Big) \\
&- \mathcal{B}\left(\ddot{t},\ddot{u},\ddot{\imath},\beta,0,(\Gamma 0)(\ddot{\imath},\beta)\right) + \mathcal{B}\left(\ddot{t},\ddot{u},\ddot{\imath},\beta,0,(\Gamma 0)(\ddot{\imath},\beta)\right)\Big|\Big\} d\beta\, d\ddot{\imath} \\
&\le \left|\mathcal{A}(\ddot{t},\ddot{u},0)\right| + \int_0^{\ddot{u}}\int_{\mathrm{e}} \left|\mathcal{B}\left(\ddot{t},\ddot{u},\ddot{\imath},\beta,0,(\Gamma 0)(\ddot{\imath},\beta)\right)\right| d\beta\, d\ddot{\imath} + g(\ddot{t},\ddot{u})|\varphi| \\
&+ \int_0^{\ddot{u}}\int_{\mathrm{e}} \left\{r(\ddot{t},\ddot{u},\ddot{\imath},\beta)[|\varphi|+|\bar{\varphi}|] + \left[\int_0^{\beta}\int_{\mathrm{e}} m(\ddot{t},\ddot{u},\ddot{z},\ddot{\zeta})|\varphi|\,d\ddot{\zeta}\,d\ddot{z}\right]\right\} d\beta\, d\ddot{\imath} \\
&\le \beta e^{\lambda(\ddot{t}+|\ddot{u}|)} + |\varphi|_{z}\Big[g(\ddot{t},\ddot{u})e^{\lambda(\ddot{t}+|\ddot{u}|)} + \int_0^{\ddot{u}}\int_{\mathrm{e}} r(\ddot{t},\ddot{u},\ddot{\imath},\beta)\Big(e^{\lambda(\ddot{\imath}+|\beta|)} \\
&+ \int_0^{\ddot{u}}\int_{\mathrm{e}} m(\ddot{\imath},\beta,\ddot{z},\ddot{\zeta})e^{\lambda(\ddot{z}+|\ddot{\zeta}|)}\,d\ddot{\zeta}\,d\ddot{z}\Big) d\beta\, d\ddot{\imath}\Big] \\
&\le \beta e^{\lambda(\ddot{t}+|\ddot{u}|)} + |\varphi|_{z}\alpha e^{\lambda(\ddot{t}+|\ddot{u}|)} \\
&\le \beta e^{\lambda(\ddot{t}+|\ddot{u}|)} + \mathcal{M}\alpha e^{\lambda(\ddot{t}+|\ddot{u}|)} \\
&\left|(\Psi\varphi)(\ddot{t},\ddot{u})\right| \le [\beta+\mathcal{M}\alpha]e^{\lambda(\ddot{t}+|\ddot{u}|)}
\end{aligned} \quad (15)$$

$$\left|(\Psi\varphi)(\ddot{t},\ddot{u})\right|_{z} \le [\beta+\mathcal{M}\alpha] \quad (16)$$

From (16), Consequently $\Psi\varphi \in \mathcal{Z}$. Let φ, $\ddot{\upsilon} \in \mathcal{Z}$. From (13) and applying the theories, we have

