Chromaticity of Normal Graphs and Cyclic Graphs

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Abstract: In our work, we will find new results about the chromaticity of Normal graphs and Cyclic Graphs of Dihedral group D_{2n}, ($G_{Nor}D_{2n}$ and $G_{Cyc}D_{2n}$, respectively). We will determine the new theorems about the chromatic uniqueness of these graphs, and we will study all cases of a positive integer number n for $G_{Nor}D_{2n}$ and $G_{Cyc}D_{2n}$. Our primary objective is to establish new theorems regarding the chromatic uniqueness of these graphs, which will significantly enhance the understanding of their structural and combinational characteristics. We will thoroughly analyze all cases of positive integers n to accurately determine the chromatic numbers and systematically explore the implications of these findings within graph theory. This research not only contributes to the theoretical framework of graph theory by enriching its foundational concepts but also opens promising avenues for future interdisciplinary studies examining the intricate interplay between group theory, combinatorial mathematics, and graph properties, thereby advancing both theoretical insights and practical computational applications.

1 INTRODUCTION

During the more than a century that the Four-colour problem was discussed, many ideas were proposed that led to solve this known problem [8]. In 1912, Birkhoff [4] defined a mapping $P\ (M, \lambda)$ which gives the number of proper λ-colourings of a map M. In 1932, The author Whitney [25] is the one who expanded the concept of chromatic polynomials, he transferred the study from maps to graphs. Also he put some basic consequence of this. In 1968, Read [23] asked two questions:

1) What is the sufficient and necessary conditions for two graphs to share the same chromatic polynomial; that means, to be chromatically equivalent?
2) What is the sufficient and necessary conditions for a graph to be chromatically unique?

In particular, Chao and Whitehead Jr. [7] in 1978 defined a graph to be chromatically unique if no other graph shares its chromatic polynomial.

Chromaticity problem of graph means the study and discuss the two questions above. There are many authors worked in this area. Dong, Koh, and Teo [11] published a useful book on chromatic polynomials and chromaticity of graphs. For more details on chromaticity of graphs, we refer to the survey papers [10], [9], [20], [23], and [25].

The literature of algebraic graph theory has grown extremely since 1974, many papers have appeared in this area next. Some researchers studied the colouring of the graph of ring, one of them the famous author, I. Beck. He introduced the idea of a zero-divisor graph in 1988 [3]. While he was mainly interested in colorings. In 2013, Sahib and Khalaf [12] introduce many results about the chromatic uniqueness of some algebraic graph like the zero-divisor graph $\Pi(z_n)$, for the cases, when n is even, $n = p^2$ or $n = p_1 \times p_2$. The chromaticity of $\Pi(z_n)$ is still an open problem for the other cases of n.

In 2022, Al-Janabi and Gabor [2] solved this problem for all cases of n.

Other researchers studied the properties of the graph for group. We will show about some of them. For more information, see these researches [18] and [19].

In 1975-1987, Cavior and Calhoun respectively [5] and [6], compute the subgroups of the dihedral group D_{2n}.

In 2013, Ma presented a definition for the cyclic graph of the Dihedral group and Dicyclic group, where the elements of G are the vertices of $\Gamma(G)$, such that two different vertices x and y are connected by one edge iff $< x, y >$ is a cyclic subgroup of G [25].

In 2022, Ohood and Hayder presented three types of graphs, which are called Normal and Cyclic, denoted by G_{Nor} and G_{Cyc}, respectively [13], [14], [15], [16], and [17].

In our work, we will discuss the new results about the chromatic uniqueness of Normal graphs for Dihedral group $D_{2n}(G_{Nor}D_{2n})$.

2 PRELIMINARIES

All graphs used in this paper are simple and the graph G defined as $G = (V, E)$ such that the vertices set of G are denoted by $V = V(G)$, and $E = E(G)$ is the set edges of G. The degree of vertex $v(v \in V(G))$ in G, is the number of all edges incident to v, and denoted by $d(v)$. A graph G is said to be complete if any tow vertices of G are adjacent and denoted by K_m. The graph $G' = (V', E')$ is subgraph of the graph $G = (V, E)$ if $\emptyset \neq V'(G) \subseteq V(G)$, and $E'(G) \subseteq E(G)$. Tow graphs G and H are isomorphic if there is a bijective function $f: V(G) \longrightarrow V(H)$ with preserves the edges, in other any tow vertices are adjacent in G iff they are adjacent in H, and denoted by $G \cong H$ [1]. The matrix degree sequence of the simple graph G of order n is defined by:

