Design and Analysis of a New Four-Dimensional Chaotic System: Study of Dynamical Properties and Chaos Verification

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Abstract: Presented a new (4D) four-dimensional hyperchaotic system with four nonlinear terms and thirteen positive parameters. The chaotic system is tested through Mathematica was used to confirm the results and prove that the system is super chaotic the complex dynamics of the new system and its basic characteristics for instance Equilibrium Point, attractors, waveform analysis, Lyapunov exponents, sensitive dependent to initial conditions (SDIC) and fractal dimension to prove the chaotic behavior of the system. A change in the initial values leads to a significant change in the chaotic system, which is vulnerable to any change that occurs. As a result of the four equations generated, two fixed points were obtained: F0, F1. These points prove the chaotic nature of the system. Also, two nonnegative Lyapunov values were obtained, which rely on these values to identify the system's sensitivity to the values between the close points. The values were used to generate phase images to prove the randomness of the system. Also, the sensitivity of the key was tested, and it was found to be large enough to resist attackers.

1 INTRODUCTION

Recently, attention has turned to chaotic systems due to their essential role in encryption, such as data and image encryption, due to the connection between encryption and randomness properties. This has necessitated the development of new, high-dimensional systems for encryption applications to prevent data decoding [1]. Chaotic systems differ from hyper-chaotic ones in terms of sensitivity, breakability, complex topology and unpredictability. Hyper-chaotic systems contain two or more Lyapunov exponents. One of the new methods for creating new systems involves using new variables to increase hyper-chaos by changing the initial values, with the values of these variables being fixed [2]. Chaos theory has been applied in various fields, including neural networks, secure communications, neuroscience, and bioengineering. For these reasons, it has been proposed to create complex dynamic systems, analyse the system, and generate attractors in various forms [3] - [5].

To create a new S-box and without the possibility of breaking the sequence of numbers of the matrix formed, resorted to using chaotic systems, especially super-chaotic ones, as they have the utmost importance in encryption and scattering, as the input values change based on the values generated in the S-box of the new system The encryption strength increases with the upsurge in the nonlinearity of the sbox To ensure the strength of the encryption, it is proposed to use new systems to generate the sbox [6]-[10]. any slight change in the values has an enormous impact on the values, which is required to predict the sequence of values and their values through sensitivity measurements, images are encrypted by generating strong and difficult-to-predict keys from the generated random values. More than one key can be used [11], [12]. The system can have no or more than one equilibrium point [13], [14]. This method prevents attacks on data and maintains its integrity, as the unpredictability of encryption helps repel statistical attacks and other attacks due to its high randomness [15] - [17].

AL-hayali et al. [18] proposed a 4D (four-dimensional) hyperchaotic autonomous system. The results obtained indicate that the system exhibits complex dynamics, with Lyapunov values L.E1 = 0.1658, L.V2 = 0, L.V3 = -0.0299, and L.V4 = -1. 1353. In 2022, Huda et al. [19] created a hyperchaotic 4-dimensional system used in S-box for substituting pixels of images. Key characteristics like Lyapunov exponents, equilibrium stability and chaos waveform are examined LV1 = 4.05761, LV2 = 0.347562, LV3 = -3.94257, LV4 = -6.61896.

2 THE CREATION OF THE NOVEL CHAOTIC SYSTEM

A new (4D) four-dimensional hyperchaotic autonomous system containing four nonlinear terms is obtained as follows:

$$\begin{aligned} \frac{dx}{dt} &= hyz - ix - jw + isin(y), \\ \frac{dy}{dt} &= kxz + lx - y - ixExp(z), \\ \frac{dz}{dt} &= mxy - nz + y^2 - pSin(w,) \\ \frac{dw}{dt} &= qyz + rxz - sw + tzExp(x). \end{aligned} \quad (1)$$

Where x,y,z,w, the values are given and these values are called the state of the (4D) four-dimensional hyperchaotic autonomous system, t ∈ R, h = 5.8, i = 3, j = 0.5, k = 0.8, l = 17, m = 29, n = 2.4, o = 9, p = 5, q = 15, r = 4, s = 2.1 and t = 2. are the parameters of system that have positive values. The chosen parameter values of the new (4D) four-dimensional hyperchaotic autonomous system (1) will show the attractor of it. The conditions of the initial value are: x(0) = 0.5, y(0) = 0.4, z(0) = 1.5 and w(0) = 0.6. Based on the emerging attractions, it can be concluded that the new hyper system is generating random values.