$$\begin{aligned}
&\left|(\Psi\varphi)(\ddot{t},\ddot{u})-(\Psi\ddot{\upsilon})(\ddot{t},\ddot{u})\right| \\
&\le \left|\mathcal{A}\left(\ddot{t},\ddot{u},\varphi(\ddot{t},\ddot{u})\right)-\mathcal{A}\left(\ddot{t},\ddot{u},\ddot{\upsilon}(\ddot{t},\ddot{u})\right)\right| \\
&+ \int_0^{\ddot{u}}\int_{\mathrm{e}} \{|\mathcal{B}\left(\ddot{t},\ddot{u},\ddot{\imath},\beta,\varphi(\ddot{\imath},\beta),\right. \\
&\left.\left[\int_0^{\beta}\int_{\mathrm{e}} \left|\mathcal{H}\left(\ddot{t},\ddot{u},\ddot{z},\ddot{\zeta},\varphi(\ddot{z},\ddot{\zeta})\right)\right| d\ddot{\zeta}\,d\ddot{z}\right]\right) \\
&- \mathcal{B}(\ddot{t},\ddot{u},\ddot{\imath},\beta,\ddot{\upsilon}(\ddot{\imath},\beta), \\
&\left.\left[\int_0^{\beta}\int_{\mathrm{e}} \left|\mathcal{H}\left(\ddot{t},\ddot{u},\ddot{z},\ddot{\zeta},\ddot{\upsilon}(\ddot{z},\ddot{\zeta})\right)\right| d\ddot{\zeta}\,d\ddot{z}\right]\right)\Big|\Big\} d\beta\, d\ddot{\imath} \\
&\le g(\ddot{t},\ddot{u})|\varphi-\ddot{\upsilon}|
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{ӵ}\int_e \Big\{ r(\ddot{ŧ},ӵ,ï,ϐ)[|\varphi-\ddot{v}|+|\bar{\varphi}-\bar{\ddot{v}}|] \\
&+\Big[\int_0^{ϐ}\int_e m(ï,ϐ,\ddot{z},\ddot{\zeta})|\varphi-\ddot{v}|d\ddot{\zeta}d\ddot{z}\Big]\Big\}dϐdï \\
&\le |\varphi-\ddot{v}|_z\Big[g(\ddot{ŧ},ӵ)e^{\lambda(\ddot{ŧ}+|ӵ|)} \\
&+\int_0^{ӵ}\int_e r(\ddot{ŧ},ӵ,ï,ϐ)\Big(e^{\lambda(ï+|ϐ|)} \\
&+\int_0^{ϐ}\int_e m(ï,ϐ,\ddot{z},\ddot{\zeta})e^{\lambda(\ddot{z}+|\ddot{\zeta}|)}d\ddot{\zeta}d\ddot{z}\Big)dϐdï\Big]
\end{aligned}
$$

$$|\Psi\varphi)(\ddot{ŧ},ӵ)-(\Psi\ddot{v})(\ddot{ŧ},ӵ)|\le|\varphi-\ddot{v}|_z\alpha e^{\lambda(\ddot{ŧ}+|ӵ|)}$$

$$|\Psi\varphi-\Psi\ddot{v}|_z\le|\varphi-\ddot{v}|_z\alpha$$

According to the Banach fixed point theorem see [14], [15], Ψ has a unique fixed point in Z. Since α < 1. (13) has a solution at the fixed point of Ψ. This evidence is comprehensive.

4 CHARACTERISTICS OF SOLUTIONS

In this section, we examine some fundamental characteristics of the solutions to (3) using the inequality established in [12] and Theorem 1.

First, we will provide the subsequent theorem regarding the estimate of the solution to (3).

Theorem 2. Assume that the functions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{H}$ in (3) and (4)" satisfy the requirements.

$$|\mathcal{A}(\ddot{ŧ},ӵ,\varphi)-\mathcal{A}(\ddot{ŧ},ӵ,\ddot{v})|\le g(\ddot{ŧ},ӵ)|\varphi-\ddot{v}| \tag{17}$$

$$
\begin{aligned}
&|\mathcal{B}(\ddot{ŧ},ӵ,ï,ϐ,\varphi,\bar{\varphi})-\mathcal{B}(\ddot{ŧ},ӵ,ï,ϐ,\ddot{v},\bar{\ddot{v}})| \\
&\le q(\ddot{ŧ},ӵ)p(ï,ϐ)[|\varphi-\ddot{v}|+|\bar{\varphi}-\bar{\ddot{v}}|]
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
&|\mathcal{H}(\ddot{ŧ},ӵ,\ddot{z},\ddot{\zeta},\varphi)-\mathcal{H}(\ddot{ŧ},ӵ,\ddot{z},\ddot{\zeta},\ddot{v})| \\
&\le q(\ddot{ŧ},ӵ)n(\ddot{z},\ddot{\zeta})|\varphi-\ddot{v}|
\end{aligned} \tag{19}
$$

where $g,q,p,n\in C(\omega,\mathcal{R}_+)$, Let

$$
c=\sup_{(\ddot{ŧ},ӵ)\in\omega}\Big|\mathcal{A}(\ddot{ŧ},ӵ,0)+\int_0^{ӵ}\int_e \mathcal{B}\big(\ddot{ŧ},ӵ,ï,ϐ,0,(\Gamma 0)(ï,ϐ)\big)dϐdï\Big| \tag{20}
$$