$$\mathcal{M} = \begin{pmatrix} deg(v_1) & deg(v_2) \dots & deg(v_n) \\ \mu(deg(v_1)) & \mu(deg(v_2)) \dots & \mu(deg(v_n)) \end{pmatrix},$$

where $deg(v)$ is the number degree of vertex v in graph G and $\mu(deg(v))$ is the multi degree of vertex v [14]. $P(G, \lambda)$ is the number of distinct $\lambda - colouring$ of G, thus $P(G, \lambda) > 0$ iff G is $\lambda - colouring$. If $P(G, \lambda) = P(H, \lambda)$, then the graphs G and H are chromatically equivalent or $\chi - equivalent$ and denoted by $G \sim H$. Moreover, if the graphs G, H are isomorphic $H \cong G$, then G is chromatically unique or $\chi - unique$.

3 AXILLARY RESULTS

Definition 3.1 [11]: A mapping of the graph G $g: V(G) \longrightarrow V\{1,2, \dots, \lambda\}$, $\lambda \in N$, is defined as $\lambda - coloring$ of G if $g(u) \neq g(v)$, for all vertices u and v are adjacent. The number of $\lambda - colorings$ of G it is indicated by $P(G, \lambda)$.

Proposition 3.2 [22]: For two different vertices x and y in a graph G. $P(G, \lambda)$ is given below:

$$P(G, \lambda) = P(G + xy, \lambda) + P(G \circ xy, \lambda),$$

where $G + xy$ is adding a new edge xy to G, such that $x, y \in V(G)$, and $xy \notin E(G)$. The graph $G \circ xy$ is given by G and identifying tow vertices x and y which incident with e and deleting all the multiple edges (except one), if they appear.

Corollary 3.3 [21]: For an edge in the graph G, $P(G, \lambda)$ is given by:

$$P(G, \lambda) = P(G - uv, \lambda) + P(G \circ uv, \lambda),$$

where the graph $G - uv$ is obtained from G by removing uv.

Corollary 3.4 [11]: For the complete graph on n vertices the chromatic polynomial it is:

$$P(K_n, \lambda) = \lambda(\lambda - 1) \ . \ . \ . \ (\lambda - (n - 1)).$$

Likewise, it is known that $P(\overline{K_n}, \lambda) = \lambda^n$.

Theorem 3.5 [11]: If $G \cap H$ is a complete graph, then

$$P(H \cup G, \lambda) = \frac{P(H, \lambda) \times P(G, \lambda)}{P(H \cap G, \lambda)} \ .$$

Definition 3.6 [24]: A group G generated by two elements a and b with given specific relations is defined a Dihedral group.

$$G = \{\langle a, b \rangle : |a| = 2, |b| = n, ba^{-1} = ab \ if \ e = a \ or \ ab = ba^{n-1}\}.$$

G is denoted by D_n or D_{2n} and called Dihedral group of degree n or $2n$.

$$D_{2n} = \{e, a, a^2, \dots, a^{n-1}, b, ab, a^2b, \dots, a^{n-1}b \},$$

where $a^n = e$, $b^2 = e$, $bab = a^{-1}$.

Definition 3.7 [24]: A subgroup H of be a group G is a subset of G and denoted by $H \leq G$, such that H is also a group with the operation of group G.

Definition 3.8 [24]: Let $H \subset G$, then H is called a normal subgroup of G and denoted by $H \trianglelefteq G$ if H is closed with respect to conjugates, in other words, for every $a \in H$ and $x \in G, xax^{-1} \in H$.

Definition 3.9 [17]: We have G is a finite group. The cyclic graph of group G is denoted by G_{Cyc}, it whose two sets, the set $E\big(G_{Cyc}\big) = \{(a, b) | \langle a, b \rangle \leq G\}$ is an edge set, and $V\big(G_{Cyc}\big) = \{a | \ a \in G\}$ is a vertex set.

Definition 3.10 [14]: Let G be a finite group. The normal graph of G is denoted by G_{Nor}, and it is defined as $E(G_{Nor}) = \{(a, b) | \langle a, b \rangle \trianglelefteq G\}$ the set edges, and $V(G_{Nor}) = \{a : a \in G\}$ the set of vertices.