3 THE SYSTEM ANALYSIS

Complex dynamics of the new system and basic properties (1) the novel dynamics system has many features explained in this section.

3.1 Fixed Point

Gain from that system (1) has two fixed points:

$$\begin{aligned} 0 &= hyz - ix - jw + isin(y), \\ 0 &= kxz + lx - y - ixExp(z), \\ 0 &= mxy - nz + y^2 - pSin(w), \\ 0 &= jqyz + rxz - sw + tzExp(x). \end{aligned} \quad (2)$$

When h = 5.8, i = 3, j = 0.5, k = 0.8, l = 17, m = 29, n = 2.4, o = 9, p = 5, q = 15, r = 4, s = 2.1 and t = 2, two fixed points become: F0 {x = 0, y = 0, z = 0, w = 0}, F1 {x=0.178759, y=0.266214, z=1.72831, w = 5.84311}

Matrix system of the Jacobian (1), gain:

$$f = \begin{cases} f_1 = \frac{dx}{dt} = hyz - ix - jw + isin(y) \\ f_2 = \frac{dy}{dt} = kxz + lx - y - ixExp(z) \\ f_3 = \frac{dz}{dt} = mxy - nz + y^2 - pSin(w) \\ f_4 = \frac{dw}{dt} = jqyz + rxz - sw + tzExp(x) \end{cases} \quad (3)$$

$$J = \begin{bmatrix} \frac{\partial.f_1}{\partial.x} & \frac{\partial.f_1}{\partial.y} & \frac{\partial.f_1}{\partial.z} & \frac{\partial.f_1}{\partial.w} \\ \frac{\partial.f_2}{\partial.x} & \frac{\partial.f_2}{\partial.y} & \frac{\partial.f_2}{\partial.z} & \frac{\partial.f_2}{\partial.w} \\ \frac{\partial.f_3}{\partial.x} & \frac{\partial.f_3}{\partial.y} & \frac{\partial.f_3}{\partial.z} & \frac{\partial.f_3}{\partial.w} \\ \frac{\partial.f_4}{\partial.x} & \frac{\partial.f_4}{\partial.y} & \frac{\partial.f_4}{\partial.z} & \frac{\partial.f_4}{\partial.w} \end{bmatrix} \quad (4)$$

For fixed point F0. {x. = 0, y. = 0, z. = 0, w. = 0}, the parameters and the F0 resulting from the system in (4) that create The Jacobian matrix as shown below:

$$J = \begin{bmatrix} -3 & 3 & 0 & -0.5 \\ 14 & -1 & -3 & 0 \\ 0 & 0 & -2.4 & -1 \\ 0 & 0 & 2 & -2.1 \end{bmatrix}. \quad (5)$$

To get an eigenvalue, assume $|\lambda I - J0| = 0$ after it is obtained that corresponds to the fixed of the eigenvalues, F0 (0, 0, 0, 0) are respectively as follows: $\lambda_1 =$ -8.55744, $\lambda_2 =$ 4.55744, $\lambda_3 =$ -2.25+1.40624 i , λ_4 =2.25-1.40624i. Thus, fixed F0 (0, 0, 0, 0) is a saddle point. So, at the F0, the system is unstable. Also, F1 is an unstable saddle point that can be proven to be a fixed point.

For fixed point F1{x = 0.178759, y = 0.266214, z = 1.72831, w = 5.84311}and h = 5.8, i = 3, j = 0.5, k = 0.8, l = 17, m = 29, n = 2.4, o = 9, p = 5, q = 15, r = 4, s = 2.1 and t = 2, The result represents the Jacobian matrix.