If $\varphi(\ddot{ŧ}\,ӵ)$ is any solution of (3) on ω, then

$$
\begin{aligned}
|\varphi(\ddot{ŧ},ӵ)|\le c\Big[g(\ddot{ŧ},ӵ)&+q(\ddot{ŧ},ӵ)\int_0^{ӵ}\int_e [p(ï,ϐ) \\
&+n(ï,ϐ)]\times exp\Big(\int_0^{ӵ}\int_e q(\ddot{z},\ddot{\zeta})[f(\ddot{z},\ddot{\zeta}) \\
&+n(\ddot{z},\ddot{\zeta})]d\ddot{\zeta}d\ddot{z}\Big)dϐdï\Big]
\end{aligned} \tag{21}
$$

for all $(\ddot{ŧ},ӵ)\in\omega$

Proof. Using the hypotheses and given that (3) can be solved by $\varphi(\ddot{ŧ},ӵ)$, we have

$$
\begin{aligned}
&|\varphi(\ddot{ŧ},ӵ)| \\
&\le\Big|\mathcal{A}\big(\ddot{ŧ},ӵ,\varphi(\ddot{ŧ},ӵ)\big)-\mathcal{A}(\ddot{ŧ},ӵ,0)+\mathcal{A}(\ddot{ŧ},ӵ,0)\Big| \\
&+\int_0^{ӵ}\int_e\Big\{\Big|\mathcal{B}\Big(\ddot{ŧ},ӵ,ï,ϐ,\varphi(\ddot{ŧ},ӵ), \\
&\Big[\int_0^{ϐ}\int_e\Big|\mathcal{H}\big(\ddot{ŧ},ӵ,\ddot{z},\ddot{\zeta},\varphi(\ddot{z},\ddot{\zeta})\big)-\mathcal{H}(\ddot{ŧ},ӵ,\ddot{z},\ddot{\zeta},0) \\
&+\mathcal{H}(\ddot{ŧ},ӵ,\ddot{z},\ddot{\zeta},0)\Big|d\ddot{\zeta}d\ddot{z}\Big]\Big) \\
&-\mathcal{B}\big(\ddot{ŧ},ӵ,ï,ϐ,0,(\Gamma 0)(ï,ϐ)\big) \\
&+\mathcal{B}\big(\ddot{ŧ},ӵ,ï,ϐ,0,(\Gamma 0)(ï,ϐ)\big)\Big|\Big\}dϐdï \\
&\le\Big|\mathcal{A}(\ddot{ŧ},ӵ,0) \\
&+\int_0^{ӵ}\int_e +\mathcal{B}\big(\ddot{ŧ},ӵ,ï,ϐ,0,(\Gamma 0)(ï,ϐ)\big)dϐdï\Big| \\
&+\Big|\mathcal{A}\big(\ddot{ŧ},ӵ,\varphi(\ddot{ŧ},ӵ)\big)-\mathcal{A}(\ddot{ŧ},ӵ,0)\Big|+ \\
&\int_0^{ӵ}\int_e\Big\{\Big|\mathcal{B}\Big(\ddot{ŧ},ӵ,ï,ϐ,\varphi(\ddot{ŧ},ӵ), \\
&\Big[\int_0^{ϐ}\int_e\Big|\mathcal{H}\big(\ddot{ŧ},ӵ,\ddot{z},\ddot{\zeta},\varphi(\ddot{z},\ddot{\zeta})\big)-\mathcal{H}(\ddot{ŧ},ӵ,\ddot{z},\ddot{\zeta},0) \\
&+\mathcal{H}(\ddot{ŧ},ӵ,\ddot{z},\ddot{\zeta},0)\Big|d\ddot{\zeta}d\ddot{z}\Big]\Big) \\
&-\mathcal{B}\big(\ddot{ŧ},ӵ,ï,ϐ,0,(\Gamma 0)(ï,ϐ)\big)\Big|\Big\}dϐdï \\
&\le c+g(\ddot{ŧ},ӵ)|\varphi(\ddot{ŧ},ӵ)| \\
&+q(\ddot{ŧ},ӵ)\int_0^{ӵ}\int_e p(ï,ϐ)\Big[|\varphi(ï,ϐ)| \\
&+q(ï,ϐ)\int_0^{ϐ}\int_e n(\ddot{z},\ddot{\zeta})|\varphi(\ddot{z},\ddot{\zeta})|d\ddot{\zeta}d\ddot{z}\Big]dϐdï.
\end{aligned} \tag{22}
$$