Proposition 3.11 [14]: The matrix degree sequence for the normal graph of the Dihedral group D_{2n} is given by the following: If n is even positive integer number, then the matrix is:

$$\mathcal{M}(G_{Nor}D_{2n}) = = \begin{pmatrix} 2(\varphi(n)+\varphi(\frac{n}{2})) & 2n-1 & n-1 \\ n & \varphi(n)+\varphi(\frac{n}{2}) & n-(\varphi(n)+\varphi(\frac{n}{2})) \end{pmatrix}$$

Proposition 3.12 [17]: The matrix degree sequence for the cyclic graph of Dihedral group D_{2n} is given by the following:

$$\mathcal{M}(G_{Cyc}D_{2n}) = \begin{pmatrix} 2n-1 & n-1 & 1 \\ 1 & n-1 & n \end{pmatrix},$$

where n is a positive integer number.

Definition 3.13 [1]: Let the complete graph K_n, such that $M = V(K_n)$, and L_k is subset of M, $\forall k \le t$ and $t \in N$. The t-clique-join graph (simply, t-CJ-graph) is obtained from t arbitrary graphs $Z_1, Z_2, \dots, Z_t$, on pairwise disjoint vertex sets by joining every vertex in Z_k with all vertices of L_k, where $Z_k \cap M = \phi$. It is denoted by $F = F\ (M, L_1, L_2, \dots, L_t, Z_1, Z_2, \dots, Z_t)$. $\mathfrak{I} = \mathfrak{I}(M, f_1, f_2, \dots, f_t, Z_1, Z_2, \dots, Z_t)$ is the set of all such graphs with $|L_1| = f_1, |L_2| = f_2, \dots, |L_t| = f_t$. For example, $F \in \mathfrak{I}$.

Corollary 3.14 [1]: We have an empty graph it is Z_k of order a_k, and let L_k be the clique subgraphs of order f_k, $L_k \subset M$, $1 \le k \le 2$ and $f_1 \ne f_2$. Let $G \in \mathfrak{I}(M, f_1, f_2, Z_1, Z_2)$ be a 2-CJ-graph, $|M| = m$, then the chromatic polynomial of G is:

$$P(G,\lambda) = \lambda(\lambda-1)\dots(\lambda-f_1)^{a_1+1}\dots(\lambda - f_2)^{a_2+1}\dots(\lambda-m+1).$$

Theorem 3.15 [1]: Let G be connected graph with chromatic polynomial of an integral-root and P be the integral-root, such that P has some roots with exponent at least 3, then G it is not $\chi - unique$.

Theorem 3.16 [1]: Let P be an integral-root chromatic polynomial of a connected graph G, such that P has one root only with exponent 2 and no more roots with the exponent, then it is $\chi - unique$.

Notion 3.17 [22]: We will denote by G_{Nor-e} for the graph $G_{Nor}D_{2n} \setminus e$, where $V(G_{Nor-e}) = \{a, a^2, \dots, a^{n-1}, b, ab, \dots, a^{n-1}b\}$.

Lemma 3.18 [22]: If $G_{Nor}D_{2n}$ be a normal graph of D_{2n}, then there is complete subgraph $K_{\frac{3n}{2}-2}$ of $G_{Nor}D_{2n}$, where $n \ge 6$ is an even number.

Theorem 3.19 [22]: Let n be an odd positive integer number. Then the chromatic polynomial for a normal graph of the Dihedral group D_{2n} is given by:

$$P(G_{Nor}D_{2n},\lambda) = \lambda(\lambda-1)\dots(\lambda-(n-1))^2\dots(\lambda - (2n-2)).$$

Theorem 3.20 [22]: The chromatic polynomial of $G_{Nor}D_{2n}$, where n be an even positive integer number, $n \ge 6$, is given by:

$$P(G_{Nor}D_{2n},\lambda) = \lambda(\lambda-1)\dots(\lambda-(n-2))^2(\lambda - (n-1))^2\dots(\lambda-(\tfrac{3n}{2}-3))P(G,\lambda).$$

4 THE MAIN AIM

4.1 Chromatic Uniqueness of Normal Graphs

Theorem 4.1.1: Let $G_{Nor}D_{2n}$ be a normal graph of the Dihedral group D_{2n}, if n is an odd positive integer number, then $G_{Nor}D_{2n}$ is $\chi - unique$.