Can also get the eigenvalues that agree with the fixed point F1 gained by acting as: $\lambda_1 =$ -14.9763, $\lambda_2 =$ 11.906, $\lambda_3 =$ -3.38476, and $\lambda_4 =$ -2.04494.Thus, the fixed point F1{x = 0.168131, y = 2.40114, z = 1.005, w = 10.6654}, }, saddle point. So, the hyperchaotic system proves that the (4D) four-dimensional hyperchaotic autonomous system at these points E1 is unstable, At the point. E1, which is the saddle point, is unstable, as proven by the hyper-chaotic system.

3.2 Dissipativity

System (1) acts as vector notation in equation (3) of this article: it delineates the vector field f on R^4 via the equation to find the divergence:

$$\nabla \cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} + \frac{\partial f_4}{\partial w} \quad (6)$$

Note that$\nabla \cdot f.$ measures the rate of change of volume underflow$\Phi_{t.}$ of f.

Let D represent a region in $R^4$with a smooth boundary and let D(t) = $\Phi_t(t)$, the image of D. under $\Phi_{t.}$, t represents the time of the flow of f. Let V(t) be the volume of D(t),), By Liouville's theorem, can gain:

$$\frac{dV}{dt}\int_{D(t)} (\nabla f) dxdydzdudw \quad (7)$$

From the system (1), generate that

$$\nabla . f = \frac{\partial f1}{\partial x} + \frac{\partial f2}{\partial y} + \frac{\partial f3}{\partial z} + \frac{\partial f4}{\partial w} = \text{-b-1-g-1}<0 \quad (8)$$

Because of the positive constant b and f, substituting (8) into (7) and simplifying, bring

$$\frac{dV}{dt} = \mathrm{b} - 1 - \mathrm{g} - 1 \int_{D(t)} dxdydzdwdv\, du \quad = (-b - 1 - g - 1)V(t) = e^{-8.5}\,\mathrm{v(t)}$$

In differential equations when solving the first-order algebraic equation linear, a unique solution

$$\mathrm{V.(t)} = \mathrm{V.(0)}e^{(-b-1-g-1)t} = \mathrm{V.(0)}e^{-8.6t} \quad (9)$$

According to the equation above (9), understand that V(t) over time any volume should shrink to zero fast. Then, the hyper system shown in system (1) is dissipative. Since system (1) is a dissipative system, that means all orbits in (1). As a result, it is limited to a specific of R^4 that has zero size. This indicates the new system (1) has the asymptotic motion that settles into an attractor.

3.3 Lyapunov Exponents and Dimensions

An average rate of divergence is calculated by calculating the Lyapunov exponent or calculating (convergence) of two nearby trajectories, according to the quantisation measure approach to the sensitive dependence of nonlinear dynamical theory and its dependence on initial conditions. So, the four Lyapunov exponents of the dynamical system (1) nonlinear, h = 5.8, i = 3, j = 0.5, k = 0.8, l = 17, m = 29, n = 2.4, o = 9, p = 5, q = 15, r = 4, s = 2.1 and t = 2, are the parameters.

$$\lambda = \lim_{t\to\infty} \frac{1}{t} ln \frac{|\ \delta X(t)\ |}{|\ \delta X(0)\ |} \quad (10)$$

Where:

- $\delta X(0)$ is an initial small perturbation in the system's state;
- $\delta X(t)$ is the perturbation after time t;
- λ measures the exponential rate of convergence or divergence of adjacent trajectories. Are obtained as follows: LV1= 6.31523, LV2= 0.951044, LV3= -6.51461 and LV4= -9.26106. Applying the above equation yielded fractional results for the Lyapunov dimension values of the new chaotic system. Due to the fractional nature obtained, it was found that the system has nonperiodic orbits, as its nearby paths diverge.