We can observe from (22) and Theorem 1 in [12] that

$$
\begin{aligned}
&|\varphi(\ddot{ŧ},ӵ)|\le c[g(\ddot{ŧ},ӵ) \\
&+q(\ddot{ŧ},ӵ)\int_0^{ӵ}\int_e [p(ï,ϐ)+n(ï,ϐ)] \\
&\times exp\Big(\int_0^{ӵ}\int_e q(\ddot{z},\ddot{\zeta})[f(\ddot{z},\ddot{\zeta}) \\
&+n(\ddot{z},\ddot{\zeta})]d\ddot{\zeta}d\ddot{z}\Big)dϐdï\Big]
\end{aligned} \tag{23}
$$

This proof is complete.

Lastly, we provide a result regarding the (3) solution's continuous dependency on the functions involved. Analyse (3) and its equivalent equation.

$$\ddot{v}(\ddot{t},\ddot{y}) = \bar{\mathcal{A}}\left(\ddot{t},\ddot{y},\ddot{v}(\ddot{t},\ddot{y})\right) + \int_0^{\ddot{y}}\int_{\mathrm{e}} \bar{\mathcal{B}}\left(\ddot{t},\ddot{y},\ddot{\imath},\beta,\ddot{v}(\ddot{\imath},\beta),(\bar{\Gamma}\ddot{v})(\ddot{\imath},\beta)\right) d\beta d\ddot{\imath}, \tag{24}$$

for all $(\ddot{t},\ddot{y}) \in \omega$ where:

$$(\bar{\Gamma}\ddot{v})(\ddot{t},\ddot{y}) = \int_0^{\ddot{y}}\int_{\mathrm{e}} \bar{\mathcal{H}}\left(\ddot{t},\ddot{y},\ddot{z},\ddot{\zeta},\ddot{v}(\ddot{z},\ddot{\zeta})\right) d\ddot{\zeta} d\ddot{z},$$

and $\bar{\mathcal{A}} \in C(\omega \times \mathcal{R},\mathcal{R})$, $\bar{\mathcal{B}} \in C(\omega^2 \times \mathcal{R}^2,\mathcal{R})$, $\bar{\mathcal{H}} \in C(\omega^2 \times \mathcal{R},\mathcal{R})$.

Theorem 3. Assume that, respectively, the functions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{H}$ in (3) fulfil the requirements (17), (18) and (19). Additionally, assume that the supplied solution to (23) on ω is $\ddot{v}(\ddot{t},\ddot{y})$.

$$\left|\mathcal{A}\left(\ddot{t},\ddot{y},\ddot{v}(\ddot{t},\ddot{y})\right) - \bar{\mathcal{A}}\left(\ddot{t},\ddot{y},\ddot{v}(\ddot{t},\ddot{y})\right)\right| + \int_0^{\ddot{y}}\int_{\mathrm{e}} \left|\mathcal{B}\left(\ddot{t},\ddot{y},\ddot{\imath},\beta,\ddot{v}(\ddot{\imath},\beta),(\Gamma\ddot{v})(\ddot{\imath},\beta)\right) - \bar{\mathcal{B}}\left(\ddot{t},\ddot{y},\ddot{\imath},\beta,\ddot{v}(\ddot{\imath},\beta),(\bar{\Gamma}\ddot{v})(\ddot{\imath},\beta)\right)\right| d\beta d\ddot{\imath} \le \varepsilon,$$

where A, B, $\Gamma\varphi$ and $\bar{A}, \bar{B}, \bar{\Gamma}\ddot{v}$ are as in (3) and (23) and $\varepsilon > 0$ is a small, arbitrary constant, respectively. Then, the functions involved in (3) determine the solution $\varphi(\ddot{t},\ddot{y})$ of (3) on ω continuously.