Proof: If n is odd positive integer number, then (by Theorem 3.19) $P(G_{Nor}D_{2n},\lambda)$ is given by $P(G_{Nor}D_{2n},\lambda) = \lambda(\lambda-1)\dots(\lambda-t)^2\dots(\lambda-(2n-2))$, since this polynomial has one root only of exponent 2, then (by Theorem 3.16) the graph $G_{Nor}D_{2n}$ is $\chi - unique$, where n is an odd number.

Theorem 4.1.2: Let $G_{Nor}D_{2n}$ be a normal graph of the Dihedral group D_{2n}, if $n \ge 6$ is an even number, then $G_{Nor}D_{2n}$ is not $\chi - unique$.

Proof: By (Proposition 3.11), we get:

$$\mathcal{M}(G_{Nor}D_{2n} \setminus a^j b = \{e, \mathrm{a}^{\mathrm{i}}\}) = \begin{pmatrix} n-1 & n-1 \\ \varphi(n)+\varphi(^n/_2) & n-\left(\varphi(n)+\varphi(^n/_2)\right) \end{pmatrix} = \begin{pmatrix} n-1 \\ n \end{pmatrix} = V(K_n),\ 0 \le j \le n-1,\ 1 \le i \le n-1,$$

such that $V(K_n) = \{e, \mathrm{a}^{\mathrm{i}}\}$, $1 \le i \le n-1$, then there is a complete subgraph $K_{\frac{3n}{2}-2}$ of $G_{Nor}D_{2n}$ (by Lemma 3.18), and $V(K_{\frac{3n}{2}-2}) = \{b, ab, \mathrm{a}^2 b, \dots, \mathrm{a}^{\frac{n}{2}-2}b, \mathrm{a}^{n-1}b, \mathrm{a}^{\mathrm{i}}\}$, $i \ne \frac{n}{2}$. Since the vertex $e \in V(K_n)$, which is connected by one edge to the all vertices $\{\mathrm{a}^{\mathrm{i}}\} \in V(K_n)$, $1 \le i \le n-1$. Since the vertex b is not adjacent with the vertex e. So, if we change the edge $\{e, \mathrm{a}^{\frac{n}{2}}\}$ with the edge $\{e, \mathrm{b}\}$, we get the graph H, such that $H \ncong G_{Nor}D_{2n}$, because the vertices $e, \mathrm{a}^{\frac{n}{2}}$ are adjacent in the graph $G_{Nor}D_{2n}$ and not in the graph H. Then (By Theorem 3.20), $P(G_{Nor}D_{2n},\lambda)$ is given by:

$$P(G_{Nor}D_{2n},\lambda) = \lambda(\lambda-1)\dots(\lambda-(n-2))^2(\lambda - (n-1))^2\dots(\lambda-(\tfrac{3n}{2}-3))P(G,\lambda).$$

Now, in the graph H, the vertex e is incident to the vertices $\{b, \mathrm{a}^{\mathrm{i}}\} \in V(K_{\frac{3n}{2}-2})$, $i \ne \frac{n}{2}$, such that

$\{a^i\} \in V(K_{n-2})$, $i \neq \frac{n}{2}$, then $P\left(K_{\frac{3n}{2}-2} + e, \lambda\right)$ is given by:

$$P\left(K_{\frac{3n}{2}-2} + e, \lambda\right) = \lambda(\lambda-1)\ldots(\lambda-(n-2))\,(\lambda-(n-1))^2\ldots(\lambda-(\tfrac{3n}{2}-3)).$$

Since the vertex $a^{\frac{n}{2}} \in H$ is also incident to the vertices $\{a^i\} \in V(K_{n-2})$, $i \neq \frac{n}{2}$, then we get the polynomial $P\left(K_{\frac{3n}{2}-2} + e + a^{\frac{n}{2}},\ \lambda\right) = \lambda(\lambda-1)\ldots(\lambda-(n-2))^2\,(\lambda-(n-1))^2\ldots(\lambda-(\frac{3n}{2}-3))$, and $K_{\frac{3n}{2}-2} + e + a^{\frac{n}{2}}$ is a subgraph of H then the polynomial of H is:

$$P(H,\lambda) = \lambda(\lambda-1)\ldots(\lambda-(n-2))^2(\lambda-(n-1))^2\ldots(\lambda-(\tfrac{3n}{2}-3))P(G,\lambda).$$

Since $H \ncong G_{Nor}D_{2n}$, and $P(G_{Nor}D_{2n},\lambda) = P(H,\lambda)$, then the graph $G_{Nor}D_{2n}$ is not $\chi - unique$, where $n \geq 6$ is an even number (see Figure 1).