Therefore, this is proof that it is a system with high genuine randomness in the nonlinear system. The system exhibits chaotic characteristics when the most prominent Lyapunov exponent value is nonnegative. Since the LV1 and LV2 Lyapunov exponents are nonnegative, and the other two Lyapunov exponents are negative.Means the system is hyper-chaotic. There is an exemplary characteristic of chaos, it also has a fractal dimension calculated by Lyapunov exponents, the Kaplan-Yorke dimension and DKY can be shown as:

$$\mathrm{D_{KY}} = \mathrm{j} + \frac{1}{|\mathrm{LV_{j+1}}|}\sum_{\mathrm{i=1}}^{\mathrm{j}} \mathrm{LV_i} \quad (11)$$

Where j is the first j Lyapunov exponent, namely, j represents the maximum value of i value which that meets both $\sum_{i=1}^{j} LV > 0$ and $\sum_{i=1}^{j+1} LV_i < 0$ in the same time. Li, the sequence is descending according to the Lyapunov exponents sequence. DKY is the upper bound (Kaplan-Yorke) of the dimension of the system information. For the system in this paper, by observing the values of four Lyapunov exponents, determined that the value of j is 3, and then can be expressed as the Kaplan-Yorke dimension from the above due to LV1+LV2+ LV3+LV4 >0 and LV1+LV2+LV3+LV4 <0, the Lyapunov dimension of the (4D) four-dimensional hyperchaotic autonomous system is:

$$D_{K.Y} = j + \frac{1}{|LV_{j+1}|}\sum_{i=1}^{j} LV_i$$

$$D_{K.Y} = 3 + \frac{1}{|LV_{j+1}|}\sum_{i=1}^{3} LV_i = 3 + \frac{\mathrm{LV_1+LV_2+LV}}{L_4}$$
$$= 3 + \frac{6.31523+0.951044+\ -6.51461}{9.26106} = 3.08116$$

3.4 Phase Portraits

The parameters h = 5.8, i = 3, j = 0.5, k = 0.8, l = 17, m = 29, n = 2.4, o = 9, p = 5, q = 15, r = 4, s = 2.1 and t = 2, . The phase portraits are displayed in figures (1, 2, 3, 4). From the values used within the system equations and the results obtained, observe that the very interesting attractors generated are hyper-chaotic and exhibit dynamic behavior. The Lyapunov exponents value of equations (1) are found to be LV1= 2.88108, LV2= 0.0828833, LV3= -2.48959 and LV4= -25.0176. There are two nonnegative Lyapunov exponent values, and it is obvious that the system is hyperchaotic.

To finish The numerical simulation, the MATHEMATICA program was utilized. dynamical performances of hyper chaotic are frequent and complex in this nonlinear system. Figures 1, 2 display attractors in three dimensions, and the strange attractors in two dimensions are shown in Figures 1, 2. As noticed in Figure 1, the butterfly effect represents the creation of a chaotic representation resembling a butterfly's flight motion through random values.

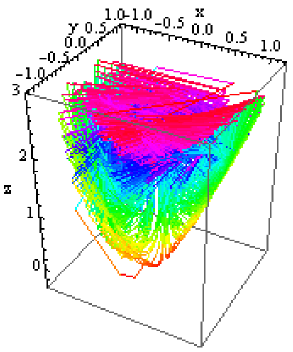

Figure 1: Attractors of hyper chaotic, 3D view (x-y-z).

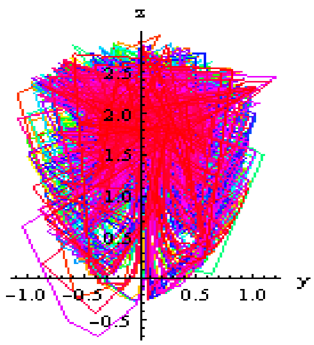

Figure 2: Attractors of chaotic phase plane (y–z).

3.5 Waveform Analysis

To prove that a system is chaotic through waves, the system's waveform must be nonperiodic. Figures 3, 4. From MATHEMATICA will simulate the state plot and through the result show the state plot against the time. the new system waveform. $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ domain of the time shown in Figures 3, 4 are aperiodic. Multiple chaotic motions can exhibit complex behavior and can be distinguished by nonperiodic properties that can be noticed through the time domain.