Proof. Let $\mathcal{D}(\ddot{t},\ddot{y}) = \left|\varphi(\ddot{t},\ddot{y}) - \ddot{v}(\ddot{t},\ddot{y})\right|$, for $(\ddot{t},\ddot{y}) \in \omega$. Using the hypotheses, we have:

$$\begin{aligned}\mathcal{D}(\ddot{t},\ddot{y}) &\le \left|\mathcal{A}\left(\ddot{t},\ddot{y},\varphi(\ddot{t},\ddot{y})\right) - \mathcal{A}\left(\ddot{t},\ddot{y},\ddot{v}(\ddot{t},\ddot{y})\right)\right| \\ &+ \left|\mathcal{A}\left(\ddot{t},\ddot{y},\ddot{v}(\ddot{t},\ddot{y})\right) - \bar{\mathcal{A}}\left(\ddot{t},\ddot{y},\ddot{v}(\ddot{t},\ddot{y})\right)\right| \\ &+ \int_0^{\ddot{y}}\int_{\mathrm{e}} \left|\mathcal{B}\left(\ddot{t},\ddot{y},\ddot{\imath},\beta,\varphi(\ddot{\imath},\beta),(\Gamma\varphi)(\ddot{\imath},\beta)\right) - \mathcal{B}\left(\ddot{t},\ddot{y},\ddot{\imath},\beta,\ddot{v}(\ddot{\imath},\beta),(\Gamma\ddot{v})(\ddot{\imath},\beta)\right)\right| d\beta d\ddot{\imath} \\ &+ \int_0^{\ddot{y}}\int_{\mathrm{e}} \left|\mathcal{B}\left(\ddot{t},\ddot{y},\ddot{\imath},\beta,\ddot{v}(\ddot{\imath},\beta),(\Gamma\ddot{v})(\ddot{\imath},\beta)\right) - \bar{\mathcal{B}}\left(\ddot{t},\ddot{y},\ddot{\imath},\beta,\ddot{v}(\ddot{\imath},\beta),(\bar{\Gamma}\ddot{v})(\ddot{\imath},\beta)\right)\right| d\beta d\ddot{\imath} \\ &\le \varepsilon + g(\ddot{t},\ddot{y})\mathcal{D}(\ddot{t},\ddot{y}) \\ &+ q(\ddot{t},\ddot{y})\int_0^{\ddot{y}}\int_{\mathrm{e}} p(\ddot{\imath},\beta)\left[\mathcal{D}(\ddot{\imath},\beta) + q(\ddot{\imath},\beta)\int_0^{\beta}\int_{\mathrm{e}} n(\ddot{z},\ddot{\zeta})\mathcal{D}(\ddot{z},\ddot{\zeta})d\ddot{\zeta}d\ddot{z}\right] d\beta d\ddot{\imath}\end{aligned} \tag{26}$$

Now, applying [6] Theorem 1, the relation (26) produces:

$$\left|\varphi(\ddot{t},\ddot{y}) - \ddot{v}(\ddot{t},\ddot{y})\right| \le \varepsilon\Big[g(\ddot{t},\ddot{y}) + q(\ddot{t},\ddot{y})\int_0^{\ddot{y}}\int_{\mathrm{e}} [p(\ddot{\imath},\beta) + n(\ddot{\imath},\beta)] \times exp\left(\int_0^{\ddot{y}}\int_{\mathrm{e}} q(\ddot{z},\ddot{\zeta})[p(\ddot{z},\ddot{\zeta}) + n(\ddot{z},\ddot{\zeta})]d\ddot{\zeta}d\ddot{z}\right) d\beta d\ddot{\imath}\Big] \tag{27}$$

For $(\ddot{t},\ddot{y}) \in \omega$. From (27) Consequently, the functions involved in (3) determine the answer in a continuous manner.

This proof is complete.