Example 4.1.3: Let $G_{Nor}D_{2n}$ be a graph, where $n = 8$, with vertex set of $G_{Nor}D_{16}$ is $\{e, a, a^2, a^3, a^4, a^5, a^6, a^7, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b\}$ (Fig. 2). It is clear that $V(K_{\frac{3n}{2}-2}) = V(K_{10}) = \{a, a^2, a^3, a^5, a^6, a^7, b, ab, a^2b, a^7b\}$ and $V(K_{n-1}) = V(K_7) = \{a, a^2, a^3, a^4, a^5, a^6, a^7\}$, the vertex $e \in G_{Nor}D_{16}$ is incident to all vertices of K_7 but the vertex $e \in H$ is incident to all vertices of $V(K_6) = \{a, a^2, a^3, a^5, a^6, a^7\}$ and the vertex $b \in K_{\frac{3n}{2}-2}$, then we can apply Theorem 4.1.2, in this graph we used Maple Program to get the following chromatic polynomial (Fig. 2):

$$P(G_{Nor}D_{16},\lambda) = P(H,\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)^2(\lambda-7)^2(\lambda-8)(\lambda-9)(\lambda^4-42\lambda^3+665\lambda^2-4700\lambda+12501).$$

Proposition 4.1.4: Let $G_{Nor}D_8$ be a normal graph of the Dihedral group D_8, where $n = 4$, then it is $\chi - unique$.

Proof: For a graph $G_{Nor}D_{2n}$ with vertex set is $\{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$, and $n = 4$ (see Fig. 3), $P(G_{Nor}D_8,\lambda)$ is given by:

$$P(G_{Nor}D_8,\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)^2(\lambda-4)(\lambda-5)(\lambda-6).$$

Then, (by Theorem 3.7) the polynomial of $G_{Nor}D_8$ has one root of multiplicity 2, so $G_{Nor}D_8$ is $\chi - unique$, where $n = 4$.

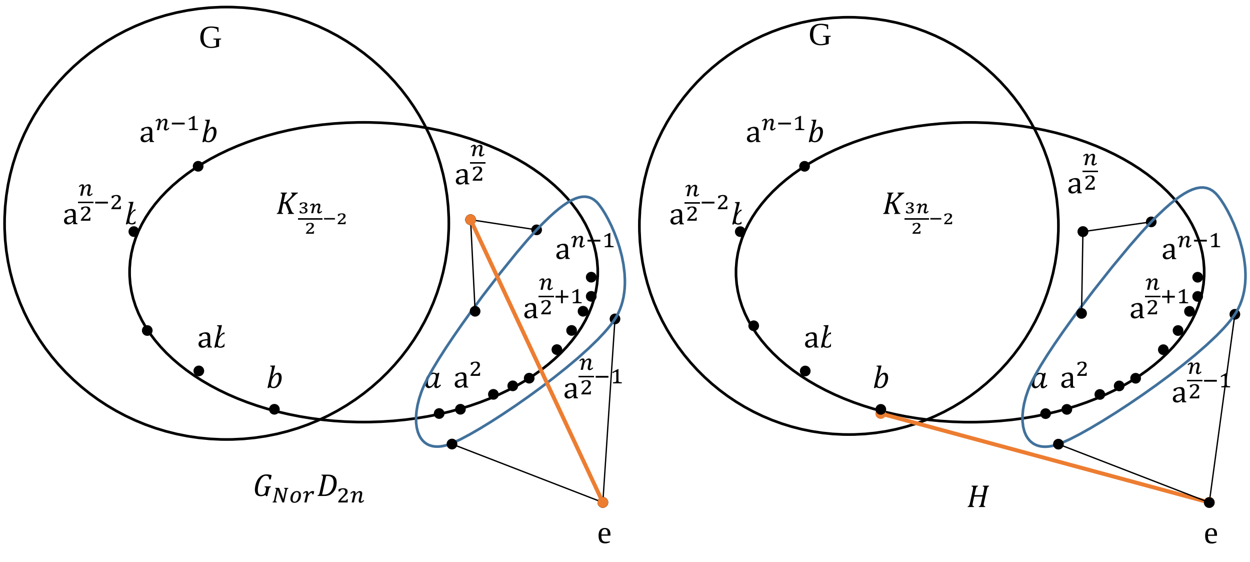

Figure 1: $G_{Nor}D_{2n}$ and H are chromatically equivalent.