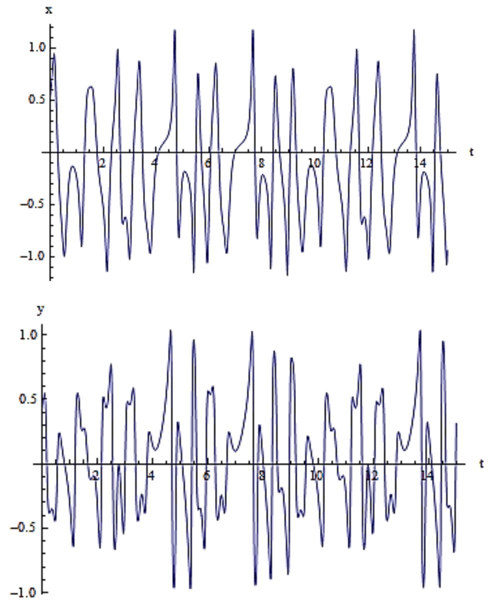

Figure 3: Time vs x, y of hyper chaotic system.

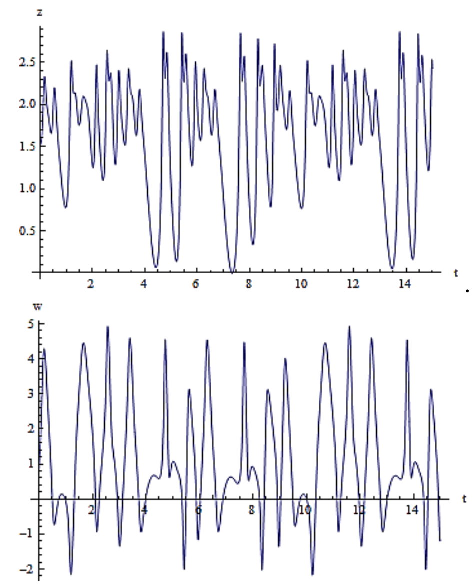

Figure 4: Time vs z, w of hyper chaotic system.

3.6 Bifurcation Diagram

The b values explain differences in behavior in a few values of b, so can find out what happens for a large range of b, Using the maximum value of z and a mathematical simulation program, the behavior of a large range of b can be studied and a numerical bifurcation diagram can be created for it and track of when retained discarded the transients the values. The Figure 5 shows the value of b where b ∈ [4.9,5], the diagram below shows the bifurcation and the enlarge of period doubling in the space 4.9≤ b ≤5.

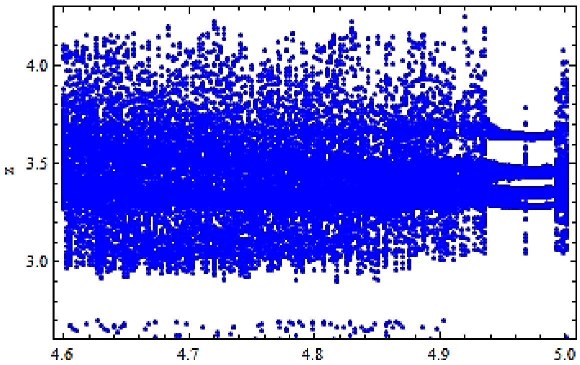

Figure 5: Bifurcation diagram of hyper syetem.

3.7 Initial Conditions Sensitivity

A chaotic system's greatest significant characteristics include its high sensitivity and the inability to predict it in the long term, even with initial conditions. Although there is a slight difference in between them, there are separate results. The state of the new system can reach a state where it is not predicted at a certain time, and depending on the initial conditions, the Figures 6, 7 show how sensitive the system is. The values are initial of the system are set to : x(0) = 0.5, y(0) = 0.9, z(0) = 2.6, w(0) = 3 for the solid line and x(0) = 0.5, y(0) = 0.90000.0000.00001, z(0) = 2.6 and w(0) = 3 for the dashed line.