5 AN EXAMPLE

The problem under consideration is described by the following nonlinear Volterra–Fredholm integral equation:

$$\varphi(\ddot{t},\ddot{y}) = w(\ddot{t},\ddot{y})\varphi(\ddot{t},\ddot{y}) + \int_0^{\ddot{y}}\int_{\mathrm{e}} \left(d(\ddot{t},\ddot{y},\ddot{\imath},\beta)\varphi(\ddot{\imath},\beta) + \int_0^{\ddot{z}}\int_{\mathrm{e}} f(\ddot{t},\ddot{y},\ddot{z},\ddot{\zeta})\varphi(\ddot{z},\ddot{\zeta})d\ddot{\zeta}d\ddot{z}\right) d\beta d\ddot{\imath} \tag{28}$$

where $w(\ddot{t},\ddot{y}) = \frac{\lambda}{13}e^{\lambda(\ddot{t}+|\ddot{y}|)}$

$$d(\ddot{t},\ddot{y},\ddot{\imath},\beta) = \frac{\lambda^2}{e^{\lambda}-1}e^{\lambda(\ddot{t}+|\ddot{y}|)}$$

$$f(\ddot{t},\ddot{y},\ddot{z},\ddot{\zeta}) = \frac{\lambda^2}{t(e^{\lambda}-1)}e^{\lambda(\ddot{t}+|\ddot{y}|)}$$

$\mathrm{e} = [0,1]$, $\lambda > 0$ Here,

$$\mathcal{A}\left(\ddot{t},\ddot{y},\varphi(\ddot{t},\ddot{y})\right) = w(\ddot{t},\ddot{y})\varphi(\ddot{t},\ddot{y})$$

$$\mathcal{H}\left(\ddot{t},\ddot{y},\ddot{z},\ddot{\zeta},\varphi(\ddot{z},\ddot{\zeta})\right) = \int_0^{\ddot{z}}\int_0^1 f(\ddot{t},\ddot{y},\ddot{z},\ddot{\zeta})\varphi(\ddot{z},\ddot{\zeta})d\ddot{\zeta}d\ddot{z}$$

$$\mathcal{B}\left(\ddot{t},\ddot{y},\ddot{\imath},\beta,\varphi(\ddot{\imath},\beta),(\Gamma\varphi)(\ddot{\imath},\beta)\right) = \int_0^{\ddot{y}}\int_0^1 \left(d(\ddot{t},\ddot{y},\ddot{\imath},\beta)\varphi(\ddot{\imath},\beta) + (\Gamma\varphi)(\ddot{\imath},\beta)\right) d\beta d\ddot{\imath}$$

We can see that

$$|\mathcal{A}(\ddot{t},\ddot{y},\varphi) - \mathcal{A}(\ddot{t},\ddot{y},\ddot{v})| \le \frac{\lambda}{13}e^{\lambda(\ddot{t}+|\ddot{y}|)}|\varphi - \ddot{v}|$$

$$|\mathcal{H}(\ddot{t},\ddot{y},\ddot{z},\ddot{\zeta},\varphi) - \mathcal{H}(\ddot{t},\ddot{y},\ddot{z},\ddot{\zeta},\ddot{v})| \le \frac{\lambda^2}{e^{\lambda}-1}e^{\lambda(\ddot{t}+|\ddot{y}|)}|\varphi - \ddot{v}|$$

$$|\mathcal{B}(\ddot{t},\ddot{y},\ddot{\imath},\beta,\varphi,\bar{\varphi}) - \mathcal{B}(\ddot{t},\ddot{y},\ddot{\imath},\beta,\ddot{v},\bar{\ddot{v}})| \le \frac{2\lambda^2}{e^{\lambda}-1}e^{\lambda(\ddot{t}+|\ddot{y}|)}|\varphi - \ddot{v}|.$$

By choosing $r(\ddot{t},\ddot{y},\ddot{\imath},\beta) = \frac{2\lambda^2}{e^{\lambda}-1}e^{\lambda(\ddot{t}+|\ddot{y}|)}$ and $m(\ddot{t},\ddot{y},\ddot{z},\ddot{\zeta}) = \frac{\lambda^2}{e^{\lambda}-1}e^{\lambda(\ddot{t}+|\ddot{y}|)}$

It is easy to see that the functions $r(\ddot{t},\ddot{y},\ddot{\imath},\beta)$ and $m(\ddot{t},\ddot{y},\ddot{z},\ddot{\zeta})$ are continuous, and so the conditions (1)-(2) are satisfied on the order hand, we get that:

$$\left|\mathcal{A}(\ddot{t},\ddot{ч},0)\right| + \int_0^{\ddot{ч}}\int_{\mathrm{e}} \left|\mathcal{B}\left(\ddot{t},\ddot{ч},\ddot{\imath},\beta,0,(\Gamma 0)(\ddot{\imath},\beta)\right)\right| d\beta d\ddot{\imath}$$
$$\leq \beta e^{\lambda(\ddot{t}+|\ddot{ч}|)}$$
$$\frac{\lambda}{13}e^{\lambda(\ddot{t}+|\ddot{ч}|)} \leq \beta e^{\lambda(\ddot{t}+|\ddot{ч}|)}$$