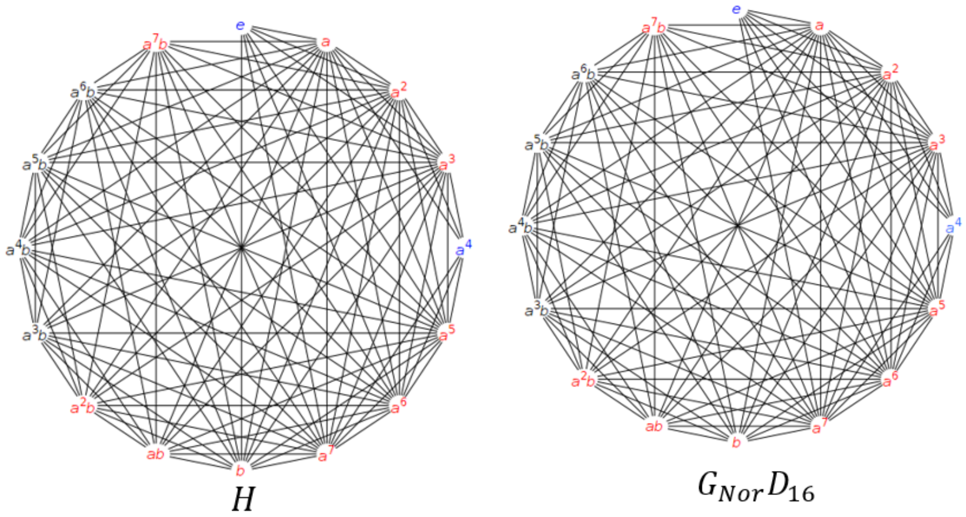

Figure 2: $G_{Nor}D_{16}$ and H are $\chi - equivalent$ but non-isomorphic graphs.

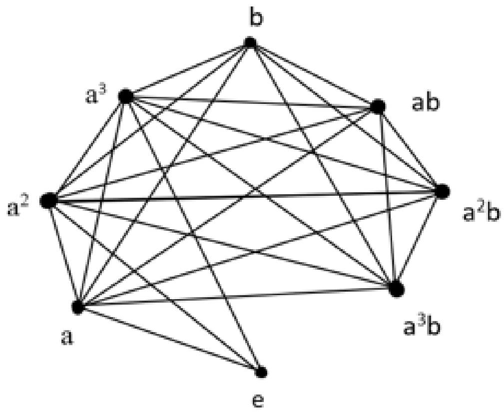

Figure 3: Normal graph of D_8 .

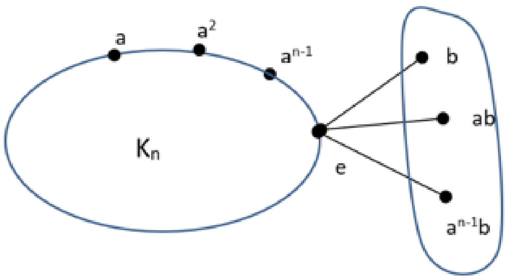

Figure 4: Cyclic graph of D_{2n}.

4.2 Chromatic Uniqueness of Cyclic Graphs

Lemma 4.2.1: For a cyclic graph of D_{2n}. If n is positive integer number, then

$$G_{Cyc}D_{2n} \setminus \{b, ab, \dots, a^{n-1}b\} = \{e, a, a^2, \dots, a^{n-1}\} \cong K_n.$$

Proof: Let $D_{2n} = \{e, a, a^2, \dots, a^{n-1}, b, ab, a^2b, \dots, a^{n-1}b\}$ and by (Proposition 3.12)

$$\mathcal{M}(\mathrm{G_{Cyc}D_{2n}}) = \begin{pmatrix} 2n-1 & n-1 & 1 \\ 1 & n-1 & n \end{pmatrix}.$$