4 ANALYSES OF THE KEY SPACE

A separate set of keys is used in the encryption process, known as the key space size. A robust algorithm must be sensitive to these secret keys. The key size is critical to reduce the risk of the encryption algorithm being brute force attacked and making it unworkable. A wider key ensures that the attack and the possibility of breaking the key code are reduced, and the key space size should be more than 2^{128} .

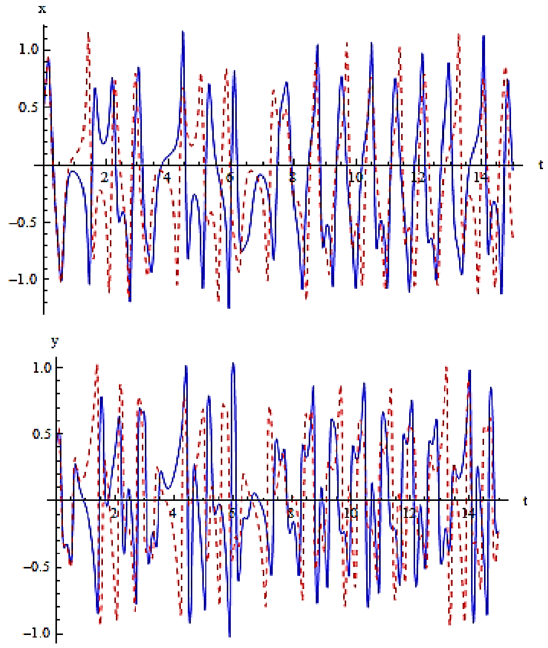

Figure 6: Sensitivity in hyper chaotic system x(t), y(t).

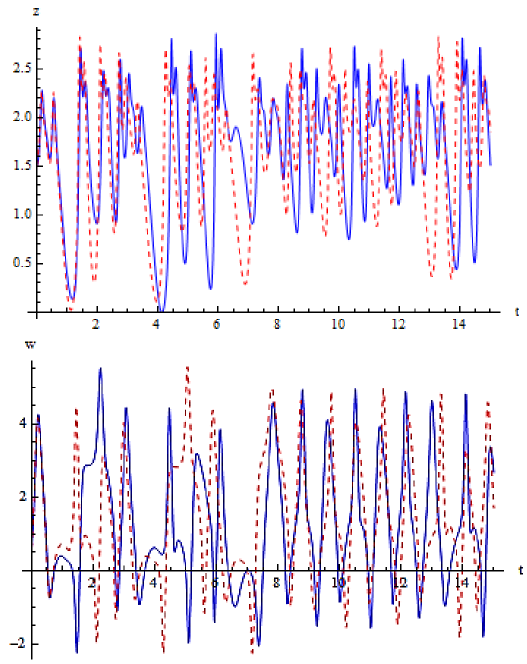

Figure 7: Sensitivity in hyper chaotic system z(t), w(t).

The secret key used in this algorithm is generated from the chaotic system, which has parameters and initial conditions. that mention in create of the hyper system

Used as a secret key, if the precision is 10^{-14} he key space can be reached about (1014)17 = $10^{238} \simeq 2^{757}$,therefore it is bigger 2^{128}The key space turned out large enough to resist brute-force exhaustion attacks.

5 CONCLUSIONS

This work is about creating a (4D) four-dimensional hyperchaotic autonomous system , its properties are presented and analyzed, and tests are conducted to prove its chaotic nature by these values h = 5.8, i = 3, j = 0.5, k = 0.8, l = 17, m = 29, n = 2.4, o = 9, p = 5, q = 15, r = 4, s = 2.1 and t = 2, also The conditions of the initial value as x(0) = 0.5, y(0) = 0.4, z(0) = 1.5 and w(0) = 0.6.

The system consists of four equations that have a Sin and exponential and thirteen different parameters, through which random values are generated and used in encryption or as encryption keys. As a result of the strong results obtained through the tests that providing impressive encryption. The random behavior appears in the form of waves. The system was simulated using Mathematica and its chaotic nature was proven through numerical simulation. Any slight change in the values has an enormous impact on the values, which is required to predict the sequence of values and their values through sensitivity measurements, and two Lyapunov values are nonnegative.

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