Which satisfied the condition (4) of Theorem 1 with $\beta = \frac{\lambda}{13}$ and $\lambda > 0$. Moreover:

$$g(\ddot{t},\ddot{ч})e^{\lambda(\ddot{t}+|\ddot{ч}|)} + \int_0^{\ddot{ч}}\int_{\mathrm{e}} r(\ddot{t},\ddot{ч},\ddot{\imath},\beta)$$
$$\left[e^{\lambda(\ddot{\imath}+|\beta|)} + \int_0^{\ddot{ч}}\int_{\mathrm{e}} m(\ddot{\imath},\beta,\ddot{z},\ddot{\zeta})e^{\lambda(\ddot{z}+|\ddot{\zeta}|)}\, d\ddot{\zeta} d\ddot{z}\right] d\beta d\ddot{\imath}$$
$$\leq \alpha e^{\lambda(\ddot{t}+|\ddot{ч}|)}$$
$$\Rightarrow \frac{\lambda}{13}e^{\lambda(\ddot{t}+|\ddot{ч}|)} + \frac{2\lambda^2}{e^{\lambda}-1}e^{\lambda(\ddot{t}+|\ddot{ч}|)}\int_0^{\ddot{ч}}\int_0^1 \Big[e^{\lambda(\ddot{\imath}+|\beta|)}$$
$$+\frac{\lambda^2}{e^{\lambda}-1}e^{\lambda(\ddot{\imath}+|\beta|)}\int_0^{\ddot{z}}\int_0^1 e^{\lambda(\ddot{z}+|\ddot{\zeta}|)}\, d\ddot{\zeta} d\ddot{z}\Big] d\beta d\ddot{\imath}$$
$$\leq \alpha e^{\lambda(\ddot{t}+|\ddot{ч}|)}$$
$$\Rightarrow \frac{\lambda}{13}e^{\lambda(\ddot{t}+|\ddot{ч}|)} + (e^{\lambda}+1)(e^{\lambda\ddot{ч}}-1)e^{\lambda(\ddot{t}+|\ddot{ч}|)}$$
$$\leq \alpha e^{\lambda(\ddot{t}+|\ddot{ч}|)}$$
$$\left[\frac{\lambda}{13} + (e^{\lambda}+1)(e^{\lambda\ddot{ч}}-1)\right]e^{\lambda(\ddot{t}+|\ddot{ч}|)} \leq \alpha e^{\lambda(\ddot{t}+|\ddot{ч}|)}$$

If we chose $\ddot{ч} = ln\left(\frac{e^{\lambda}+2}{e^{\lambda}+1}\right)\frac{1}{\lambda}, \lambda > 0$

$$\alpha = \frac{\lambda}{13} + (e^{\lambda}+1)(e^{\lambda\ddot{ч}}-1) < 1$$

It follows that the condition (3) of Theorem 1 is satisfied for $\alpha < 1$.

From the results above, we infer that the problem (28) satisfied the all conditions of Theorem 1, which guarantees the problem (28) has a unique solution.

6 CONCLUSIONS

The existence and uniqueness of solutions for the nonlinear Volterra-Fredholm integral equation in two variables have been rigorously established through the application of the Banach contraction principle and the imposition of Lipschitz conditions within a Banach space. The qualitative properties of these solutions have been thoroughly analyzed using specific integral inequalities, providing a comprehensive understanding of their fundamental behavior and structural characteristics. Furthermore, the solutions exhibit continuous dependence on the governing functions and parameters, demonstrating the stability of the model under small perturbations and ensuring its robustness for practical applications. Illustrative examples confirm the mathematical consistency and effectiveness of the proposed framework, highlighting its potential for extension to a broader class of multidimensional nonlinear integral equations and its applicability in complex mathematical modeling scenarios.

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