$deg(e) = 2n-1$, $\mu(deg(e)) = 1$, $deg(a^i) = n-1$, $\mu(deg(a^i)) = n-1$, $deg(a^i b) = 1$ and $\mu(deg(a^i b)) = n$, $\mathrm{i} = 1,2,\dots,\mathrm{n}-1$. Since $a^i b$ incident to 1 vertex is e then in $G_{Cyc}D_{2n} \setminus \{b, ab, \dots, a^{n-1}b\}$, $deg(e) = n-1$. Then

$$\mathcal{M}(\mathrm{G_{Cyc}}D_{2n} \setminus \{b, ab, \dots, a^{n-1}b\}) = \begin{pmatrix} n-1 & n-1 \\ 1 & n-1 \end{pmatrix} = \begin{pmatrix} n-1 \\ n \end{pmatrix} = V(K_{\mathrm{n}}).$$

Theorem 4.2.2: Let $n \geq 2$ be a positive integer number. Then the chromatic polynomial in cyclic graph of the Dihedral group D_{2n} is given as:

$$P\big(G_{Cyc}D_{2n}, \lambda\big) = \lambda(\lambda-1)^{n+1} \dots (\lambda-(n-1))$$

Proof: Let D_{2n} be Dihedral group of order $2n$ such that n is positive integer number, and $D_{2n} = \{e, a, a^2, \dots, a^{n-1}, b, ab, a^2b, \dots, a^{n-1}b\}$ and by (Lemma 4.2.1), then $G_{Cyc}D_{2n} \setminus \{b, ab, \dots, a^{n-1}b\} = \{e, a, a^2, \dots, a^{n-1}\} \cong K_n$, and the chromatic polynomial of K_n is given by the following:

$$P(K_n, \lambda) = \lambda(\lambda-1)\dots(\lambda-(n-1)).$$

Since $a^i b$ incident to one vertex (e), then by (Corollary 3.14), see (Figure 4)

$$P\big(G_{Cyc}D_{2n}, \lambda\big) = \lambda(\lambda-1)^{n+1} \dots (\lambda-(n-1)).$$

Example 4.2.3: Let $G = D_{10} = \{e, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\}$, where $n = 5$. By (Theorem 4.2.2) the chromatic polynomial of this graph is given by:

$$P(G_{Cyc}D_{10}, \lambda) = \lambda(\lambda-1)^6(\lambda-2)(\lambda-3)(\lambda-4).$$

It's clear that, there is a subgraph K_5, and the vertex e in K_5, it incident to $\{b, ab, a^2b, a^3b, a^4b\}$.

Theorem 4.2.4: Let $G_{Cyc}D_{2n}$ be a cyclic graph of the Dihedral group D_{2n}, if n is positive integer number, then it is not $\chi - unique$.

Proof: If n is positive integer number, and the vertex set is:

$$V(G_{Cyc}D_{2n}) = \{e, a, a^2, \dots, a^{n-1}, b, ab, a^2b, \dots, a^{n-1}b\},$$

then by (Theorem 4.2.2) the chromatic polynomial of the cyclic graph is:

$$P\big(G_{Cyc}D_{2n}, \lambda\big) = \lambda(\lambda-1)^{n+1} \dots (\lambda-(n-1)),$$

then (by Theorem 3.15) the cyclic graph, $G_{Cyc}D_{2n}$, is not $\chi - unique$, where $n = p$.

5 CONCLUSIONS

In this study, we explored new results concerning the chromaticity of Normal graphs and Cyclic Graphs associated with the Dihedral group D_{2n} (G_Nor D_{2n} and G_Cyc D_{2n}, respectively). We successfully established new theorems related to the chromatic uniqueness of these graphs, providing detailed analyses for all cases of positive integers n. Our findings significantly enhance the existing understanding of the structural and combinational attributes of these graphs, accurately determining their chromatic numbers and elucidating critical insights into their properties. The results presented contribute substantially to the theoretical foundations of graph theory, facilitating a deeper comprehension of graph coloring concepts. Furthermore, this research opens exciting pathways for future investigations, encouraging interdisciplinary approaches that integrate group theory, combinatorial mathematics, and advanced graph analytics. Ultimately, these theoretical advancements pave the way for novel computational applications and provide valuable frameworks for further academic exploration.

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